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1 The six trigonometric functions

Given a right triangle, once we select one of its acute angles θ, we can describe the sides as O (*opposite* of θ), A (*adjacent* to θ), and H (hypotenuse). There are six possible ways to divide one of the sides by an other side, and each of these ratios has a name:

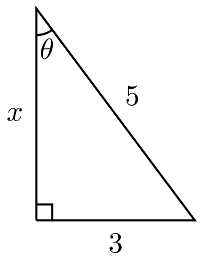
Table 1: **The six trigonometric functions of θ**

Name of function	**Abbreviation**		**Definition**		
sine θ	$\sin\ \theta$	$=$	$\dfrac{\text{opposite side}}{\text{hypotenuse}}$	$=$	$\dfrac{O}{H}$
cosine θ	$\cos\ \theta$	$=$	$\dfrac{\text{adjacent side}}{\text{hypotenuse}}$	$=$	$\dfrac{A}{H}$
tangent θ	$\tan\ \theta$	$=$	$\dfrac{\text{opposite side}}{\text{adjacent side}}$	$=$	$\dfrac{O}{A}$
cosecant θ	$\csc\ \theta$	$=$	$\dfrac{\text{hypotenuse}}{\text{opposite side}}$	$=$	$\dfrac{H}{O}$
secant θ	$\sec\ \theta$	$=$	$\dfrac{\text{hypotenuse}}{\text{adjacent side}}$	$=$	$\dfrac{H}{A}$
cotangent θ	$\cot\ \theta$	$=$	$\dfrac{\text{adjacent side}}{\text{opposite side}}$	$=$	$\dfrac{A}{O}$

All these ratios are not independent of each other. If we know one of them we can figure out the rest.

Example 1. The acute angle θ as $\sin\theta = \frac{3}{5}$. Find $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

Answer. We can use any right triangle that has an angle with $\sin\theta = \frac{3}{5}$ to calculate the other trig functions. If we have a right triangle with hypotenuse $H = 5$, and the leg opposite to θ to be $O = 3$, then $\sin\theta = \frac{O}{H} = \frac{3}{5}$. Such a triangle is shown below:



Using the Pythagorean Theorem we can calculate:

$$x^2 = 5^2 - 3^2 = 16 \Longrightarrow x = 4$$

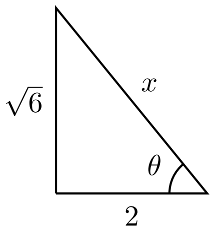
So we have all three sides and we can calculate:

$$\cos\theta = \frac{4}{5}$$
$$\tan\theta = \frac{3}{4}$$
$$\cot\theta = \frac{4}{3}$$
$$\sec\theta = \frac{5}{4}$$
$$\csc\theta = \frac{5}{3}$$

□

Example 2. An acute angle θ has $\tan\theta = \frac{\sqrt{6}}{2}$. Find the other trigonometric ratios of θ.

Answer. The angle θ in the following triangle has tangent equal to $\frac{\sqrt{6}}{2}$.



Using the Pythagorean theorem we can calculate:

$$x^2 = \left(\sqrt{6}\right)^2 + 2^2 \Longrightarrow x^2 = 10 \Longrightarrow x = \sqrt{10}$$

So we have:

$$\sin\theta = \frac{\sqrt{6}}{\sqrt{10}} = \frac{\sqrt{60}}{10} = \frac{2\sqrt{15}}{10} = \frac{\sqrt{15}}{5}$$
$$\cos\theta = \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$$
$$\cot\theta = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$
$$\sec\theta = \frac{\sqrt{10}}{2}$$
$$\csc\theta = \frac{\sqrt{10}}{\sqrt{6}} = \frac{\sqrt{60}}{6} = \frac{2\sqrt{15}}{6} = \frac{\sqrt{15}}{3}$$

2 Exercises

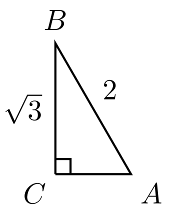
1. Find the values of the other five trigonometric ratios of the acute angle θ given the indicated value of one of the ratios.
   (a) $\sin\ \theta = \dfrac{3}{4}$
   (b) $\cos\ \theta = \dfrac{2}{3}$
   (c) $\cos\ \theta = \dfrac{\sqrt{10}}{5}$
   (d) $\sin\ \theta = \dfrac{2}{4}$
   (e) $\tan\ \theta = \dfrac{5}{9}$
   (f) $\tan\ \theta = 3$
   (g) $\sec\ \theta = \dfrac{7}{3}$
   (h) $\csc\ \theta = 3$

3 Solving right triangles

A triangle has three sides and three angles a total of six unknown quantities. To *solve* a triangle means to find the measurements of all its sides and all its angles.

Example 3. In a right triangle ABC, with $C = 90°$, we have $a = \sqrt{3}$ units and $c = 2$ units. Solve this triangle.

Answer. First recall the convention that when we name the corners and the sides of a triangle we use upper case letters for the corners and then we name each side with the lower case letter of the corner opposite it. So in our triangle c is the side opposite to C, and a is opposite the corner A. So we have the following:



Using the Pythagorean theorem we can calculate:

$$b^2 + \left(\sqrt{3}\right)^2 = 2^2 \Longrightarrow b^2 = 4 - 3 \Longrightarrow b^2 = 1 \Longrightarrow b = 1$$

So we have $\sin B = \frac{1}{2} \Longrightarrow B = 30°$ Then $A = 90° - 30° \Longrightarrow A = 60°$

□

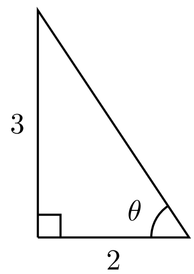
In this example we made use of the fact that we know that the sine of 30° is $\frac{1}{2}$[1]. What would we have done if we didn't get a sine that we recognize? Well, that's what calculators are for! All scientific calculators have buttons labeled $\sin^{-1}$ (as well as $\cos^{-1}$, and $\tan^{-1}$). These buttons give you what angle has a given sine. For example, I can use the builtin calculator in my editor to find:

$$\sin^{-1} 0.83 \approx 56.0987380031$$

[1] We actually cheated a bit. We assumed that if an angle has the same sine as an angle of 30°, then this angle must be 30°. That turns out to be true: if two acute angles have the same sine then they are equal

This means that the angle that has sine 0.83 is (approximately) 56.0987380031°. Usually in the class we round to the nearest hundredth (i.e. the second decimal place), so if we need to use an angle that has sine .83 we will use 56.1°.

Example 4. Find θ in the triangle below:



Answer. We have from the figure that

$$\tan\theta = \frac{3}{2}$$

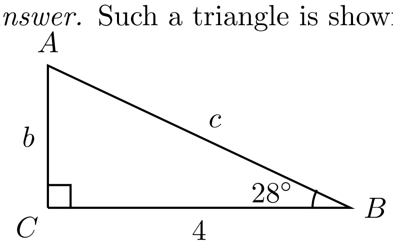
Using our calculator we get:

$$\tan^{-1}\frac{3}{2} = 56.309932474$$

Rounding to the nearest hundredth, we have $\theta \approx 56.3°$. □

Example 5. In a right triangle ABC with $C = 90°$ we have that $b = 4$ and $B = 28°$. Solve the triangle.

Answer. Such a triangle is shown below:



We have that $A = 90° - 28° = 62°$.
We can calculate b using $\tan 28^{circ} = \frac{b}{4}$. The calculator gives $\tan 28° \approx 0.53$. So

$$\frac{b}{4} = 0.53 \Longrightarrow b = 2.12$$

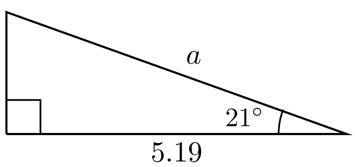
We also have $\cos 28° = \frac{4}{c}$. The calculator gives $\cos 28° \approx 0.88$. So we have:

$$\frac{4}{c} = .88 \Longrightarrow c = \frac{4}{.88} \Longrightarrow c = 4.55$$

□

4 Exercises

1. In the following figure find a:



2. In a right triangle KLM with $K = 90°$ have that $M = 22°$, and $k = 3$. Find the lengths of l, m.
3. Solve the triangle ABC where $A = 90°$, $B = 33°$, and $b = 2.5$
4. Solve the triangle ABC where $B = 90°$, $a = \sqrt{2}$, and $b = 2$.
5. In a right triangle ABC with $A = 90°$ we have $a = 2$ and $b = 1$. Find the angle B.
6. In a right triangle ABC with $B = 90°$ we have that $a = 2.1$, and $b = 3$. Find the angle C.
7. In a right triangle PQR we have $P = 90°$, $r = 5$ and $q = 6$. Solve the triangle.
8. In a triangle with $A = 90°$ and $\cos B = .32$ find $\sin B$.
9. For an acute angle θ of a right triangle we have $\sin\theta = \frac{\sqrt{5}}{3}$. Find $\cos\theta$ and $\tan\theta$.
10. The acute angles of a right triangle are ϕ and θ. If $\tan\theta = 4.3$ find $\cos\phi$.

□