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1. Eigenvalues and Eigenvectors

Definition 1. If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, we say that the *non-zero* vector $\mathbf{v} \in \mathbf{R}^n$ is an *eigenvector* of T with *eigenvalue* λ, if $T(\mathbf{v}) = \lambda\mathbf{v}$.

So an eigenvector is a vector that is mapped to a multiple of itself. Of course since $T(\mathbf{0}) = \mathbf{0}$, the zero vector always has that property (for all λ), but we are interested in non-trivial solutions to this problem.

Let $\lambda \in \mathbf{R}$, we want to know if λ is an eigenvalue of T. In other words, whether the equation

$$T(\mathbf{v}) = \lambda\mathbf{v}$$

has non-zero solutions. If A is the matrix of T we can write this equation as

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} &\iff A\mathbf{v} = \lambda(I\mathbf{v}) \\ &\iff A\mathbf{v} - \lambda(I\mathbf{v}) = \mathbf{0} \\ &\iff (A - \lambda I)\,\mathbf{v} = \mathbf{0} \end{aligned}$$

So λ is an eigenvalue if and only if the homogeneous system $(A - \lambda I)\,\mathbf{v} = \mathbf{0}$ has non-zero solutions. This can happen if and only if the matrix $A - \lambda I$ is singular[1]. So we have:

Proposition 2. *λ is an eigenvalue of T if and only* $\det(A - \lambda I) = 0$.

Example 3. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Then find the eigenspace of each eigenvalue.

Answer. We have:

$$A - \lambda I = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}$$

So

$$\det(A - \lambda I) = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4$$

So the eigenvalues of A are the solutions of the equation:

$$\lambda^2 - 5\lambda + 4 = 0 \iff \lambda = 1, \text{ or } \lambda = 4$$

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[1] Why?

The eigenspace that corresponds to the eigenvalue $\lambda = 1$, is the space of solutions of the homogeneous system

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution has parametric expression $\mathbf{x} = (-2t, t), \quad t \in \mathbf{R}$ or, $\mathbf{x} = t(-2, 1)$. So the eigenspace for $\lambda = 1$ is $\langle(-2, 1)\rangle$.

The eigenspace that corresponds to the eigenvalue $\lambda = 4$, is the space of solutions of the homogeneous system

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution has parametric expression $\mathbf{x} = (t, t), \quad t \in \mathbf{R}$ or, $\mathbf{x} = t(-2, 1)$. So the eigenspace for $\lambda = 4$ is $\langle(1, 1)\rangle$. □

Example 4. Find the eigenvalues and eigenspaces of the matrix

$$\begin{pmatrix} -3 & 4 \\ 3 & 1 \end{pmatrix}$$

Answer. The eigenvalues are the solutions of the equation:

$$\det \begin{pmatrix} -3-\lambda & 4 \\ 3 & 1-\lambda \end{pmatrix} = 0$$

Or, equivalently

$$(-3-\lambda)(1-\lambda) - 12 = 0 \Longleftrightarrow \lambda^2 + 2\lambda - 15 = 0 \Longleftrightarrow \lambda = 3, \text{ or } \lambda = -5$$

The eigenspace for $\lambda = 3$ is the space of solutions to:

$$\begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution has parametric expression $\mathbf{x} = (3t, 2t), \quad t \in \mathbb{R}$, or $\mathbf{x} = t(3, 2)$. So the eigenspace for $\lambda = 3$ is $\langle(2, 3)\rangle$.

The eigenspace for $\lambda = -5$ is the space of solutions to:

$$\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The solution has parametric expression $\mathbf{x} = (t, 2t), \quad t \in \mathbb{R}$, or $\mathbf{x} = t(1, 2)$. So the eigenspace for $\lambda = 5$ is $\langle(1, 2)\rangle$. □

One of the many application of eigenvalues is in solving systems of linear differential equations. Here is a basic example:

Example 5. Solve the system of differential equations:

$$\begin{cases} y_1' &= -3y_1 + 4y_2 \\ y_2' &= \quad 3y_1 + \ y_2 \end{cases}$$

where $y_i = y_i(t)$ are functions, and y_i' their derivatives, $i = 1, 2$.

Answer. The system can be written as $\mathbf{y}' = A\mathbf{y}$ where $\mathbf{y} = (y_1, y_2)$ is a vector valued function, and A is the matrix of Example 4.

The key observation is that if $\mathbf{y} = \mathbf{c}e^{\lambda t}$, where $\lambda \in \mathbb{R}$ and $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$, then $\mathbf{y}' = \lambda e^{\lambda t}\mathbf{c}$, so that $\mathbf{y}' = \lambda \mathbf{y}$. So if a solution of such form exists we must have

$$\begin{aligned} A\mathbf{y} = \lambda \mathbf{y} &\iff Ae^{\lambda t}\mathbf{c} = \lambda e^{\lambda t}\mathbf{c} \\ &\iff A\mathbf{c} = \lambda \mathbf{c} \end{aligned}$$

So that the constants c_1, c_2 form an eigenvector of A with eigenvalue λ.

From Example 4 then, we get two solutions:

$$\mathbf{y}_3 : \begin{cases} y_1 = & 2e^{3t} \\ y_2 = & 3e^{3t} \end{cases}, \quad \mathbf{y}_5 : \begin{cases} y_1 = & e^{5t} \\ y_2 = & 2e^{5t} \end{cases}$$

Or using vector notation $\mathbf{y}_3 = e^{3t}(2, 3)$, $\mathbf{y}_5 = e^{5t}(1, 2)$. It turns out that these two solutions form a *basis* of the space of all solutions, so that every other solution can be uniquely expressed as a linear combination of $\mathbf{y}_3$ and $\mathbf{y}_5$. This is usually expressed by saying that the *general solution* of the system is

$$\mathbf{y}(t) = a\mathbf{y}_3(t) + b\mathbf{y}_5(t), \quad a, b \in \mathbb{R}$$

□

You might be wandering about using the expression "linearly independent" for two vector valued *functions*. It turns out that functions share many important properties with vectors, we can add them and we can multiply them with scalars, and these operations obey the same laws as vector addition and multiplication of vectors by scalars. This is expressed by saying that the space of vector valued real functions form a *vector space*. Linear Algebra is not really about Euclidean spaces but vector spaces in general. All the theory and methods we have developed so far work, mutatis mutandis, for all vector spaces. The set of all vector valued real functions form an *infinite-dimensional* vector space, and the set of solutions to the system of differential equations in the example above, is a two-dimensional subspace[2]!

[2]If you want to learn more take MTH 42: Linear Algebra.