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Review of fractions

Rational expressions are fractions with numerator and denominator polynomials. We need to remember how we work with fractions (a.k.a. rational numbers) before we start working with rational expressions. Work through the following exercises:

1. Evaluate:
   (a) $2 - \frac{3}{5}$
   (b) $\frac{12}{5} \cdot \frac{10}{3}$
   (c) $\dfrac{\frac{3}{4}}{\frac{8}{9}}$
   (d) $\frac{2}{3}\left(\frac{1}{2} - \frac{3}{4}\right)$
   (e) $x^2 - x$, when $x = \frac{3}{5}$
   (f) $4x - 3y$, when $x = -\frac{7}{8}$ and $y = \frac{2}{5}$
   (g) $\frac{x+y}{z}$, when $x = 2$, $y = -\frac{3}{4}$, and $z = \frac{1}{2}$

Rational expressions

A *rational expression* is an algebraic expression that can be expressed as a division of two polynomials, in other words, it's a fraction whose numerator and denominator are polynomials.

Example 1. The following are examples of rational expressions:

$$x^2 - 3x + 2, \qquad \frac{x^3 - 1}{x^5 - 5x^2 + 3x - 7}, \quad \frac{2x+7}{4x-1}, \quad \frac{2x^2 - 3x + 5}{x^2 + 6x - 2}$$

The following are not rational expressions:

$$\sqrt{x^2 - 5}, \qquad x^{2/3} + \frac{1}{x^2}, \qquad 3^{x^2 - 1}$$

because the first one involves a square root, the second has a rational exponent (and so it involves a third root), and the third one involves variable exponents.

One characteristic of rational expressions is that there may be some real numbers for which the expression cannot be evaluated.

Example 2. Consider the following rational expression:

$$\frac{x^2 + 5x - 6}{x - 2}$$

If we try to evaluate this expression for $x = 2$ we get:

$$\frac{(2)^2 + 5(2) - 6}{(2) - 2} = \frac{25 + 10 - 6}{2 - 2} = \frac{29}{0}$$

Since division by zero is undefined, the expression is undefined when $x = 2$.

In this case, $x = 2$ is the only value of x for which the expression is undefined. Substituting any other real number for x will give us a real number in the numerator and a *non-zero* real number in the denominator, and so the division will make sense.

In general:

A rational expression $\dfrac{P(x)}{Q(x)}$ is undefined at the *roots* of the denominator $Q(x)$. In other words the expression is defined for all real numbers except those that make the denominator equal to 0.

Example 3. Determine the values of x for which the following expression is defined:

$$\frac{2x+3}{5x-6}$$

Solution. The expression is not defined when the denominator $5x - 6$ is 0. We have:

$$5x - 6 = 0 \iff 5x = 6 \iff x = \frac{6}{5}$$

So the expression is defined for all real numbers except $\frac{6}{5}$. □

Example 4. Determine the values of x for which the following expression is defined:

$$\frac{x^3 + 5x^2 - 4x - 1}{x^2 - 5x + 6}$$

Answer. The expression is undefined when

$$x^2 - 5x + 6 = 0$$

This quadratic equation has two real solutions, $x = 2$ and $x = 3$. Therefore the expression is defined for all real numbers except 2 and 3. □

Example 5. Determine the values of x for which the following expression is defined:

$$\frac{x^2 - 3}{x^2 - 2x + 5}$$

Answer. The expression is defined for all real numbers except the solutions of the equation

$$x^2 - 2x + 5 = 0$$

The discriminant of this quadratic equation is $D = (-2)^2 - 4 \cdot (1)(4) - 16$. It follows that the denominator has no real roots. Therefore there are no exceptions: the expression is defined for all real numbers. □

Exercises

1. Determine the values of x for which the following expression is defined.

   (a) $\dfrac{x^3 - 8}{x + 2}$

   (b) $\dfrac{x^2 - 5}{3x - 4}$

   (c) $\dfrac{3x - 5}{x^3 - 8}$

(d) $\dfrac{2x-1}{x^3+x^2-2x}$

(e) $\dfrac{x^2-4}{x^2+5}$

(f) $\dfrac{x^2+7x-44}{-2x^2+5x+12}$

(g) $\dfrac{x^3+27}{x^2+9x+18}$

(h) $\dfrac{x^2-6x+5}{x^2-2x-2}$

(i) $\dfrac{3x}{x^4-81}$

Simplifying Rational Expressions

Example 6. Simplify: $\dfrac{x^2+x}{2x+2}$

Answer. We have to factor both the numerator and the denominator of the expression and then cancel any common factors.

$$\begin{aligned}\frac{x^2+x}{2x+2} &= \frac{x(x+1)}{2(x+1)} \\ &= \frac{x}{2}\end{aligned}$$

□

Example 7. Simplify: $\dfrac{x^2+3x-18}{x^2-36}$

Answer. Again we factor numerator and denominator and cancel common factors:

$$\begin{aligned}\frac{x^2+3x-18}{x^2-36} &= \frac{(x-3)(x+6)}{(x-6)(x+6)} \\ &= \frac{x-3}{x-6}\end{aligned}$$

□

Example 8. Simplify: $\dfrac{x-3}{3-x}$

Answer. In this example the numerator and the denominator are *opposite expressions*. So after taking -1 as a common factor from one of them, say the denominator, we get:

$$\begin{aligned}\frac{x-3}{3-x} &= \frac{x-3}{-1(x-3)} \\ &= \frac{1}{-1} \\ &= -1\end{aligned}$$

□

1. Simplify each of the following. Assume that all rational expressions that appear are defined.

   (a) $\dfrac{2a^2b^3}{ab^4c}$

(b) $\dfrac{-45x^4y}{75x^5y^3}$

(c) $\dfrac{x^5 - 3x^4 + 7x^3 + x^2}{x^3}$

(d) $\dfrac{x^2 - 4}{3x + 6}$

(e) $\dfrac{a^4 - 16}{a^3 + 4a}$

(f) $\dfrac{xy - xy^2}{x - 1}$

(g) $\dfrac{x^4 - 13x^2 + 36}{x^2 - x - 6}$

(h) $\dfrac{10x^3 - 10x}{5x^3 + 5x^2 + 5x}$

(i) $\dfrac{a - b}{b - a}$

(j) $\dfrac{x - 5}{-x^2 - 2x + 35}$

(k) $\dfrac{x^2 + 3x - 10}{5x - x^2 - 6}$

(l) $\dfrac{2x^2 - 7x + 3}{9 - x^2}$

Multiplication and Division of Rational Expressions

1. Perform the indicated operations. Simplify your answers as much as possible.

(a) $\dfrac{10y^2}{5x^2} \cdot \dfrac{15x^3}{2y^4}$

(b) $\dfrac{18mn^3}{16m^2n^2} \div \dfrac{8mn^3}{9m^2n}$

(c) $\dfrac{5x^2y^5}{7xy} \cdot \dfrac{21x^3}{10y^4} \div \dfrac{3x^3y^2}{2x^5y^3}$

(d) $24x^4y^3 \cdot \dfrac{5x^2}{8x^3y^5}$

(e) $\dfrac{\frac{2a^4b^3}{c^3}}{\frac{a^5b^2c}{abc^4}}$

(f) $(3a + 3b) \div \dfrac{3}{a + b}$

(g) $(2x - 5) \cdot \dfrac{x^2 + 1}{4x - 10}$

(h) $\dfrac{x^2 - 6x - 55}{x^2 + 12x + 35} \cdot \dfrac{x + 5}{3x - 33}$

(i) $\dfrac{x^3 + 2x^2}{x^2 - 4x + 4} \cdot \dfrac{x^2 + 13x + 22}{3x^2 + 5x - 2}$

(j) $\dfrac{27t^3 + 8}{9t^3 - 6t^2 + 4t} \cdot \dfrac{t^2 - t}{3t^2 - t - 2}$

(k) $\dfrac{x^2-5x-24}{-x^2-x+30}\cdot\dfrac{x^2-7x+10}{x^2-15x+56}$

(l) $\dfrac{x^4-81}{x^2-x-20}\div\dfrac{x^2-x-6}{3x-x^2+10}$

(m) $\dfrac{x+3}{4x^2-12x+9}\div(3x^2+11x+6)$

(n) $\dfrac{\dfrac{ac+ad+bc+bd}{ac+ad-bc-bd}}{a^2-b^2}$

(o) $\dfrac{x^2-6xy+9y^2}{x^2-4y^2}\cdot\dfrac{x^2-5xy+6y^2}{(x-3y)^2}\div\dfrac{x^2-9y^2}{x^2-xy-6y^2}$

(p) $\dfrac{\dfrac{1}{x^2-16}}{\dfrac{1}{(x-4)^2}}$

(q) $\dfrac{\dfrac{a^3-a^2b}{ac-a}}{\left(\dfrac{a-b}{c-1}\right)^2}$

Addition and subtraction of rational expressions

1. Perform the following operations and simplify your answers as much as possible:

A. $\dfrac{3x+5}{x+1}+\dfrac{2x-5}{x+1}$ B. $\dfrac{3x}{x+2}+\dfrac{6}{x+2}$ C. $\dfrac{x^2}{3x-12}-\dfrac{16}{3x-12}$ D. $\dfrac{3a+1}{2a-6}+\dfrac{a+2}{3-a}$

E. $\dfrac{4}{x}+\dfrac{4}{y}$ F. $2+\dfrac{3b}{2b-1}$ G. $\dfrac{2x^2+3x-9}{x+3}-x$ H. $\dfrac{3x-1}{x-2}+\dfrac{3x}{2-x}$ I. $\dfrac{5}{x}-\dfrac{5x-1}{x^2}$

J. $\dfrac{3x}{5}-\dfrac{5}{3x}$ K. $\dfrac{10x^2-x+11}{x^2-81}-\dfrac{9x^2-x+20}{x^2-81}$ L. $\dfrac{5}{3x-3}-\dfrac{2}{x-1}$ M. $\dfrac{3x+7}{x^2-x-12}+\dfrac{4x+3}{12+x-x^2}$

N. $\dfrac{4}{x-2}+\dfrac{8}{x^2-4x+4}$ O. $\dfrac{a-7}{a^2-4}-\dfrac{a}{a^2+3a+2}$ P. $\dfrac{3x}{x^2+x-6}+\dfrac{2}{2x^2+7x+3}$

Q. $\dfrac{1}{x-2}-\dfrac{2x+8}{x^3-8}$ R. $\dfrac{1}{x+1}-\dfrac{1}{x}+\dfrac{2}{x^2}$ S. $t-\dfrac{5}{2t-1}+2$ T. $\dfrac{5z-12}{z^2-8z+15}-\dfrac{3z-2}{z-3}+\dfrac{3}{z-5}$

U. $\dfrac{2}{y^2-9}-\dfrac{3}{y^2-4y+3}+\dfrac{y-1}{y^2+2y-3}$ V. $-\dfrac{2x}{x^2-4}+\dfrac{3}{x-2}+\dfrac{2x+1}{x+2}$ W. $\left(\dfrac{b}{2a}\right)^2-\dfrac{c}{a}$

Complex Fractions

1. Evaluate each expression:

A. $1+\dfrac{1}{2}$ B. $1+\dfrac{1}{1+\dfrac{1}{2}}$ C. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{2}}}$ D. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$

2. Simplify each of the following as much as possible:

A. $\dfrac{\frac{1}{a}}{1-\frac{1}{a}}$ B. $\dfrac{2-\frac{3}{a}}{2+\frac{3}{a}}$ C. $\dfrac{\frac{x}{y}-3}{\frac{x^2}{y^2}-9}$ D. $\dfrac{1+\frac{1}{a+3}}{1+\frac{7}{a-3}}$ E. $\dfrac{\frac{1}{x+h}-\frac{1}{h}}{h}$

F. $\dfrac{\frac{2}{x}-1}{\frac{3}{x^2}+2}$ G. $\dfrac{\frac{1}{a-b}+\frac{1}{a+b}}{\frac{1}{a^2-b^2}}$ H. $\dfrac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x}+\frac{1}{y}}$ I. $\dfrac{\frac{1}{x+2}-\frac{1}{x+3}}{\frac{2}{x^2+5x+6}}$ J. $\dfrac{\frac{3}{2x+1}-\frac{x}{x+2}}{\frac{2x^2-2x-6}{2x^2+5x+2}}$

K. $\dfrac{\frac{2}{x-2}+3}{1-\frac{3}{x+1}}$ L. $\dfrac{1+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}}$ M. $\dfrac{1-\frac{3}{x}-\frac{10}{x^2}}{\frac{5}{x}-1}$ N. $\dfrac{\frac{z+3}{z-3}+\frac{z-3}{z+3}}{\frac{z+3}{z-3}-\frac{z-3}{z+3}}$