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1. Recall from last time

Upper triangular matrices.

Definition 1. An $n \times n$ matrix A is called *upper triangular* if all the entries below the main diagonal are 0. In other words, if $i > j$ then $a_{ij} = 0$.

Examples:

Example 2. The following 4×4 matrix is upper triangular

$$\begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Example 3. The following 4×4 matrix is *not* upper triangular:

$$\begin{pmatrix} 0 & 2 & 1 & -2 \\ 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 5 & -3 & 0 & 0 \end{pmatrix}$$

The determinant of an UT matrix is very easy to calculate:

Proposition 4. *The determinant of an UT matrix equals the product of its* diagonal *entries. In other words, if A is UT then*

$$|A| = a_{11}a_{22}\cdots a_{nn}$$

For example the determinant of the matrix in Example 2 above is $1 \cdot 3 \cdot 2 \cdot (-3) = -18$.

We can use the above result to calculate determinants, for all matrices: even if a matrix is not UT we can turn it in to UT by using *row operations* (defined bellow).

Row operations. We consider the following three row operations on a matrix:

(1) Interchange (swap) $\mathbf{r}_i$ and r_j.
(2) Replace $\mathbf{r}_i$ with $\lambda\mathbf{r}_i$, for some $\lambda \in \mathbb{R}$
(3) Add a a multiple of a row to another row, or equivalently replace the row $\mathbf{r}_j$ with the sum $\lambda\mathbf{r}_i + \mathbf{r}_j$, for some $\lambda \in \mathbb{R}$.

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Proposition 5. *Any $n \times n$ matrix can be turned to an UT matrix by using the above three row operations.*

Example 6. Consider the matrix in Example 3 above.

$$\begin{pmatrix} 0 & 2 & 1 & -2 \\ 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 5 & -3 & 0 & 0 \end{pmatrix}$$

We will use a sequence of row operations to turn it into an UT matrix. First we swap the first and second row:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 2 & 1 & 1 \\ 5 & -3 & 0 & 0 \end{pmatrix}$$

Now want to make all the elements of the first column that are below the main diagonal 0. In our case we need to worry only about the entry in the fourth row and first column.

We use operation 3) and add -5 times the first row to the fourt:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & -3 & -10 & -15 \end{pmatrix}$$

Now we want to have all elements in the second column that are below the diagonal to be 0. We do this by first adding -1 times the second row to the third:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & -3 & -10 & -15 \end{pmatrix}$$

Now we need to make the second entry in the fourth column 0. We do this by first multiplying the fourth by 2 (so that in the next step we'll add to it 3 times the second row):

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & -6 & -20 & -30 \end{pmatrix}$$

Next we add the 3 times the second row to the fourth:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -17 & -36 \end{pmatrix}$$

Finally we need to make the entry in the fourth row and third column 0. We can accomplish that by adding -17 times the second row to the fourth:

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

And we are done.

The following proposition allows us to calculate determinants by using row operations to turn a matrix to UT.

Proposition 7. *Let A be an $n \times n$ matrix, and A' the result of applying a row operation to A. Then*

(1) *If the operation was a row swap then the determinant changes sign. In other words $|A'| = -|A|$.*
(2) *If we multiply a row by a number λ the determinant is multiplied by that number, in other words: $|A'| = \lambda|A|$.*
(3) *If we add the multiple of a row to another row the determinant stays the same, in other words $|A'| = |A|$.*

Here is an example:

Example 8. Compute the determinant of the following matrix:

$$A = \begin{pmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{pmatrix}$$

We will transform the matrix to UT, by a sequence of row operations keeping track of the change in determinants:
We first swap the first two rows, to get a matrix A_1:

$$A_1 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{pmatrix}$$

The determinant changes sign: $\det(A) = -\det(A_1)$
Then we add -2 times the first row to the second:

$$A_2 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{pmatrix}$$

The determinant doesn't change: $\det(A) = -\det(A_2)$

Then we add the first row to the third.

$$A_3 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 3 & 5 & 4 & 0 \end{pmatrix}$$

The determinant doesn't change: $\det(A) = -\det(A_3)$
Then we add -3 times the first row to the fourth.

$$A_4 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{pmatrix}$$

The determinant doesn't change: $\det(A) = -\det(A_4)$
Now we add the third row to the second:

$$A_5 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{pmatrix}$$

The determinant doesn't change: $\det(A) = -\det(A_5)$
Next we add 2 times the first row to the third, and -2 times the first row to the fourth.

$$A_7 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 3 & -11 \end{pmatrix}$$

The determinant doesn't change: $\det(A) = -\det(A_7)$
Next we multiply the last row with -2.

$$A_8 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & -6 & 22 \end{pmatrix}$$

The determinant is multiplied by -2. $\det(A) = \frac{1}{2}\det(A_8)$
Finaly we add 3 times the third row to the fourth.

$$A_9 = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 0 & 55 \end{pmatrix}$$

The determinant doesn't change: This is an UT matrix and we have:

$$\det(A) = \frac{1}{2}\det(A_9) = \frac{1}{2}(-2) \cdot 2 \cdot 55 = -110$$

2. Subspaces, spans, systems

Definition 9. A subset V of $\mathbb{R}^n$ is called a /linear subspace of $\mathbb{R}^n$ if

(1) $\mathbf{0} \in V$
(2) If $\mathbf{a} \in V$ and $\lambda \in \mathbb{R}$ then $\lambda\mathbf{a} \in V$.
(3) If $\mathbf{a} \in V$ and $\mathbf{b} \in V$ then $\mathbf{a}+\mathbf{b} \in V$

Example 10. The set $V = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z = 0\}$ is a linear subspace.

Example 11. The set $V = \{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z = 2\}$ is *not* a linear subspace.

Definition 12. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ be vectors in a Euclidean space $\mathbb{R}^n$. The vector $\mathbf{b} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a_2} + \ldots + \lambda_m\mathbf{a}_m$ is called the *linear combination* of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ with *coefficients* $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Proposition 13. *If V is a subspace of $\mathbb{R}^n$ and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in V$ then any linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ is in V.*

Definition 14. Let $S = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ be a set of vectors in $\mathbb{R}^n$. The span of S is the set of all possible linear combinations of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. In symbols:

$$\langle S\rangle = \langle \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\rangle = \{\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a_2} + \ldots + \lambda_m\mathbf{a}_m : \lambda_i \in \mathbb{R}, 1 \le i \le m\}$$

Proposition 15. *For any set of vectors S of $\mathbf{R}^n$, $\langle S\rangle$ is a linear subspace of $\mathbb{R}^n$.*

Example 16. The span of $\{\mathbf{0}\}$ is the zero-subspace.

Example 17. The span of $\{\mathbf{a}\}$ where $\mathbf{a} \neq \mathbf{0}$ is the line through the origin determined by $\mathbf{a}$.

What about the span of two vectors? It's either a line or a plane.

Example 18. If $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$ then $\langle \mathbf{a}, \mathbf{b}\rangle$ is a plane.

Example 19. If however, $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ then $\langle \mathbf{a}, \mathbf{b}\rangle$ is a actually a line, namely the same as $\langle \mathbf{a}\rangle$. This is so because $\mathbf{b} = 2\mathbf{a}$, so any linear combination of $\mathbf{a}$ and $\mathbf{b}$ is actually just a multiple of $\mathbf{a}$.

We now have the basic question:

***Question* 20.** When is a given vector $\mathbf{b}$ in the span of a set of vectors S?

Example 21. In $\mathbb{R}^4$ consider the set $S = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ where, $\mathbf{a}_1 = (1, 1, 3, 1), \mathbf{a}_2 = (2, 1, 2, -1), \mathbf{a}_3 = (7, 3, 5, -5), \mathbf{a}_4 = (1, 1, -1, 2), \mathbf{a}_5 = (-1, 0, 9, 0)$

Are the vectors $\mathbf{w} = (-4, 2, 28, 10)$, and $\mathbf{u} = (3, 1, 2, -1)$ in $\langle S\rangle$?

Let's do $\mathbf{w}$ first. We need to solve the following *vector equation*:

$$\mathbf{w} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} + x_4\mathbf{a_4} + x_5\mathbf{a_5}$$

for real numbers x_1, x_2, x_3, x_4.

This translates to a system:

$$\begin{cases} x_1 + 2x_2 + 7x_3 + x_4 - x_5 & = -4 \\ x_1 + x_2 + 3x_3 + x_4 & = 2 \\ 3x_1 + 2x_2 + 5x_3 - x_4 + 9x_5 & = 28 \\ x_1 - x_2 - 5x_3 + 2x_4 & = 10 \end{cases}$$

which has solution $x_1 = 2, x_2 = 1, x_3 = -1, x_4 = 2, x_5 = 3$. So $\mathbf{w}$ is in $\langle S \rangle$.

For $\mathbf{u}$ on the other hand we get the system

$$\begin{cases} x_1 + 2x_2 + 7x_3 + x_4 - x_5 & = 3 \\ x_1 + x_2 + 3x_3 + x_4 & = 1 \\ 3x_1 + 2x_2 + 5x_3 - x_4 + 9x_5 & = 2 \\ x_1 - x_2 - 5x_3 + 2x_4 & = -1 \end{cases}$$

which is *incosistent*. So $\mathbf{u}$ is *not* in $\langle S \rangle$.

3. Next Time

Solving systems of linear equations.

(1) We can interchange two equations.
(2) We can multiply any equation by a scalar.
(3) We can add to an equation the scalar multiple of another.

Reduced Row Echelon Form. Lea M be an $m \times n$ matrix. We can consider M as a row of *column vectors* or a column of *row vectors.*

Definition 22. A matrix is in *Reduced Row-Echelon Form* (or in short *Echelon Form*) when the following conditions are met:

(1) If there is a row where every entry is 0 then this row is bellow any row that contains non-zero entries.
(2) The leftmost non-zero entry of a row is 1.
(3) The leftmost non-zero entry of any row is the only non-zero entry in its column.
(4) If m_{ij} and m_{st} are two leftmost non-zero entries in their rows then if $s > i$ then $t > j$. In other words, the leftmost non-zero entry of a row, is to the right of the leftmost non-zero entry of any row below it.

Terminology: The leftmost non-zero entry of a row in RREF is called the *leading* 1. A column that contains a leading 1 is called a *pivot column.* The number of non-zero rows (which equals the number of leading 1's, and also the number of pivot columns) is called the *rank* of the matrix and is denoted by $r(M)$, or when the matrix is understood simply by r.

Example 23. The 5×8 matrix

$$\begin{pmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is in RREF. It has three non-zero rows, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. So it's rank is $r = 3$.

Example 24. The 6×9 matrix

$$\begin{pmatrix} 1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\ 0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is *not* in RREF. In fact it fails all the requirements listed above.

RREF is very usefull as we will keep seen. For one, if we have a matrix in RREF it's very easy to calculate its determinant. Also a system whose matrix is in RREF is very easy to solve.

Proposition 25. *Any matrix can be turned into RREF using row operations.*