MTH 35, SPRING 2017

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1. Injections, surjections, and Isomorphisms

Definition 1. A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is called:

- *Injective*, or *one-to-one* if for any vector $\mathbf{b} \in \mathbb{R}^m$ there is *at most one* vector $\mathbf{x} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{b}$. Equivalently if

$$T(\mathbf{x}) = T(\mathbf{y}) \Longrightarrow \mathbf{x} = \mathbf{y}$$

- *Surjective*, or *onto* if for any vector $\mathbf{b} \in \mathbb{R}^m$ there is *at least one* vector $\mathbf{x} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{b}$.
- an *isomorphism* if it's both injective *and* surjective. In other words, if for any vector $\mathbf{b} \in \mathbb{R}^m$ there is *exactly one* vector $\mathbf{x} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{b}$.

Example 2. Consider the following linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}$:

$$(x, y) \mapsto x - y$$

This *is* a surjective linear transformation. Indeed given any real number (say 3) we can find a vector $\mathbf{x} = (x, y) \in \mathbb{R}^2$ so that $T(\mathbf{x}) = 3$. For example we can take $(x, y) = (1, -2)$. In fact, there is a whole line in $\mathbb{R}^2$, the line with equation $x - y = 3$, whose each point is mapped to 5. Of course, there is nothing special about 3, for any real number b there is a whole line of points (x, y) so that $T\left((x, y)\right) = b$.

This means that T is *not* injective. Many vectors of $\mathbb{R}^2$ are mapped to the same vector in $\mathbb{R}$.

Example 3. Consider now the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ where A is the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ 3 & -2 & 3 & -1 \end{pmatrix}$$

in other words $T\colon (x_1, x_2, x_3, x_4) \mapsto (x_1 + 2x_2 + x_3 - 3x_4, 3x_1 - 2x_2 + 3x_3 - x_4)$.

Is this transformation surjective? This question translates to: "Does the system $A\mathbf{x} = \mathbf{b}$ has at least one solution for all $\mathbf{b} \in \mathbb{R}^2$?. If $\mathbf{b} = (b_1, b_2)$ then the augmented matrix of the system is

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & b_1 \\ 3 & -2 & 3 & -1 & b_2 \end{array}\right)$$

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whose RREF is:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 1 & -1 & \frac{b_1+b_2}{4} \\ 0 & \boxed{1} & 0 & -1 & \frac{3b_1-b_2}{8} \end{array}\right) \tag{1}$$

So the system has solutions for every $\mathbf{b}$ and the transformation is injective. Since there are non-pivot columns in the RREF the solutions are not unique, and so the transformation is not injective.

Example 4. Consider the linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ whose matrix is

$$[S] = \begin{pmatrix} -2 & 11 \\ 3 & -13 \\ 1 & -5 \end{pmatrix}$$

Look at the system $[T](\mathbf{x}) = \mathbf{b}$. The augmented matrix is

$$\left(\begin{array}{cc|c} -2 & 11 & b_1 \\ 3 & -13 & b_2 \\ 1 & -5 & b_3 \end{array}\right)$$

Te RREF is:

$$\left(\begin{array}{cc|c} 1 & 0 & 5b_1 + 11b_3 \\ 0 & 1 & b_1 + 2b_3 \\ 0 & 0 & -2b_1 + b_2 - 7b_3 \end{array}\right) \tag{2}$$

The third row tell us that in order for the system to be consistent we need

$$-2b_1 + b_2 - 7b_3 = 0 \tag{3}$$

So this transformation is not surjective.

On the other hand as long as Condition (3) is satisfied the system has a unique solution. So this transformation is injective.

We now define two important subspaces associated with a linear transformation.

Definition 5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

- The *kernel* or *null space* of T is the subspace of $\mathbb{R}^n$ that consists of all vectors that are mapped to $\mathbf{0}$:
$$\ker T = \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\}$$
The notation $\mathcal{N}(T)$ is often also used for this subspace.
- The *range* or *image* of T is the subspace of $\mathbb{R}^m$ that consists of all vectors $\mathbf{b}$ so that the system $T(\mathbf{x}) = \mathbf{b}$ has a solution:
$$\operatorname{im} T = \{\mathbf{b} \in \mathbb{R}^m : T(\mathbf{x}) = \mathbf{b}, \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$
The notation $\mathcal{R}(T)$ is also often used for this subspace.

The following holds:

Proposition 6. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.*

- *T is surjective if and only if,* $\operatorname{im} T = \mathbb{R}^m$

- *T is injective if and only if,* $\ker T = \{\mathbf{0}\}$

Proof. The first assertion is obvious, so let's just prove the second.
The only if part: Clearly $\mathbf{0}$ is always in the kernel of T since $T(\mathbf{0}) = \mathbf{0}$. If T is injective there can be no other vector $\mathbf{x}$ with $T(\mathbf{x}) = \mathbf{0}$.
The if part: Assume $\ker T = \{\mathbf{0}\}$. We prove that then T is injective: let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} T(\mathbf{x}) = T(\mathbf{y}) &\Longrightarrow T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0} \\ &\Longrightarrow T(\mathbf{x} - \mathbf{y}) = \mathbf{0} \\ &\Longrightarrow \mathbf{x} - \mathbf{y} \in \ker T \\ &\Longrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0} \\ &\Longrightarrow \mathbf{x} = \mathbf{y} \end{aligned}$$

□

Example 7. Let's revisit Example 2. Since T is surjective, we have $\operatorname{im} T = \mathbb{R}$.

To find the kernel of T we need to look at the set of vectors that are mapped to the zero vector. In other words look at the solution of the linear system

$$T(\mathbf{x}) = 0$$

This is the line K with equation $x - y = 0$. Now the set of solutions to this equation is described in parametric form as:

$$\mathbf{x} = s(1, 1), \quad s \in \mathbf{R}$$

In other words

$$\ker T = \langle (1, 1) \rangle$$

Since $\{(1, 1)\}$ is a linearly independent set, it follows that $\{(1, 1)\}$ is a basis of $\ker T$ so $\ker T$ is one-dimensional.

Now when we look at the set of vectors that are mapped to any other vector in $\mathbb{R}$, say 3, we get again a line, in this case the line $L_3 \colon x - y = 3$, and this line is a translation of the line K. In fact L_3 is the translation of L_3 by the vector $(1, -2)$ that is a particular solution of the equation $x - y = 3$. The same happens with L_{-2} the set of vectors that are mapped to -2, L_{-2} is a translation of K by the vector, $(0, -2)$ that is a particular solution of $x - y = -2$.

In general, the set L_b of vectors that are mapped to $b \in \mathbb{R}$ is a translation of K by any particular solution, for example we could take the particular solution $\mathbf{p}_b = (0, b)$. Then we can write

$$L_b = \mathbf{p}_b + K := \{\mathbf{p}_b + \mathbf{x} : \mathbf{x} + K\}$$

Example 8. Let's now look at Example 3. Again since T is surjective we have that $\operatorname{im} T = \mathbb{R}^2$.

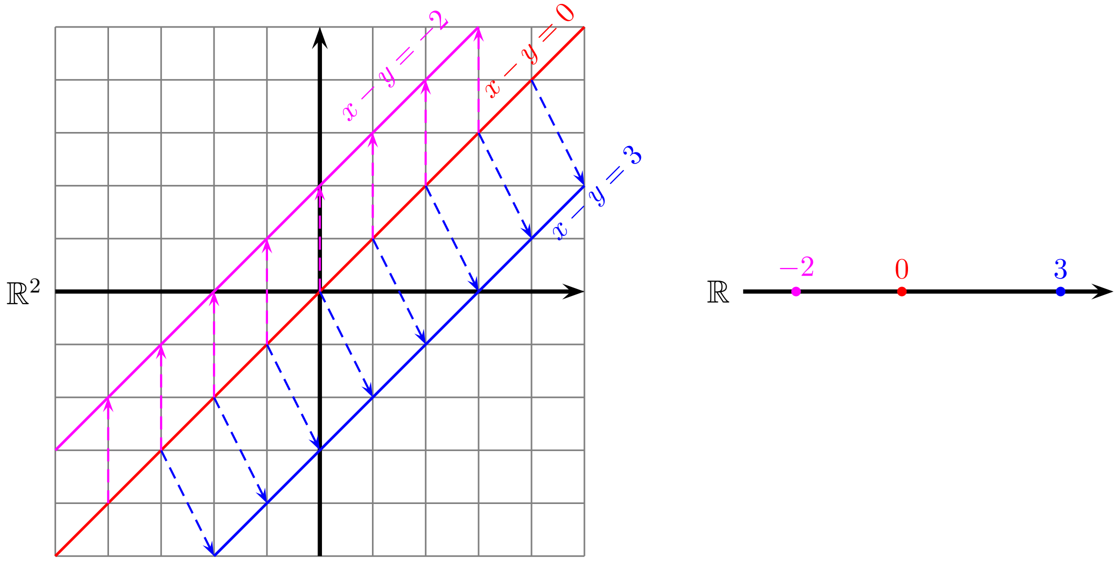

FIGURE 1. The transformation T of Example 2

To find the kernel we have to solve the system $[T](\mathbf{x}) = \mathbf{0}$. From the RREF of its augmented matrix we see that x_3 and x_4 are free variables, so by setting $x_3 = s$ and $x_4 = s$ we get the parametric form:

$$(x_1, x_2, x_3, x_4) = (s - t, -t, s, t)$$

or equivalently:

$$\mathbf{x} = s(1, 0, 1, 0) + t(-1, -1, 0, t), \quad s, t \in \mathbb{R}$$

In other words

$$\ker T = \langle (1, 0, 1, 0), (-1, -1, 0, 1) \rangle$$

and since $\{(1, 0, 1, 0), (-1, -1, 0, 1)\}$ is a linear independent set it is a basis of $\ker T$. So the kernel is 2-dimensional.

Notice that the non-pivot columns of the RREF gave us a basis of $\ker T$ by the above procedure. Each non-pivot column gives a free variable, and then a vector in the basis of $\ker T$.

What about the set $L_{\mathbf{b}}$ of vectors that are mapped to a vector $\mathbf{b}$, that is the solutions of the system $[T]\mathbf{x} = \mathbf{b}$?

From (1) we see that

$$L_{\mathbf{b}} = \mathbf{p}_{\mathbf{b}} + \ker T$$

where, $\mathbf{p}_{\mathbf{b}} = (\frac{b_1+b_2}{4}, \frac{3b_1-b_2}{8})$. Of course, $\mathbf{p}_{\mathbf{b}}$ is a particular solution.

In general we have that

Proposition 9. *For a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^m$, and a vector $\mathbf{b} \in \mathbb{R}^M$, the set of vectors $L_{\mathbf{b}}$ that are mapped to $\mathbf{b}$, is a translation of* $\ker T$ *by a (*any*) particular vector in $L_{\mathbf{b}}$.*

Example 10. Let's look at Example 4. Now we have an injective transformation, so $\ker T = \{\mathbf{0}\}$.

To find $\operatorname{im} T$ we have to find for which $\in \mathbb{R}^3$ the system $[T](\mathbf{x}) = \mathbf{b}$ has a solution. From the discussion in Example 4 we know that this subspace consists of those vectors $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$ whose coordinates satisfy

We can solve Condition (3) for b_2 and we get

$$b_2 = 2b_1 + 7b_3$$

so that $\operatorname{im} T$ has the parametric description

$$\mathbf{b} = s(1, 2, 0) + t(0, 7, 1), \quad s, t \in \mathbb{R}$$

In other words,

$$\operatorname{im} T = \langle (1, 2, 0), (0, 7, 1) \rangle$$

and since $\{(1, 2, 0), (0, 7, 1)\}$ is linearly independent, it is a basis of $\operatorname{im} T$. So $\operatorname{im} T$ is 2-dimensional.

In the last example we found a basis for the range of T, by explicitly figuring under what conditions on $\mathbf{b}$ the system $[T]\mathbf{x} = \mathbf{b}$ has solutions. We could have avoided all that work by reasoning as follows: The columns of the matrix $[T]$ are the (coordinates of the) images of the standard basis of $\mathbb{R}^n$, so $\operatorname{im} T$ is generated by (the vectors whose coordinates are) the columns of T. This means that the pivot columns of $[T]$ form a basis of $\operatorname{im} T$.[1] So a basis for the range of T in Example 4 is formed by the columns of $[T]$ (since all columns are pivot). So a basis for $\operatorname{im} T$ is $\{(-2, 3, 1), (11, -13, 5)\}$.

In general we have:

Proposition 11. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then The columns of $[T]$ that correspond to the pivot columns of the RREF form a basis of the subspace* $\operatorname{im} T$.

A basis of $\ker T$*, is obtained by putting the solution of the homogeneous system $[T]\mathbf{x} = \mathbf{0}$ in to parametric form. There is one basic vector for every non-pivot column.*

Definition 12. The *rank* of a linear transformation T is the dimension of its range:

$$r(T) = \dim \operatorname{im} T$$

The *nullity* of a linear transformation T is the dimension of its kernel:

$$n(T) = \dim \ker T$$

We immediately have:

Proposition 13. *The rank of a linear transformation T equals the number of the pivot columns of $[T]$. The nullity of T equals the number of the non-pivot columns of $[T]$.*

If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ then

$$r(T) + n(T) = n$$

[1] Why?

Example 14. Find the rank and nullity of the following 3×5 matrix:

$$\begin{pmatrix} 1 & 0 & -3 & 2 & 1 \\ 2 & 1 & 3 & -4 & 0 \\ 3 & 1 & 0 & -2 & 1 \end{pmatrix}$$

Answer. The RREF is

$$\begin{pmatrix} 1 & 0 & -3 & 2 & 1 \\ 0 & 1 & 9 & -8 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two pivot and three non-pivot columns. Therefore $r(T) = 2$ and $n(T) = 3$. □

***Question* 15.** Find a basis for $\operatorname{im} T$ and $\ker T$ for the linear transformation of Example 14.

Finally, we give an example of a linear transformation that is both injective and surjective, in other words an isomorphism.

Example 16. The linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose matrix is given below is an isomorphism:

$$[T] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & 0 & 2 \end{pmatrix}$$

Indeed the RREF is the 3×3 identity matrix. So every system $[T]\mathbf{x} = \mathbf{b}$ has a unique solution.

We will talk more about isomorphisms in the next lecture.

2. Project: Orthogonal Complements

Start with the first example from Section 1. Recall that the kernel of the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}$ is the line K with equation $x - y = 0$. The line passing through the origin and orthogonal to K is the line with equation $x + y = 0$. Let's denote that line with $K^{\perp}$. Notice that $K^{\perp}$ is a subspace of R^2.

Notice also that for any $b \in \mathbb{R}$ there are infinitely many vectors $\mathbf{x} \in \mathbb{R}^2$ with $T(\mathbf{x}) = b$, but *only one* of all these vectors lies in $K^{\perp}$. This allow us to chose a unique canonical solution to the equation $x - y = b$: just chose the vector that is perpendicular to the kernel.

Notice also that if we restrict T to have domain $K^{\perp}$, i.e. if we consider $T\colon K^{\perp} \to \mathbb{R}$ then T is an isomorphism.

We have an analogous situation for any linear transformation. In this project we develop this idea.

Definition 17. Let V be a linear subspace of $\mathbb{R}^n$. The *orthogonal complement* of V is defined as

$$V^{\perp} = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{v} = 0\}$$

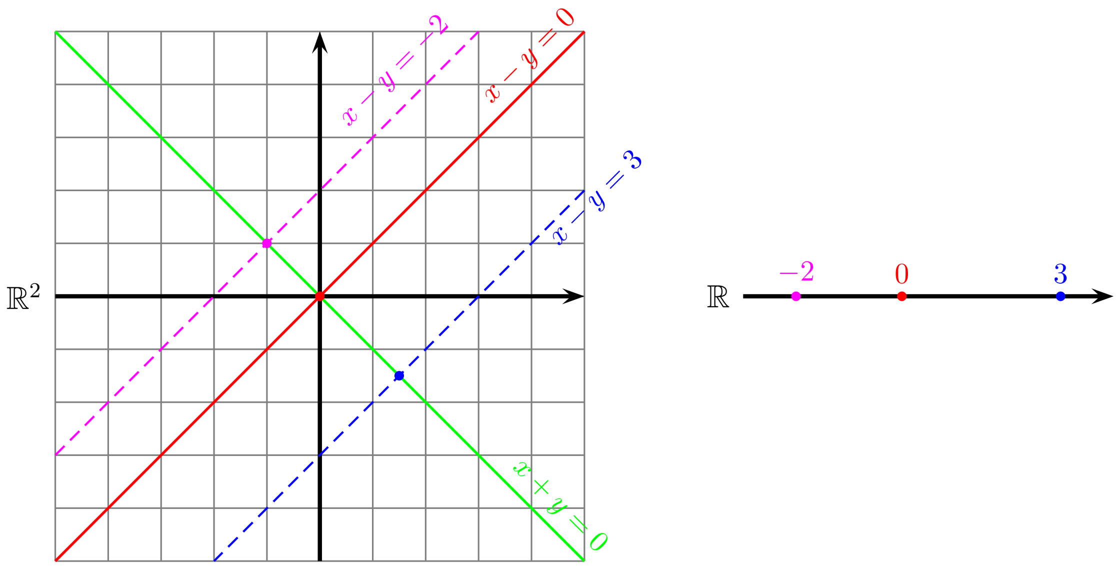

FIGURE 2. The transformation T of Example 2, again

Exercises.

(1) Prove that $V^\perp$ is a linear subspace of $\mathbb{R}^n$.

(2) Prove that the only vector that is common to V and $V^\perp$ is the zero vector:
$$V \cap V^\perp = \{\mathbf{0}\}$$

(3) Prove that if $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a basis of V then
$$V^\perp = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \cdot \mathbf{v}_i = 0, \quad i = 1, \dots, k\}$$

(4) Prove that $\dim V^\perp = n - \dim V$

Hint. The previous question gives a system whose solution set is $V^\perp$

(5) If $T\colon \mathbb{R} \to \mathbb{R}^m$ is a linear transformation, then if we restrict T to the orthogonal complement of the kernel, T becomes injective. In other words $T\colon (\ker T)^\perp \to \mathbb{R}^m$ is injective.

(6) Find a basis for $(\ker T)^\perp$ where T is the linear transformation of Example 3.

(7) This exercise also refers to Example 3. For $\mathbf{b} = (3, 4)$ find the unique $\mathbf{x} \in (\ker T)^\perp$ so that $T(\mathbf{x}) = \mathbf{b}$.