Using the quadratic formula

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Recall 1. Recall that last time we discussed the *quadratic formula:* The solution to the quadratic equation, in *standard form*,

$$ax^2 + bx + c = 0$$

where $a \neq 0$ is given by the formula:

$$x = \frac{-b \pm \sqrt{D}}{2a} \tag{1}$$

where,

$$D = b^2 - 4ac \tag{2}$$

is the *discriminant* of the equation.

Example 1. Solve the quadratic equation:

$$6x^2 - 5x - 6 = 0$$

Answer. The LHS of the equation is 0 so this equation is in standard form. To use the quadratic formula we need to identify the coefficients, a, b, and c: a is the coefficient of the quadratic term so $a = 6$, b is the coefficient of the linear term so $b = -5$, and c is the constant term, so $c = -6$.

The next step is to find the value of the discriminant:

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-5)^2 - 4(6)(-6) \\ &= 25 + 144 \\ &= 169 \end{aligned}$$

So the two solutions are given by

$$\begin{aligned} x &= \frac{-(-5) \pm \sqrt{169}}{2(6)} \\ &= \frac{5 \pm 13}{12} \\ &= \begin{cases} \frac{18}{12} \\ \frac{-8}{12} \end{cases} \\ &= \begin{cases} \frac{3}{2} \\ -\frac{2}{3} \end{cases} \end{aligned}$$

□

Example 2. Solve the quadratic equation:

$$x^2 + 2x = 2$$

Answer. In this case the equation is *not* in standard form because the LHS is not 0. So before we can apply the quadratic formula we need to put the equation into standard form, we do that by transfering the constant term 2 from the LHS to the RHS. Remember that when we transfer terms from one side of the equation to the other we need to change their sign! The equation in standard form is then:

$$x^2 + 2x - 2 = 0$$

So for this equation, $a = 2$, $b = 2$, and $c = -2$. As always we first compute the discriminant:

$$\begin{aligned} D &= b^2 - 4ac \\ &= (2)^2 - 4(1)(-2) \\ &= 4 + 8 \\ &= 12 \end{aligned}$$

So the two solutions are given by

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{12}}{2(1)} \\ &= \frac{-2 \pm 2\sqrt{3}}{2} \\ &= -1 \pm \sqrt{3} \end{aligned}$$

Recall that the above is a shortcut notation for two solutions:

$$x = -1 + \sqrt{3} \quad \text{or} \quad x = -1 - \sqrt{3}$$

□

Example 3. The hypotenuse of a right triangle is 5 inches long, and one of its legs is 1 inche longer than the other leg. Find the lengths of the two legs of the triangle.

Answer. If x stands for the length of the smaller leg in inches then the larger leg is $x + 1$ inches long. Since the hypotenuse is 5 inches long we have the folowing equation by the Pythagorean Theorem:

$$x^2 + (x+1)^2 = 5^2$$

Before we can solve we need to put the equation into standard form. We start by performing the operations in the LHS and then transfer the constant term from the RHS. We have:

$$\begin{aligned} x^2 + (x+1)^2 = 5^2 &\iff x^2 + x^2 + 2x + 1 = 25 \\ &\iff 2x^2 + 2x + 1 = 25 \\ &\iff 2x^2 + 2x + 1 - 25 = 0 \\ &\iff 2x^2 + 2x - 24 = 0 \end{aligned}$$

Now we have a quadratic equation in standard form so we can apply the quadratic formula. We have $a = 2$, $b = 2$, and $c = -24$. So:

$$\begin{aligned} D &= b^2 - 4ac \\ &= (2)^2 - 4(2)(-24) \\ &= 4 + 192 \\ &= 196 \end{aligned}$$

The two solutions are given by

$$\begin{aligned} x &= \frac{-(2) \pm \sqrt{196}}{2(2)} \\ &= \frac{-2 \pm 14}{4} \\ &= \begin{cases} \frac{12}{4} \\ \frac{-16}{4} \end{cases} \\ &= \begin{cases} 3 \\ -4 \end{cases} \end{aligned}$$

Since the length has to be positive we take the positive solution and we have that the length of the smaller leg is 3 inches. The longer leg is then $3 + 1 = 4$ inches. □

Sometimes the two solutions turn out to be the same. This happens when the discriminant is zero. Here is an example:

Example 4. Solve the following equation:

$$9x^2 - 6x + 1 = 0$$

Answer. We have $a = 9$, $b = -6$, and $c = 1$. So,

$$\begin{aligned} D &= b^2 - 4ac \\ &= (-6)^2 - 4(9)(1) \\ &= 36 - 36 \\ &= 0 \end{aligned}$$

So the two solutions are

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{0}}{2(9)} \\ &= \frac{6 \pm 0}{18} \\ &= \begin{cases} \frac{6+0}{18} \\ \frac{6-0}{18} \end{cases} \\ &= \begin{cases} \frac{1}{3} \\ \frac{1}{3} \end{cases} \end{aligned}$$

So we have one *double* solution $x = \frac{1}{3}$. □

1 Exersices

1. In a right triangle one of the legs is 2 inches shorter than the other. Find the lengths of the two legs if the hypotenuse is 3 inches long.
2. The y coordinate of a point is one more than twice its x coordinate. Find the coordinates of the point if its distance from the origin $(0,0)$ is 1 and it lies in the fourth quadrant.
3. The length of a rectangle is 3 centimeters less that its width. Find the dimensions of the rectangle if its area is 18 square centimeters.

2 The significance of the discriminant

The nature of the solutions to a given quadratic equation can be determined by its discriminant, see Equation (2). Indeed, we have that

- if $D > 0$ then we have two distinct *real* solutions.
- if $D = 0$ then we have one *double* real solution, $x = -\dfrac{b}{2a}$
- if $D < 0$ then we have *no real solutions*: both solutions are complex numbers.

Example 5. Find the real number b so that the equation

$$x^2 + bx + 5 = 0$$

has a double solution.

Answer. In order for the equation to have a double solution we need the discriminant to be 0. But $D = b^2 - 4 \cdot 5 = b^2 - 20$. So we need

$$b^2 - 20 = 0$$

Which gives two possible values: $b = \pm 2\sqrt{5}$. □

3 Exercises

1. Find the real number b so that the following equation:

$$9x^2 + bx + 25 = 0$$

has exactly one (double) real solution.

2. For which real numbers a the equation $ax^2 - 4x + 7 = 0$ has real solutions?
3. For which real numbers c the equation:

$$3x^2 - 5x + c = 0$$

has no real solutions?

4. Find the real number a if the equation: $ax^2 - 12x + 2a + 1 = 0$ has a double solution.