MTH 35, SPRING 2017

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1. Isomorphisms, invertible matrices, and determinants again

Recall that a linear transformation is called an *isomorphism* if it is both injective and surjective. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be an isomorphism. Then observe first that since $\ker T = \{\mathbf{0}\}$ the RREF of $[T]$ doesn't have any non-pivot columns so all n columns of $[T]$ are pivot. On the other hand, we know that the pivot columns form a basis of the range of T, so $\operatorname{im} T$ has dimension n. But since T is surjective, $\operatorname{im} T = \mathbb{R}^m$. It follows that $m = n$. So:

Proposition 1. *If $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is an isomorphism, then $m = n$. In particular the matrix of an isomorphism is always a square matrix.*

An isomorphism has many properties. We list here a few equivalent definitions:

Proposition 2. *The following are equivalent for a map $T\colon \mathbb{R}^n \to \mathbb{R}^n$*

(1) *T is an isomorphism.*
(2) *T is* full-rank, *i.e. $r(T) = n$.*
(3) *T has trivial kernel, i.e. $\ker T = \{\mathbf{0}\}$.*
(4) *The system $T(\mathbf{x}) = \mathbf{b}$ has a unique solution for each vector $\mathbf{b} \in \mathbb{R}^n$.*
(5) *The RREF of $[T]$ is the $n \times n$ identity matrix I_n.*
(6) *T is* invertible, *i.e. there is an inverse linear transformation $T^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ with the property:*

$$T \circ T^{-1} = T^{-1} \circ T = I_n$$

where I_n stands for the identity transformation $\mathbf{x} \mapsto \mathbf{x}$.
(7) *$[T]$ the matrix of T is invertible, i.e. there is an $n \times n$ matrix $[T]^{-1}$ with the property that*

$$[T][T]^{-1} = [T]^{-1}[T] = I_n$$

where now I_n stands for the $n \times n$ identity matrix.
(8) *The matrix $[T]$ is* non-singular, *i.e.*

$$\det[T] \neq 0$$

Sketch of proof: [1] By the rank-nullity theorem the first three assertions are equivalent. The fourth is just a restatement of what it means to be both injective and surjective. The fifth just translates the second and third in

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[1] Fill in the details of this sketch!

terms of the RREF of T: indeed every column has to be pivot, so it will be one of the standard basic vectors, furthermore by the definition of RREF the k-th column has to be the $\mathbf{e}_k$.

The sixth and seventh are also equivalent since we know that the matrix of a composition of linear transformations is the product of the matrices of the linear transformations.

The fourth statement implies the sixth. Indeed, for a $\mathbf{b} \in \mathbf{R}^n$ define $T^{-1}(\mathbf{b})$ to be the unique solution of the system $T(\mathbf{x}) = \mathbf{b}$. Then by definition $T(T^{-1}(\mathbf{b})) = \mathbf{b}$. To see that $T^{-1}(T(\mathbf{x})) = \mathbf{x}$ notice that $T^{-1}(T(\mathbf{x}))$ is the unique solution to the system $T(\mathbf{x}) = T(\mathbf{x})$. One can also check that T^{-1} defined this way is linear: $T^{-1}(\lambda\mathbf{a}+\mu\mathbf{b})$ is the unique solution to the system $T(\mathbf{x}) = \lambda\mathbf{a} = \mu\mathbf{b}$. Since T is linear $\lambda T^{-1}(a) + \mu T^{-1}(b)$ is such a solution.

Similarly the sixth statement implies the fourth.

Finally by the method we have for computing determinants via row operations we can see that the fifth and eight statements are equivalent. □

Proposition 3. *We have:*

- *If A is invertible then A^{-1} is also invertible and*
$$\left(A^{-1}\right)^{-1} = A$$
- *If A and B are invertible $n \times n$ matrices, their product AB is also invertible. Indeed*
$$(AB)^{-1} = B^{-1}A^{-1}$$

Today we will concentrate on how we find the inverse of a matrix (or a linear transformation) when it exists.

Finding the inverse using row operations. If A is invertible then its RREF is the identity matrix. We will use the following fact:

Proposition 4. *Let A be an invertible matrix. Then the row operations that transform A to the identity matrix, transform the identity matrix to A^{-1}.*

So to find the inverse matrix of A then we construct an "augmented" matrix $[A|I_n]$, by concatenating A and the identity matrix. Then we concentrate on the left part A and use row operations to transform it to I_n. When finished the right part has been transformed to A^{-1}.

Example 5. Use row operations to find the inverse of
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Answer. We start with the augmented matrix:
$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)$$

We multiply the first row by -3 and add it to the second:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

Then add the second row to the first:

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right)$$

finally we divide the second row by -2;

$$\left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

We can verify:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1(-2) + 2\left(\frac{3}{2}\right) & 1(1) + 2\left(-\frac{1}{2}\right) \\ 3(-2) + 4\left(\frac{3}{2}\right) & 3(1) + 3(1) + 4\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As an exercise, verify that the other multiplication also gives the identity, i.e. that

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

□

Example 6. Find the inverse of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}$$

Answer. We start with the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

We add -2 times the first row to the second and -3 times the first row to the third:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -4 & -7 & -2 & 1 & 0 \\ 0 & -5 & -9 & -3 & 0 & 1 \end{array}\right)$$

Next we multiply the first row with 2 and the third with -4:

$$\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 2 & 0 & 0 \\ 0 & -4 & -7 & -2 & 1 & 0 \\ 0 & 20 & 36 & 12 & 0 & -4 \end{array}\right)$$

Next we add the second row to the first and 5 times the second row to the third

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 0 & 1 & 0 \\ 0 & -4 & -7 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 & -4 \end{array}\right)$$

Next we add 7 times the third row to the second, and add the third column to the first:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 6 & -4 \\ 0 & -4 & 0 & 12 & 36 & -28 \\ 0 & 0 & 1 & 2 & 5 & -4 \end{array}\right)$$

Finally we divide the first row by 2 and the second by -4:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 0 & 3 & 9 & -7 \\ 0 & 0 & 1 & 2 & 5 & -4 \end{array}\right)$$

So

$$A^{-1} = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 9 & -7 \\ 2 & 5 & -4 \end{pmatrix}$$

□

1.1. **A formula using determinants.** If $A = (a_{ij})$ is an $n \times n$ matrix then we denote by A_{ij} the $(n-1) \times (n-1)$matrix obtained from A by deleting the i-th row and the j-th column (these are the same matrices we saw in the inductive definition of the determinant).

Proposition 7. *The (i, j)-entry of the inverse matrix is given by the formula:*

$$\left(A^{-1}\right)_{ij} = (-1)^{i+j} \frac{1}{\det A} \det A_{ji}$$

Notice that the indices i and j have been interchanged in the RHS of the formula. That's not a typo, it works that way.

Example 8. Find the inverse of the matrix:

$$A = \begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}$$

Answer. The determinant is

$$\det A = 3 \cdot 5 - 4(-2) = 23$$

The minors are just 1×1 matrices.

$$\begin{array}{ll} A_{11} = 5 & A_{12} = -2 \\ A_{21} = 4 & A_{22} = 3 \end{array}$$

So,

$$A^{-1} = \frac{1}{23} \begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{23} & -\frac{4}{23} \\ \frac{2}{23} & \frac{3}{23} \end{pmatrix}$$

□

Example 9. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ -2 & -3 & 6 \end{pmatrix}$$

Answer. We can calculate:

$$\begin{array}{lll} \det A_{11} = 24 & \det A_{12} = 26 & \det A_{13} = -5 \\ \det A_{21} = 0 & \det A_{22} = 10 & \det A_{23} = -5 \\ \det A_{31} = -8 & \det A_{32} = -2 & \det A_{33} = 5 \end{array}$$

So we can calculate the determinant:

$$\det A = 1(24) - (-1)(26) + (2)(-5) = 40$$

So, we have:

$$A^{-1} = \frac{1}{40} \begin{pmatrix} 24 & 0 & -8 \\ -26 & 10 & 2 \\ -5 & 5 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & -\frac{1}{5} \\ -\frac{13}{20} & \frac{1}{4} & \frac{1}{20} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

□

Solving systems using inverse matrices. Consider the system

$$A\mathbf{x} = \mathbf{b}$$

where A is a non-singular matrix. Since A has an inverse A^{-1} we can write:

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\iff A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \\ &\iff (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \\ &\iff I\mathbf{x} = A^{-1}\mathbf{b} \\ &\iff I\mathbf{x} = A^{-1}\mathbf{b} \end{aligned}$$

So:

Proposition 10. *If A is a non-singular $n \times n$ matrix, then the solution of the system*

$$A\mathbf{x} = \mathbf{b}$$

is given by

$$x = A^{-1}\mathbf{b}$$

Example 11. Solve the system:

$$\begin{cases} x + 2y + 3z &= 2 \\ 2x \qquad - z &= -2 \\ 3x + \ y &= 5 \end{cases}$$

Answer. Notice that this system

$$A\mathbf{x} = \mathbf{b}$$

where the matrix A is the same as in Example 6, and $\mathbf{b} = (2, -2, 5)$.

In Example 6 we found A^{-1}. So the solution is

$$\begin{aligned}\mathbf{x} &= A^{-1}\mathbf{b} \\ &= \begin{pmatrix} 1 & 3 & -2 \\ 3 & 9 & -7 \\ 2 & 5 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -14 \\ -47 \\ -26 \end{pmatrix}\end{aligned}$$

□

Change of basis. In applications, often isomorphisms appear as *changes of basis.*

Proposition 12. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isomorphism, and $S \subset \mathbb{R}^n$. By $T(S)$ we mean the set of all the images of the vectors in S, in other words*

$$T(S) = \{T(\mathbf{v}) : \mathbf{v} \in S\}$$

(1) *S is linearly independent if and only if $T(S)$ is linearly independent.*
(2) *S is a basis if and only if $T(S)$ is a basis.*

Proposition 13. *If $B_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{u}_i, \ldots, \mathbf{u}_n\}$ are two bases of $\mathbb{R}^n$, then there is a unique isomorphism $T_{B_1,B_2}\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$T(v_i) = u_i, \quad i = 1, \ldots, n$$

This isomorphism is called the change of basis from B_1 to B_2.

Recall that the *coordinates* of a vector $\mathbf{v}$ with respect to a basis B are the coefficients of the expression of $\mathbf{v}$ as a linear combination of the vectors from B. For example if $B = \{\mathbf{i}+\mathbf{j}, -\mathbf{j}\}$ the coordinates of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$ with respect to B are $(3, 7)$.

Proposition 14. *If $(x_1, \ldots, x_n)$ are the coordinates of a vector $\mathbf{v}$ with respect to the basis B_1 then the coordinates of $\mathbf{v}$ with respect to the basis B_2 are given by the multiplication:*

$$[T_{B_1,B_2}]^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Example 15. Let $B_2 = \{\mathbf{i} + 2\mathbf{j}, 3\mathbf{i} + \mathbf{j}\}$. What are the coordinates of $\mathbf{v} = 5\mathbf{i} - 6\mathbf{j}$ with respect to the basis B_2?

Answer. The change of basis from B_1, the standard basis of $\mathbb{R}^2$ to B_2 has matrix

$$[T_{B_1,B_2}] = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$$

We calculate:

$$[T_{B_1,B_2}]^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix}$$

So the coordinates of $\mathbf{v}$ in the new basis are:

$$\frac{1}{5}\begin{pmatrix}-1 & 3\\ 2 & -1\end{pmatrix}\begin{pmatrix}5\\ -6\end{pmatrix} = \begin{pmatrix}5\\ -6\end{pmatrix} = \begin{pmatrix}-\frac{23}{5}\\ \frac{16}{5}\end{pmatrix}$$

We can verify this:

$$-\frac{23}{5}(\mathbf{i}+2\mathbf{j}) + \frac{16}{5}(3\mathbf{i}+\mathbf{j}) = 5\mathbf{i} - 6\mathbf{j}$$

□

Example 16. Find the coordinates of $\mathbf{v} = (-3, 5, 1)$ with respect to the basis $B = \{\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{j}+\mathbf{k}, \mathbf{k}\}$ or $\mathbf{R}^3$.

Answer. The matrix of the change of basis from the standard basis is

$$M = \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{pmatrix}$$

We calculate:

$$M^{-1} = \begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}$$

So the coordinates of $\mathbf{v}$ with respect to B are

$$\begin{pmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}-3\\ 5\\ 1\end{pmatrix} = \begin{pmatrix}-3\\ 8\\ -4\end{pmatrix}$$

□

Properties of determinants.

Proposition 17. *We have:*

(1) $\det(AB) = \det(A)\det(B)$
(2) $\det A^{-1} = \dfrac{1}{\det A}$
(3) $\det(\lambda A) = \lambda \det A$
(4) $\det O = 0$
(5) $\det I = 1$

Project: Row Operations via matrix multiplication. In this project we will see that row operations can be effected by multiplying a given matrix A with special kind of matrices called *elementary reduction matrices* or simply *elementary matrices*. There are three kinds of elementary matrices, and they are all result by performing a row operation to the identity matrix.
Scalar Matrices: A *scalar matrix* is obtained by multiplying one of the rows of the identity matrix by a scalar. More specifically given $\lambda \in \mathbb{R}$, and i with $1 \le i \le n$ we have the $n \times n$ matrix $M_i(\lambda)$, whose off diagonal entries

are 0, and all diagonal entries are 1 except the i-th which is λ. For example, for $n = 3$ we have:

$$M_2(5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposition Matrices: A *transposition matrix* is obtained by interchanging two rows of the identity matrix. We denote by τ_{ij} the matrix obtained by interchanging the i-th and j-th rows of the identity matrix. For example for $n = 4$, τ_{24} is the following matrix:

$$\tau_{24} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Combining Matrices: A *combining matrix* is obtained by adding a scalar multiple of one row of I_n to an other row. More specifically $C_{ij}(\lambda)$ is the matrix obtained by adding λ times the i-th row of I_n to the j-th row. For example for $n = 3$, $C_{3\,1}(-3)$ is the following matrix:

$$C_{3\,1}(-3) = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(1) Prove that all elementary matrices are non-singular. In particular prove that

$$\det M_i(\lambda) = \lambda, \quad \det \tau_{ij} = -1, \quad \det C_{ij}(\lambda) = 1$$

(2) Prove that multiplying an $n \times m$ matrix A *on the left* by $M_i(\lambda)$, has the same effect as multiplying the i-th row of A by λ.
(3) Prove that multiplying an $n \times m$ matrix A *on the left* by τ_{ij} has the same effect as interchanging the i-th and j-th rows of A.
(4) Prove that multiplying an $n \times m$ matrix A *on the left* by $C_{ij}(\lambda)$ has the same effect as adding λ times the i-th row of A to the j-th row.
(5) Prove that

$$M_i(\lambda)^{-1} = M_i(\lambda^{-1}), \quad \tau_{ij}^{-1} = \tau_{ij}, \quad C_{ij}(\lambda)^{-1} = C_{ij}(-\lambda)$$

(6) Use the above to explain why the method of finding the inverse of a matrix using row operations works.
(7) Use the above to explain why the method of calculating the determinant using row operations works.
(8) What happens when we multiply an $m \times n$ matrix by an elementary matrix *on the right?*