BRONX COMMUNITY COLLEGE
of the City University of New York

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 35 — Take Home Part of Final
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Directions: Write your answers in a separate piece of paper. Please staple all sheets together. To get full credit you *must* show all your work. Simplify your answers whenever possible. Be certain to indicate your final answer clearly.

1. Consider the following matrix:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

   (a) Find the inverse matrix A^{-1}.

   (b) Use the inverse matrix to solve the system;

$$\begin{cases} x - 2y + 3z &= 1 \\ -x + y - 2z &= -2 \\ 2x - y - z &= -1 \end{cases}$$

2. Use row operations to find the determinant of the following matrix:

$$\begin{pmatrix} 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \\ 10 & 20 & 30 & 40 \end{pmatrix}$$

3. Consider the matrix:

$$A = \begin{pmatrix} 2 & 9 & 4 & 6 \\ 5 & 22 & 9 & 14 \\ 1 & 4 & 1 & 2 \end{pmatrix}$$

   (a) Compute the rank and nullity of A.

   (b) Give a basis for the range and the null space of A.

   (c) Use these calculations to describe the solution set of each of the systems:

$$\begin{cases} 2x + 9y + 4z + 6w &= 10 \\ 5x + 22y + 9z + 14w &= 28 \\ x + 4y + z + 2w &= 4 \end{cases} \qquad \begin{cases} 2x + 9y + 4z + 6w &= 10 \\ 5x + 22y + 9z + 14w &= 24 \\ x + 4y + z + 2w &= 4 \end{cases}$$

4. Consider the 2×2 matrix

$$A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

(a) Find the eigenvalues and the corresponding eigenvectors of A.

(b) Use the result of part (a) to solve the system of linear differential equations:

$$\begin{cases} y_1' &= 3y_1 + 4y_2 \\ y_2' &= -y_1 - 2y_2 \end{cases}$$

subject to the initial conditions $y_1(0) = 1,\ y_2(0) = 2$.

5. Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a given mapping and write

$$f(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$$

. Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $g(u, v, w) = (u - v, u + w, w + v)$ and let $h = g \circ f$.

(a) Write a formula for the derivative matrix $\mathbf{D}h$.

(b) Show that $\mathbf{D}h$ cannot have rank 3 at any point (x, y, z).

(c) Show that $\mathbf{D}h$ has an eigenvalue zero at every (x, y, z).

6. Use Lagrange multipliers to find the maximum and minimum values of

(a) $f(x, y, z) = x^4 + y^4 + z^4$ subject to the constraint $x^2 + y^2 + z^2 = 1$

(b) $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $4x^2 + 2y^2 + z^2 = 4$

(c) $f(x, y, z) = x$ subject to the constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$

7. Suppose that x and y are functions of a and b, given by

$$x = a + b^2, \quad y = ab + 2b^3.$$

(a) Use the Inverse Function Theorem to decide near which points (a, b) can we solve for a and b in terms of x and y.

(b) Show that we can't solve for a and b in terms of x and y near $(a, b) = (-4, -1)$.

(c) Show that we can solve near $(a, b) = (1, 1)$, and compute

$$\frac{\partial(a, b)}{\partial(x, y)}$$

at the point $(x, y) = (2, 3)$

8. Let S be the part of the cone $z^2 = x^2 + y^2$ lying above the unit square $0 \le x, y \le 1$. Find the surface area of S.

9. Let $\mathbf{G}(x, y) = \left(xe^{x^2+y^2} + 2xy\right)\mathbf{i} + \left(ye^{x^2+y^2} + x^2\right)\mathbf{j}$

(a) Show that $\mathbf{G} = \nabla f$ for some f; find such an f

(b) Use part (a) to show that the line integral of $\mathbf{G}$ around the edges of the triangle with vertices $(0, 0), (0, 1), (1, 0)$ is zero.

(c) State Greens theorem for the triangle part (b) and a vector field $\mathbf{F}$ and verify it for the vector field $\mathbf{G}$ above

10. Let W be the three dimensional region under the graph of $f(x,y) = e^{x^2+y^2}$ and over the region in the plane defined by $1 \leq x^2 + y^2 \leq 2$.

   (a) Find the volume of W.

   (b) Find the flux of the vector field $\mathbf{F} = (2x - xy)\,\mathbf{i} - y\,\mathbf{j} + yz\,\mathbf{k}$ out of the region W.

11. Let C be the curve $x^2 + y^2 = 1$ lying in the plane $z = 1$. Let $\mathbf{F} = (z - y)\,\mathbf{i} + y\,\mathbf{k}$.

   (a) Calculate $\nabla \times \mathbf{F}$.

   (b) Calculate $\int_C \mathbf{F} \cdot d\,\mathbf{s}$ using a parametrization of C and a chosen orientation for C.

   (c) Write $C = \partial S$ for a suitably chosen surface S and, applying Stokes theorem, verify your answer in b

   (d) Consider the sphere with radius $\sqrt{2}$ and center the origin. Let S' be the part of the sphere that is above the curve (i.e., lies in the region $z \geq 1$), and has C as boundary. Evaluate the surface integral of $\nabla \times \mathbf{F}$ over S'. Specify the orientation you are using for S'.

12. Let $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

   (a) Prove that $\mathbf{F}$ is a conservative vector field.

   (b) Let γ be the path $t \to \left(\sqrt{1+t^2} + \sqrt[3]{1+t^3} + \sqrt[4]{1+t^4}\right)$, defined for $0 \leq t \leq 1$. Evaluate

$$\int_\gamma \mathbf{F} \cdot d\,\mathbf{s}$$

13. Consider the region D bound by the curves $y = x^2 - 6$ and $y = 2 - 2x$. Let C by the boundary curve of D oriented counter clockwise. Verify Green's theorem for the integral

$$\int_C -y\,dx + x\,dy$$

14. Let C be the intersection of the surfaces $z = 2x$ and $x^2 + y^2 = 4$, oriented counterclockwise as seen from above. Verify Stoke's theorem for the following integral

$$\int_C (1 - 4z)\,dx + 2x\,dy + (1 - 5y)\,dz$$

15. Let R be the part of the solid ball $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = -1$, and let $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. Consider the boundary of the surface to be oriented by the outward pointing normal vector. Verify Gauss' theorem for the integral

$$\int_R \mathbf{F} \cdot d\mathbf{S}$$