First Homework

Due: Thursday, February 16

1. In $\mathbb{R}^4$, let $\mathbf{a} = (-1,1,-1,1)$, $\mathbf{b} = (0,2,0,0)$ and $c = (0,1,2,2)$

   (a) Compute $3\mathbf{a} - 4\mathbf{b} + \mathbf{c}$. $= (-3,3,-3,3) + (0,-8,0,0) + (0,1,2,2)$

   $= (-3,-4,-1,5)$

   (b) Find $|\mathbf{a}|$, $|\mathbf{b}|$, and $|\mathbf{c}|$.

   $|\vec{a}| = \sqrt{(-1)^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{4} = 2$

   $|\vec{b}| = \sqrt{0^2 + 2^2 + 0^2 + 0^2} = \sqrt{4} = 2$

   $|\vec{c}| = \sqrt{0^2 + 1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

   (c) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

   $\vec{a} \cdot \vec{b} = 2$ If θ is the angle then: $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{2}{2\cdot 2} = \dfrac{1}{2}$

   So $\theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

2. In $\mathbb{R}^3$, find the equation of a plane that passes through the origin, and is parallel to both vectors $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ $= \vec{a}$ and $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. $:= \vec{b}$

The plane will be orthogonal to $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= (2-6)\vec{i} - (-6-2)\vec{j} + (9+1)\vec{k}$$

$$= -4\vec{i} + 8j + 10\vec{k}$$

Let $\vec{x} = (x,y,z)$, be on the plane. Then $\vec{x} \cdot (\vec{a} \times \vec{b}) = 0 \iff -4x + 8y + 10z = 0$

$\iff \boxed{2x - 4y + 5y = 0}$

3. Use the method or row operations to calulate the following determinant:

$$\begin{vmatrix} 0 & 8 & 3 & -4 \\ -1 & 2 & -2 & 5 \\ -2 & 8 & 4 & 3 \\ 0 & -4 & 2 & -3 \end{vmatrix} = - \begin{vmatrix} -1 & 2 & -2 & 5 \\ 0 & 8 & 3 & -4 \\ -2 & 8 & 4 & 3 \\ 0 & -4 & 2 & -3 \end{vmatrix} \times -2$$

$$= - \begin{vmatrix} -1 & 2 & -2 & 5 \\ 0 & 8 & 3 & -4 \\ 0 & 4 & 8 & -7 \\ 0 & -4 & 2 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 & -2 & 5 \\ 0 & 4 & 8 & -7 \\ 0 & 8 & 3 & -4 \\ 0 & -4 & 2 & -3 \end{vmatrix} \times -2 \quad \times 1$$

$$= \begin{vmatrix} -1 & 2 & -2 & 5 \\ 0 & 4 & 8 & -7 \\ 0 & 0 & -13 & 10 \\ 0 & 0 & 10 & -10 \end{vmatrix} \times 13$$

$$= \frac{1}{13} \begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 8 & -7 \\ 0 & 0 & -13 & 10 \\ 0 & 0 & 130 & -130 \end{vmatrix} \times 10$$

$$= \frac{1}{13} \begin{vmatrix} 1 & 2 & -2 & 5 \\ 0 & 4 & 8 & -7 \\ 0 & 0 & -13 & 10 \\ 0 & 0 & 0 & -30 \end{vmatrix}$$

$$= \frac{1}{13} \cdot 4 \cdot (-13) \cdot (-30)$$

$$= \boxed{-120}$$

4. What is the area of the paralellogram whose sides are the vectors $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$, and $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$?

$$A = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 6 - 6 = 0$$

5. What is the volume of the parallepiped whose sides are the vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 6\mathbf{k}$?

$$\begin{vmatrix} 3 & -1 & 1 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{vmatrix} = -\begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 1 \\ 2 & 0 & 6 \end{vmatrix} \times 2$$

$$= -\frac{1}{3}\begin{vmatrix} 2 & 5 & 1 \\ 6 & -2 & 2 \\ 2 & 0 & 6 \end{vmatrix} \times -3 \quad \times -1$$

$$= -\frac{1}{2}\begin{vmatrix} 2 & 5 & 1 \\ 0 & -17 & -1 \\ 0 & -5 & 5 \end{vmatrix} \times -1$$

$$= \frac{1}{2}\begin{vmatrix} 2 & 5 & 1 \\ 0 & 17 & 1 \\ 0 & -5 & 5 \end{vmatrix} \times 17$$

$$= \frac{1}{34}\begin{vmatrix} 2 & 5 & 1 \\ 0 & 17 & 1 \\ 0 & -85 & 85 \end{vmatrix} \times 5$$

$$= \frac{1}{34}\begin{vmatrix} 2 & 5 & 1 \\ 0 & 17 & 1 \\ 0 & 0 & 90 \end{vmatrix}$$

$$= \frac{1}{34} \cdot 2 \cdot 17 \cdot 90$$

$$= 90$$

So volume is $\boxed{90}$.