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NIKOS APOSTOLAKIS

1. Linear independence

Example 1. Recall the set $S = \{\mathbf{a}_i : i = 1, \dots, 5\} \subset \mathbb{R}^4$ of the last two lectures, where

$$\begin{aligned} \mathbf{a}_1 &= (1, 1, 3, 1) \\ \mathbf{a}_2 &= (2, 1, 2, -1) \\ \mathbf{a}_3 &= (7, 3, 5, -5) \\ \mathbf{a}_4 &= (1, 1, -1, 2) \\ \mathbf{a}_5 &= (-1, 0, 9, 0) \end{aligned}$$

We want to look at $\langle S \rangle$ in more detail. The question whether a given vector $\mathbf{b} \in \mathbb{R}^4$ is in $\langle S \rangle$ reduces to solving the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 + x_5\mathbf{a}_5 = \mathbf{b}$$

which in turn reduces to solving the system of linear equations:

$$A\mathbf{x} = \mathbf{b}$$

where A is the coefficient matrix of the system.

Last time we saw that the RREF of A is the matrix:

$$\begin{pmatrix} \boxed{1} & 0 & -1 & 0 & 3 \\ 0 & \boxed{1} & 4 & 0 & -1 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 3. So *if the system is consistent*, the solution set of the system has two parameters (or *degrees of freedom*) coming from the free variables x_3 and x_5 that correspond to the non-pivot columns. We can write the solution set in parametric form as:

$$\begin{aligned} x_1 &= b_1' + s - 3t \\ x_2 &= b_2' - 4s + t \\ x_3 &= s \\ x_4 &= b_4' + 2t \\ x_5 &= t \end{aligned}$$

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where b'_i, $i = 1, 2, 4$ are the entries of the last column of the RREF of the *augmented* matrix of the system.

This means, that if a vector $\mathbf{b}$ is in $\langle S \rangle$ then it can be written as a linear combination of elements from S in ifinitely many ways: just chose arbitrary real values for s and t, any such choice gives a way to write $\mathbf{b}$ as a linear combination of elements of S.

Now, one particular choice of the parameters is $s = t = 0$. We get

$$\mathbf{b} = b'_1\mathbf{a}_1 + b'_2\mathbf{a}_2 + b'_4\mathbf{a}_4$$

So, any vector that is in $\langle S \rangle$ can actually be written as a linear combination of only $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$. In other words, $\langle S \rangle = \langle \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \rangle$.

Furthermore, the coefficients b'_1, b'_2, b'_4 are *uniquely* determined[1]. So any vector in $\langle S \rangle$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ in a *unique* way.

Now since the "superfluous" vectors $\mathbf{a}_3, \mathbf{a}_5$ are in $\langle S \rangle$, we can express them as linear combination of $\mathbf{a}_i$, $i = 1, 2, 4$. Indeed we have:

$$\mathbf{a}_3 = -\mathbf{a}_1 + 4\mathbf{a}_2$$

and

$$\mathbf{a}_5 = 3\mathbf{a}_1 - \mathbf{a}_2 - 2\mathbf{a}_4$$

In sum: the set S contains "superfluous" vectors, that are already linear combinations of the other vectors in S. If a vector can be expressed at all, as a linear combination of vectors from S, it can be expressed in infinitely many ways. When we remove the "superfluous" vectors $\mathbf{a}_3$, and $\mathbf{a}_5$ from S, the remaining vectors span the same subspace. Now however, a vector in that subspace can be expressed uniquely as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$.

Example 2. Now let's look at the last problem from the homework. We now have the set $B = \{(1, 1, 1), (1, 1, 0), (1, 0, -1)$ in $\mathbb{R}^3$. When we set up the system that decides whether a given vector $\mathbf{b}$ is in $\langle B \rangle$ we have:

$$A\mathbf{x} = \mathbf{b}$$

where,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

The RREF of the coefficient matrix is now:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This means that for any $\mathbf{b}$ the system has a *unique* solution. So any vector in $\mathbb{R}^3$ can be written as as a linear combination of vectors from B, in a *unique* way.

[1] Why?

Proposition 3. *For a set of vectors S in a Euclidean space $\mathbb{R}^n$ the following conditions are all equivalent (in other words if one of the conditions holds they all hold):*

(1) *Any vector in $\langle S \rangle$ can be expressed as a linear combination of vectors from S in a unique way.*
(2) *No vector from S can be expressed as linear combination of the other vectors from S.*
(3) *The only way to express $\mathbf{0}$ as a linear combination of vectors from S is to have all coefficients equal to 0.*
(4) *All the columns of the RREF of the matrix that has as columns the coordinates of the vectors of S, are pivot.*

Definition 4. A set of vectors in $\mathbb{R}^n$ is called *linearly independent* if one (and hence all) of the conditions in Proposition 3 holds. As set that is not linearly independent is called *linealy dependent*.

Example 5. In $\mathbb{R}^4$, the set S from Example 1 is linearly dependent. It's subset $S' = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is linearly independent.

Example 6. In $\mathbb{R}^3$, the set B from Example 2 is linearly independent.

Example 7. In $\mathbb{R}^4$, the vectors $\mathbf{u} = (2, 3, 1, 0)$, $\mathbf{v} = (0, -1, 0, -2)$, and $\mathbf{w} = (0, 0, 0, 1)$ form a linearly independent set.

Indeed, the matrix that has these vectors as columns is:

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

and its RREF is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that every column contains a leading 1.

Example 8. In $\mathbb{R}^3$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (3, 4, 5))$, $\mathbf{v}_2 = (2, 9, 2)$, and $\mathbf{v}_3 = (4, 18, 4)$ is linearly dependent.

Indeed we get the matrix:

$$\begin{pmatrix} 3 & 2 & 4 \\ 4 & 9 & 18 \\ 5 & 2 & 4 \end{pmatrix}$$

with RREF

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that the third row is not pivot.

1.1. **Exercises.**

(1) Complete all the details in this section

2. Basis and Dimension

Definition 9. A subset B of a linear subspace V of $\mathbb{R}^n$ is called a *basis* of V, if the following two conditions hold:

- B spans V, in other words, $\langle B \rangle = V$
- B is linearly independent.

Example 10. The *basic vectors* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are called basic because they form a basis of $\mathbb{R}^3$.

- In $\mathbb{R}^2$ the vectors $\mathbf{i} = \mathbf{e}_1 = (1, 0)$ and $\mathbf{j} = \mathbf{e}_2 = (0, 1)$ form a basis of $\mathbf{R}^2$. This basis is called the *standard basis* of $\mathbb{R}^2$.
- In $\mathbb{R}^3$ the vectors $\mathbf{i} = \mathbf{e}_1 = (1, 0, 0)$, $\mathbf{j} = \mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{k} = \mathbf{e}_3 = (0, 0, 1)$ form a basis of $\mathbf{R}^3$. This basis is called the *standard basis* of $\mathbb{R}^3$.
- In $\mathbb{R}^4$ the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ form a basis of $\mathbf{R}^4$. This basis is called the *standard basis* of $\mathbb{R}^4$.
- In general, in $\mathbb{R}^n$ the vectors $\mathbf{e}_i$, $i = 1, 2, \ldots, n$ form a a basis of $\mathbb{R}^n$. The vector $\mathbf{e}_i$ has all coordinates but the ith equal to 0 and the ith coordinate equal to 1. This basis is called the *standard basis* of $\mathbb{R}^n$.

In general a subspace will have infinitely many bases. We've already seen a basis of $\mathbb{R}^3$ different than the standard one in Example 2. Indeed in that example $\langle B \rangle = \mathbb{R}^3$ and B is linearly independent. Notice that B has three vectors, as many as the standard basis. This is more generally true, and it means that the *dimension* of $\mathbb{R}^3$ is 3.

Proposition 11. *For any subspace V of $\mathbb{R}^n$ we have:*

- *V has a basis.*
- *All bases of the same subspace have the same number of vectors.*

Definition 12. The number of vectors in a basis of a subspace V is called the *dimension* of the subspace.

Example 13. The dimension of $\mathbb{R}^n$ is n.

Example 14. Let $V = \langle S \rangle$, the linear span of the set S in Example 1. We show in Example 1 that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ spans V, and from Example 5 we know that it is linearly independent. So $\dim V = 3$.

In general it turns out that:

Proposition 15. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a finite subset of $\mathbb{R}^n$. Then the dimension of $\langle S \rangle$ equals the rank of the matrix whose columns are the vectors of S. Actually the vectors of S that correspond to the pivot columns of the RREF of the matrix form a basis of $\langle S \rangle$.*

Example 16. In $\mathbb{R}^4$ consider the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where $\mathbf{v}_1 = (1, 1, 2, 1)$, $\mathbf{v}_2 = (2, 2, 4, 2)$, $\mathbf{v}_3 = (2, 0, -1, 1)$, $\mathbf{v}_4 = (7, 1, -1, 4)$, and $\mathbf{v}_5 = (0, 2, 5, 1)$.

The RREF of the matrix $[\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3\, \mathbf{v}_4\, \mathbf{v}_5]$ is

$$\begin{pmatrix} \boxed{1} & 2 & 0 & 1 & 2 \\ 0 & 0 & \boxed{1} & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis of $\langle S \rangle$ is $\{\mathbf{v}_1, \mathbf{v}_3\}$ and $\dim \langle S \rangle = 2$.

***Question* 17.** What about the *zero-subspace*, i.e. the subspace that contains only $\mathbf{0}$ the zero vector? What is its dimension? Does it even have a basis?

Answer. This is actually a subtle issue. It turns out that it is logically consistent to posit that the empty set $\emptyset$ is a basis of $\{\mathbf{0}\}$. So $\dim\{\mathbf{0}\} = 0$. □

Definition 18. If B is a basis of a subspace V then every vector $\mathbf{v} \in V$ can be written uniquelly as a linear combination of elements of B. The coefficients of this linear combination are called the *coordinates of* $\mathbf{v}$ *with respect to the basis* B

Example 19. The coordinates of a vector in $\mathbb{R}^n$ are actually its coordinates with respect to the standard basis of $\mathbb{R}^n$.

Example 20. This refers to Example 1. The coordinates of $\mathbf{a}_3$ with respect to the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ are $(-1, 2, 0)$ and the coordinates of $\mathbf{a}_5$ with respect to the same basis are $(3, -1, -2)$.

2.1. Exercises.

(1) Complete all the details in this section.
(2) Find the coordinates of the vector $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ with respect to the basis of $\mathbb{R}^3$ given in Example 2.

2.2. **Project: Homogenuous Systems.** A linear system where all constants are 0, in other words a system of the form $A\mathbf{x} = \mathbf{0}$ is called a *homogeneous* system.

(1) Prove that a homogeneous system is always consistent.
(2) Let N be the solution set of an $m \times n$ homogeneous system. Prove that N is a linear subspace of $\mathbf{R}^n$.
(3) Prove that the columns of A that correspond to the free variables form a basis for N. What that means if there are no free variables?
(4) Conclude that the dimension of N is equal to $n - r$ where r is the rank of A.
(5) Prove that if $\mathbf{p}$ is any solution of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is vector in $\mathbb{R}^n$, then the solution set of the system is

$$S = \{\mathbf{p} + \mathbf{v} : \mathbf{v} \in N\}$$

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