Vertex form of a quadratic equation

Nikos Apostolakis

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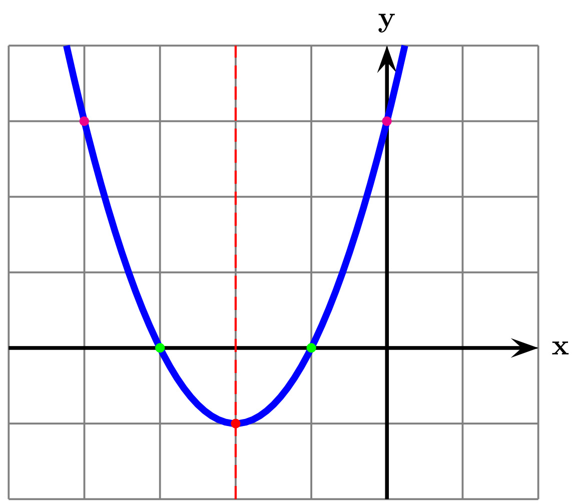
Recall 1. Last time we looked at the graphs of quadratic equations in two variables. The upshot was that the graph of the equation:

$$y - k = a(x - h)^2 \tag{1}$$

is a parabola and looks exactly like the graph of $y = ax^2$, but with *vertex* translated at (h, k). For example here is the graph of the equation

$$y + 1 = (x + 2)^2$$



Vertex: $(-2, -1)$

Axis of symmetry: $x = -2$

x-intercepts: $(-3, 0)$, $(-1, 0)$

y-intercept: $(0, 3)$

Symmetric to y-intercept: $(-4, 3)$

The axis of symmetry is the *vertical* line passing through the vertex. In this example the axis of symmetry is $x = -2$

The x-intercepts are obtained by solving the equation we get by setting $y = 0$. In our case:

$$\begin{aligned} 1 = (x+2)^2 &\iff x + 2 = \pm\sqrt{1} \\ &\iff x + 2 = \pm 1 \\ &\iff x + 2 = 1, \text{ or } x + 2 = -1 \\ &\iff x = -1, \text{ or } x = -3 \end{aligned}$$

The y-intercept is the point with $x = 0$. Plugging $x = 0$ in the equation we get

$$y + 1 = (2)^2 \iff y + 1 = 4 \iff y = 3$$

To find the point symmetric to the y-intercept, we notice that 0 is 2 units to the right of the vertex. So the x-coordinate of the point is 2 units to the left of the x-coordinate of the vertex, i.e. at -4.

Example 1. Graph the equation: $y + 2 = -2(x-1)^2$

Answer. The vertex is at $(1, -2)$, so the axis of symmetry is $x = 1$.

To find the x-intercepts we put $y = 0$, and solve:

$$2 = -2(x-1)^2 \iff -1 = (x-1)^2$$

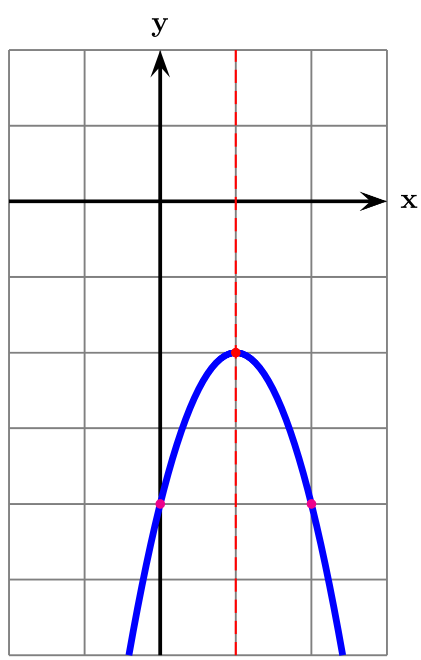
This equation has *no real solutions*, because negative numbers have imaginary square roots. It follows that the parabola has no x-intercepts.

The y-intercept is obtained by setting $x = 0$:

$$y + 2 = -2(-1)^2 \iff y + 2 = -2 \iff y = -4$$

To find the point symmetric to the y-intercept, we observe that 0 is 1 unit to the right of the vertex, so the symmetric point will be 1 unit to the right of the vertex. So the point symmetric to the y-intercept is $(2, -4)$.

In sum the graph is as follows:



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Now the question is:

Question. If we start with an equation in *standard* form

$$y = ax^2 + bx + c \tag{2}$$

how can we find the vertex form (1)?

Answer. We complete the square, of course! Instead of doing it every time we have an equation in standard form we will do it once for the general form (2), and obtain formulas. That's the power of algebra! So here goes:

$$
\begin{aligned}
y = ax^2 + bx + c &\Longleftrightarrow y - c = a\left(x^2 + \frac{b}{a}x\right) \\
&\Longleftrightarrow \frac{y-c}{a} = x^2 + \frac{b}{a}x \\
&\Longleftrightarrow \frac{y-c}{a} + \left(\frac{b}{2a}\right)^2 = x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 \\
&\Longleftrightarrow \frac{y-c}{a} + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2 \\
&\Longleftrightarrow y - c + \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 \\
&\Longleftrightarrow y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2 \\
&\Longleftrightarrow y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2
\end{aligned}
$$

where D stands for our old friend, the *discriminant*. □

The calculations above show that:

Fact 1. *The coordinates of the vertex of an equation in standard form* (2) *are given by:*

$$\boxed{h = -\frac{b}{2a},\ k = -\frac{D}{2a}} \tag{3}$$

Example 2. Graph the parabola $y = x^2 + 6x - 1$. Accurately identify the vertex, the axis of symmetry, the x-intercepts if present, the y-intercept, and the point symmetric to the y-intercept.

Answer. We first find the coordinates of the vertex, using the formulas (3):

$$h = -\frac{b}{2a} = -\frac{6}{2} = -3$$

For k we need first to calculate the discriminant:

$$D = b^2 - 4ac = 6^2 - 4 \cdot 1 \cdot (-1) = 36 + 4 = 40$$

so

$$k = -\frac{D}{4a} = -\frac{40}{4} = -10$$

The vertex is thus at $(-3, -10)$.

To find the x-intercepts we solve the equation:

$$x^2 + 6x - 1 = 0$$

By the quadratic formula the solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-6 \pm \sqrt{40}}{2} = \frac{-6 \pm 2\sqrt{10}}{2} = -3 \pm \sqrt{10}$$

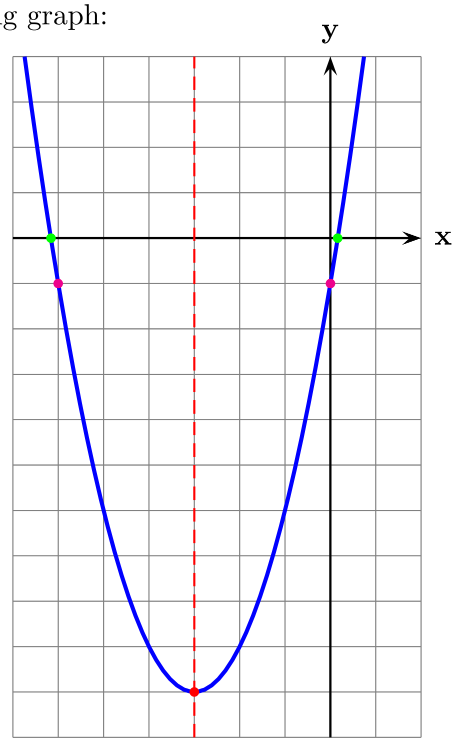
These are irrational numbers. We can use a calculator to get an approximation:

$$x \approx 0.16227766, \text{ or } x \approx -6.16227766$$

To find the y-intercept we put $x = 0$. We get $y = -1$.

0 is 3 units to the right of the vertex, so the point symmetric to the y-intercept is 3 units to the left of the vertex. So it has coordinates $(-6, -1)$.

In sum, we have the following graph:



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Example 3. Graph the parabola $y = x^2 - 2x - 2$. Accurately identify the vertex, the axis of symmetry, the x-intercepts if present, the y-intercept, and the point symmetric to the y-intercept.

Answer. First we find the discriminant:

$$D = b^2 - 4ac = (-2)^2 - 4(1)(-2) = 4 + 8 = 12$$

Now we calculate the vertex:

$$h = -\frac{b}{2a} = -\frac{-2}{2} = 1$$
$$k = -\frac{D}{4a} = -\frac{12}{4} = -3$$

So the vertex is at $(1, -3)$ and the axis of symmetry is $x = 1$.

The x-intercepts:

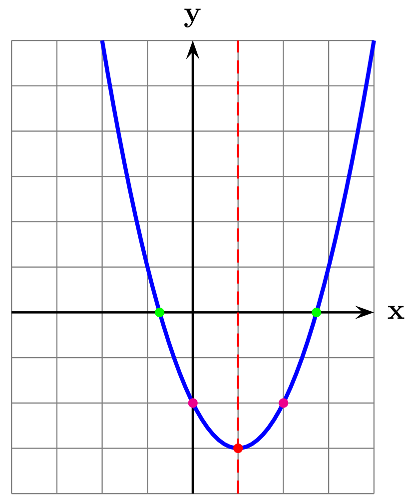
$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-2) \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

We use the calculator to find approximations to the x-intercepts:

$$1-\sqrt{3}\approx -0.73205, \quad 1+\sqrt{3}\approx 2.73205$$

The y-intercept is $y=-2$. 0 is 1 unit to the left of the vertex so the point symmetric to the y-intercept is 1 unit to the right of the vertex, so it is $(2,-2)$.

In sum we have the following:



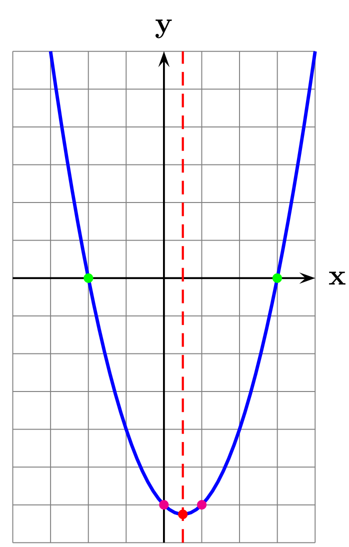
□

Now you practice: For each of the following equations identify the vertex, the axis of symmetry, the x-intercepts if present, the y-intercept, and the point symmetric to the y-intercept. Then sketch a rough graph.

1. $y=2x^2-4x+6$
2. $y=x^2-x-6$
3. $y=x^2+5x+6$

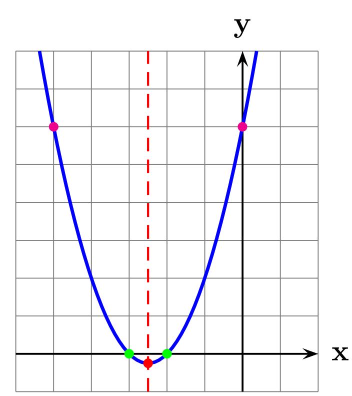
The graphs are shown below:

$y=2x^2-4x+6$



$x=\frac{1}{2}$

$y=x^2-x-6$



$x=-\frac{5}{2}$

$y=x^2+5x+6$

Exercises

1. Graph each of the following quadratic equations:
   (a) $y = 2x^2 + 4x - 6$
   (b) $y = -x^2 + 2x + 4$
   (c) $y = -x^2 + x + 6$
   (d) $y = x^2 - 2x - 8$
   (e) $y = x^2 - 5x + 6$
   (f) $y = 3x^2 - 6x$
   (g) $y = -2x^2 + 4$
2. A parabola has axis of symmetry $x = 3$, meets the x-axis at the points $(1, 0)$ and $(5, 0)$, and the y-axis at the point $(0, 10)$. Find an equation of the parabola in standard form.