Solving Radical Equqtions

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1 Solving Radical equations

Example 1. Solve the equation:

$$\sqrt{x} = 3$$

Answer. We just square both sides:

$$\left(\sqrt{x}\right)^2 = 3^2 \iff x = 9$$

□

Example 2. Solve the equation:

$$\sqrt{x-3} = 2$$

Answer. Again, we first square both sides to get rid of the radical sign:

$$\left(\sqrt{x-3}\right)^2 = 2^2 \iff x - 3 = 4 \iff x = 7$$

□

Example 3. Solve the equation:

$$\sqrt{x-3} - 5 = -2$$

Answer. Now if we just square both sides we won't get rid of the radical. So we need to *isolate the terms with radicals* first. We do that by transfering -5 to the RHS:

$$\sqrt{2x+3} - 5 = -2 \iff \sqrt{2x+3} = -2 + 5 \iff \sqrt{2x+3} = 3$$

Now we can square both sides and then solve the linear equation we get.

$$\begin{aligned}\left(\sqrt{2x+3}\right)^2 = 3^2 &\iff 2x + 3 = 9 \\ &\iff 2x = 6 \\ &\iff x = \frac{6}{2} \\ &\iff x = 3\end{aligned}$$

□

Caution: When we square both sides of an equation we don't get an equivalent equation. The new equation may have more solutions. For example, if we start with the equation

$$x = 3$$

that obviously has only one solution, and we square it we get

$$x^2 = 9$$

and this equation has *two* solutions $x = 3$ and $x = -3$. So by squaring both sides of the equation we introduced the *extraneous* solution $x = -3$, that is not a solution of the original equation.

To remedy this, we have to make sure that after we find the solutions of the squared equation, we check whether they are really solutions of the original equation and discard the extraneous ones.

Example 4. Solve the equation:

$$3 - \sqrt{x+1} = 5$$

Answer. We first separate the radical expression:

$$-\sqrt{x+1} = 2$$

Next we square both sides and solve:

$$\left(-\sqrt{x+1}\right)^2 = 2^2 \Longleftrightarrow x + 1 = 4 \Longleftrightarrow x = 3$$

But when we substitute this solution to the *original* equation we get:

$$3 - \sqrt{3+1} = 5 \Longleftrightarrow 3 - 2 = 5$$

which is *not* a true equation. So this is an *extraneous* solution. Since the only solution of the squared equation is extraneous, the original equation doesn't have solutions. □

Example 5. Solve the equation:

$$\sqrt{x+2} = x$$

Answer. Again we start by squaring both sides:

$$\left(\sqrt{x+2}\right)^2 = x^2 \Longleftrightarrow x + 2 = x^2$$

Now we have a quadratic equation. To solve it we bring it to standard form:

$$x^2 - x - 2 = 0$$

We can easily factor the LHS of this equation to get:

$$(x-2)(x+1) = 0$$

which gives us two solutions:

$$x = 2, \text{ or } x = -1$$

We are not done yet! We have to check that these solutions are actually solutions of the original equation.

We first check whether $x = 2$ is a solution, by substituting it in the original equation. We get:

$$\sqrt{2+2} = 2 \iff \sqrt{4} = 2$$

This is a true equation, so $x = 2$ is a solution.

Next we check $x = -1$:

$$\sqrt{-1+2} = -1 \iff \sqrt{1} = -1$$

This is a false equation. So $x = -1$ is an extraneous solution.

So the only solution of the original equation is $x = 2$. □

Sometimes we have equations with more than one radical expressions.

Example 6. Solve:

$$\sqrt{x+12} = 2 + \sqrt{x}$$

Answer. We will start again by squaring both sides, but this won't get rid of the radicals immediately:

$$\left(\sqrt{x+12}\right)^2 = \left(2+\sqrt{x}\right)^2 \iff x + 12 = 4 + 4\sqrt{x} + x$$

So we still have radical expressions but we did accomplish something: now we have only one radical expression. So we can continue, we first isolate the radical expression:

$$x + 12 - x - 4 = 4\sqrt{x} \iff 8 = 4\sqrt{x}$$

Then square both sides:

$$64 = 16x \iff \frac{64}{16} = x \iff 4 = x$$

We now check whether this solution is also a solution of the original equation:

$$\sqrt{4+12} = 2 + \sqrt{4} \iff \sqrt{16} = 2 + 2$$

which is true. So the solution is $x = 4$. □

Example 7. Solve:

$$\sqrt{2x-2} - \sqrt{x} = 1$$

Answer. First we arrange so that we have one radical expression in each side.

$$\sqrt{2x-2} = 1 + \sqrt{x}$$

Then we square both sides:

$$\begin{aligned}
\left(\sqrt{2x-2}\right)^2 = \left(1+\sqrt{x}\right)^2 &\iff 2x - 2 = 1 + 2\sqrt{x} + x \\
&\iff 2x - 2 - 1 - x = 2\sqrt{x} \\
&\iff x - 3 = 2\sqrt{x} \\
&\implies (x-3)^2 = \left(2\sqrt{x}\right)^2 \\
&\iff x^2 - 6x + 9 = 4x \\
&\iff x^2 - 6x - 4x + 9 = 0 \\
&\iff x^2 - 10x + 9 = 0 \\
&\iff (x-9)(x-1) = 0 \\
&\iff x = 9, \text{ or } x = 1
\end{aligned}$$

Now we check whether these solutions are solutions of the original equation.

For $x = 9$ we have:

$$\sqrt{2 \cdot 9 - 2} = 1 + \sqrt{9} \Longleftrightarrow \sqrt{16} = 1 + 3$$

which is true. So $x = 9$ is a solution.

For $x = 1$ we have:

$$\sqrt{2 \cdot 1 - 2} = 1 + \sqrt{1} \Longleftrightarrow \sqrt{0} = 1 + 1$$

which is false. So $x = 1$ is extraneous.

So the equation has only one solution $x = 9$. □

2 Exercises

1. Solve each of the following equations:
   (a) $\sqrt{x+5} = 4$
   (b) $\sqrt{5x+36} = 9$
   (c) $\sqrt{3x-1} = -3$
   (d) $\sqrt{2x+5} - 3 = 2$
   (e) $\sqrt{3x+1} + 4 = 2$
   (f) $\sqrt{x+5} = x - 5$
   (g) $\sqrt{2x+5} = x + 3$
   (h) $-\sqrt{x+3} = x - 9$
   (i) $\sqrt{2x+4} = \sqrt{1-x}$
   (j) $\sqrt{3x-4} = \sqrt{2x+7}$
   (k) $\sqrt{x+3} = \sqrt{x} - 3$
   (l) $\sqrt{x+5} - \sqrt{10x-4} = -3$
   (m) $\sqrt{\sqrt{x-3}+5} = \sqrt{x+2}$