**MTH 35, SPRING 2017**

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## 1. Solving systems of linear equations.

We continue with the examples from last time. Let's start by recalling that when we try to solve a linear system of equations we can use any of the following three operations:

(1) We can interchange two equations.
(2) We can multiply any equation by a scalar.
(3) We can add to an equation the scalar multiple of another.

Notice that these are analogous to the row operations we show already. This will became clearer later. For the moment we clarify:

**Definition 1.** Two systems of linear equations are equivalent if they have the same solution set.

**Proposition 2.** *Performing the three operations above does not change the solution set of the system.*

**Example 3.** Recall from last time that we wanted to see whether a given vector $\mathbf{w} = (-4, 2, 28, 10)$ is in the span of the following five vectors in $\mathbb{R}^4$:

$\mathbf{a}_1 = (1, 1, 3, 1), \mathbf{a}_2 = (2, 1, 2, -1), \mathbf{a}_3 = (7, 3, 5, -5), \mathbf{a}_4 = (1, 1, -1, 2), \mathbf{a}_5 = (-1, 0, 9, 0)$

This tranlates to the following *vector equation*:

$$\mathbf{w} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} + x_4\mathbf{a_4} + x_5\mathbf{a_5}$$

for real numbers $x_1, x_2, x_3, x_4, x_5$, and this in turn translates to the system:

$$\begin{cases} x_1 + 2x_2 + 7x_3 + x_4 - x_5 = -4 \\ x_1 + x_2 + 3x_3 + x_4 = 2 \\ 3x_1 + 2x_2 + 5x_3 - x_4 + 9x_5 = 28 \\ x_1 - x_2 - 5x_3 + 2x_4 = 10 \end{cases}$$

We will use the first equation to eliminate $x_1$ from the other three equations: we subtract the first equation from the second and the fourth and we add $-3$ times the first equation to the third. We get:

$$\begin{cases} x_1 + 2x_2 + 7x_3 + x_4 - x_5 = -4 \\ - x_2 - 4x_3 + x_5 = 6 \\ - 4x_2 - 16x_3 - 4x_4 + 12x_5 = 40 \\ - 3x_2 - 12x_3 + x_4 + x_5 = 14 \end{cases}$$

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Next we'll use the second equation to eliminate $x_2$ from the other equations. We multiply it by 2 and add it to the first, by $-4$ and add it to third equation, and by $-3$ and add it to fourth:

$$\begin{cases} x_1 \quad - x_3 + x_4 + x_5 = 8 \\ x_2 - 4x_3 \quad + x_5 = 6 \\ - 4x_4 + 8x_5 = 16 \\ x_4 - 2x_5 = -4 \end{cases}$$

Next we interchange the last two equations:

$$\begin{cases} x_1 \quad - x_3 + x_4 + x_5 = 8 \\ - x_2 - 4x_3 \quad + x_5 = 6 \\ x_4 - 2x_5 = -4 \\ - 4x_4 + 8x_5 = 16 \end{cases}$$

Next, trying to eliminate $x_4$ from the last equatio we mulitly the third equation by 4 and add it to the fourth:

$$\begin{cases} x_1 \quad - x_3 + x_4 + x_5 = 8 \\ - x_2 - 4x_3 \quad + x_5 = 6 \\ x_4 - 2x_5 = -4 \\ 0 = 0 \end{cases}$$

Now use the third equation to eliminate $x_4$ from the first. So we multiply the third equation by $-1$ and add it to the first:

$$\begin{cases} x_1 + \quad - x_3 \quad + 3x_5 = 12 \\ - x_2 - 4x_3 \quad + x_5 = 6 \\ x_4 - 2x_5 = -4 \\ 0 = 0 \end{cases}$$

Finally we multiply the second equation by $-1$:

$$\begin{cases} x_1 + \quad - x_3 \quad + 3x_5 = 12 \\ x_2 + 4x_3 \quad - x_5 = -6 \\ x_4 - 2x_5 = -4 \\ 0 = 0 \end{cases}$$

At this point we can consider the system solved. Once we give values to the "free variables" $x_3$ and $x_5$ we can find the other variables. For example if we set $x_3 = x_5 = 0$, we get the solution:

$$x_1 = 12, x_2 = -6, x_3 = 0, x_4 = -4, x_5 = 0$$

In general, we have that the general solution is:

$$\{(12 + x_3 - 3x_5, -6 - 4x_3 + x_5, x_3, -4 - 2x_5, x_5) : x_3, x_5 \in \mathbb{R}\}$$

If we set $x_3 = -1$ and $x_5 = 3$ we get the solution:

$$x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 2, x_5 = 3$$

that I gave you last time.

So since the system has solutions it follows that $\precapprox \in \langle S \rangle$.

On the other hand when we try to see if $\mathbf{u} = (3, 1, 2, -1)$ is in $\langle S \rangle$ we trying to express it as a linear combination of $\mathbf{a}_i$, $i = 1, \ldots, 5$ we have the system:

$$\begin{cases} x_1 + 2x_2 + 7x_3 + x_4 - x_5 & = 3 \\ x_1 + x_2 + 3x_3 + x_4 & = 1 \\ 3x_1 + 2x_2 + 5x_3 - x_4 + 9x_5 & = 2 \\ x_1 - x_2 - 5x_3 + 2x_4 & = -1 \end{cases}$$

Now the "variable part" of the system is the same as before, only the constants differ. So we can perform the same operations to bring the system in a "solved" form. When we perform the same operations though, we get the system:

$$\begin{cases} x_1 + \quad - x_3 \quad + 3x_5 & = 0 \\ \quad x_2 + 4x_3 \quad - \ x_5 & = 0 \\ \quad x_4 - 2x_5 & = 0 \\ 0 & = 1 \end{cases}$$

The last equation now is impossible. So the system doesn't have any solutions. So $\mathbf{u}$ is not in $\langle S \rangle$.

Notice that we don't need to keep writing the whole system: the coefficients are the important thing. There are two matrices that keep track of the system: the *coefficient matrix*, $a$ and the augmented matrix. A system is written in *matrix form* as

$$A\mathbf{x} = \mathbf{b}$$

the augmented matrix is $[A \,|\, \mathbf{b}]$. To solve a system we try to bring its matrix to a special form, as we did above.

**Reduced Row Echelon Form.** Lea $M$ be an $m \times n$ matrix. We can consider $M$ as a row of *column vectors* or a column of *row vectors*.

**Definition 4.** A matrix is in *Reduced Row-Echelon Form* (or in short *Echelon Form*) when the following conditions are met:

(1) If there is a row where every entry is 0 then this row is bellow any row that contains non-zero entries.
(2) The leftmost non-zero entry of a row is 1.
(3) The leftmost non-zero entry of any row is the only non-zero entry in its column.
(4) If $m_{ij}$ and $m_{st}$ are two leftmost non-zero entries in their rows then if $s > i$ then $t > j$. In other words, the leftmost non-zero entry of a row, is to the right of the leftmost non-zero entry of any row below it.

**Terminology:** The leftmost non-zero entry of a row in RREF is called the *leading* 1. A column that contains a leading 1 is called a *pivot column*. The number of non-zero rows (which equals the number of leading 1's, and also the number of pivot columns) is called the *rank* of the matrix and is denoted by $r(M)$, or when the matrix is understood simply by $r$.

**Example 5.** The $5 \times 8$ matrix

$$\begin{pmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is in RREF. It has three non-zero rows, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. So it's rank is $r = 3$.

**Example 6.** The $6 \times 9$ matrix

$$\begin{pmatrix} 1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\ 0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is *not* in RREF. In fact it fails all the requirements listed above.

RREF is very usefull as we will keep seen. For one, if we have a matrix in RREF it's very easy to calculate its determinant. Also a system whose matrix is in RREF is very easy to solve.

**Proposition 7.** *Any matrix can be turned into RREF using row operations. Furthermore the RREF of a matrix is unique.*

**Definition 8.** Two matrices are called *row equivalent* if we can turn one to the other using row operations.

A linear system is *consistent* if it has solutions.

**Proposition 9.** *A system is inconsistent if and only if, the last column of the RREF of its augmented matrix is pivot. Equivalently, the RREF of its augmented matrix contains a row* $[\mathbf{0} \,|\, 1]$.

*For a consistent system the pivot columns correspond to* dependent *variables and the non-pivot columns correspond to* independent *(or* free*) variables. In other words, if we have non-pivot columns the variables that correspond to those columns can be assigned arbitrary values, and each such assignment gives a solution of the system.*

*In particular, a system has a* unique *solution, if and only if the RREF of its coefficient matrix has only pivot columns. This can only happen if the $A$ is a square matrix (i.e. we have as many equations as unknowns) and its RREF is*

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

**Example 10.** The system:

$$\begin{cases} x+y+z &= 6 \\ x-y+z &= 2 \\ x+y-z &= 0 \end{cases}$$

has augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

Which has the following RREF:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

So it has a unique solution: $x = 1, y = 2, z = 3$.

**Example 11.** The system:

$$\begin{cases} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{cases}$$

has augmented matrix:

$$\left(\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right)$$

with RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

Due to the last row the system is incosistent

**Example 12.** Consider the system:

$$\begin{cases} x_1 - x_2 - 2x_3 + x_4 + 11x_5 &= 13 \\ x_1 - x_2 + x_3 + x_4 + 5x_5 &= 16 \\ 2x_1 - 2x_2 + x_4 + 10x_5 &= 21 \\ 2x_1 - 2x_2 - x_3 + 3x_4 + 20x_5 &= 38 \\ 2x_1 - 2x_2 + x_3 + x_4 + 8x_5 &= 22 \end{cases}$$

Its augmented matrix has RREF

$$\left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Notice that since the last column is not pivot, the system is consistent. This matrix has rank $r = 3$, and so three dependent variables: $x_1$, $x3$, and $x_4$. The non-pivot columns correspond to the free variables $x_2$ and $x_5$.

What is the solution set then? The non-zero rows of the matrix give the equations

$$\begin{cases} x_1 - x_2 + 3x_5 &= 6 \\ x_3 - 2x_5 &= 1 \\ x_4 + 4x_5 &= 9 \end{cases}$$

which we can solve in terms of the free variables:

$$x_1 = 6 + x_2 - 3x_5, \quad x_3 = 1 + 2x_5, \quad x_4 = 9 - 4x_5$$

Using $s$ for $x_2$ and $t$ for $x_5$ we can write the solution set as:

$$\{(6 + s - 3t, s, 1 + 2t, 9 - 4t, t) : s, t \in \mathbb{R}\}$$

Notice that we can think of a solution as a vector $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$. We can now write down the following parametric equation for the solution set:

$$\mathbf{x} = (6, 0, 1, 9, 0) + s(1, 1, 0, 0, 0) + t(-3, 0, 2, -4, 1), \quad s, t \in \mathbb{R}$$

We'll see that this is a (2-dimensional) plane in $\mathbb{R}^5$, namely the plane that passes through the point $(6, 0, 1, 9, 0)$ and is parallel to the vectors $(1, 1, 0, 0, 0)$ and $(-3, 0, 2, -4, 1)$

## 2. Exercises

(1) Fill all the details in the calculations in the examples in these notes.

(2) Revisit the set $S = \{\mathbf{a}_i : i = 1, \ldots, 5\}$ of Example 3.

   (a) Show that any vector in $\langle S \rangle$ can be written as a linear combination of vectors in $S$ in infinietly many ways.

   (b) Show that $\langle S \rangle = \langle S' \rangle$ where $S' = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$.

   (c) Show that any vector in $\langle S \rangle$ can be written as a linear combination of vectors in $S'$ in a *unique* way.