Friday February 10, 2017

Nikos Apostolakis

1 Pythagorean Theorem and distance formula

1. The legs of a right triangle have length 3 cm and 4 cm. Find the length of the hypotenuse.
2. One leg of a right triangle is 4 inches and the hypotenuse is 7 inches. Find the length of the other leg.
3. The hypotenuse of a right triangle is 6 cm and one of its legs is $\sqrt{6}$ cm. Find the length of the other leg.
4. The legs of a right triangle have lengths $\sqrt{2}$ inches and $1+\sqrt{3}$ inches. Find the length of the hypotenuse.
5. What is the length of the diagonal of a square of side 3?
6. One leg of a right triangle is 2 inches more than the other leg. The hypotenuse is 10 inches. Find the three sides of the triangle.
7. The sum of the lengths of the two sides of a right triangle is 4 cm while the length of the hypotenuse is 12 cm. Find the lengths of the two legs.
8. Verify that the triangle ABC in Figure 1 is isosceles.

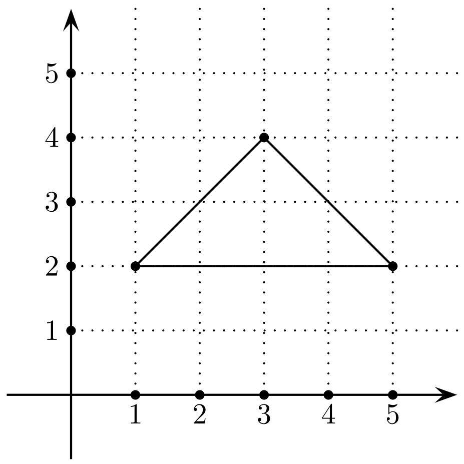
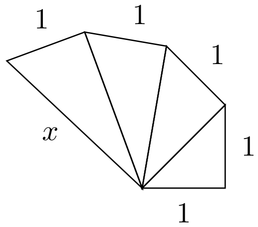


Figure 1: An allegedly isosceles triangle

9. Find x.



10. Find the distance between the points with coordinates
    (a) $(0, 0)$ and $(3, 4)$.
    (b) $(-2, 1)$ and $(-2, -4)$.
    (c) $(1, 3)$ and $(-1, -2)$.
    (d) $(2, 5)$ and $(1, 7)$.
    (e) $(-1, -3)$ and $(-4, -5)$

11. An unknown point lies in the line with equation $x = 3$ and its distance from the point $(1, 2)$ is $2\sqrt{5}$. What are the coordinates of the unknown point?

12. **Extra Credit:** Use the picture in Figure 2 to prove the Pythagorean theorem.

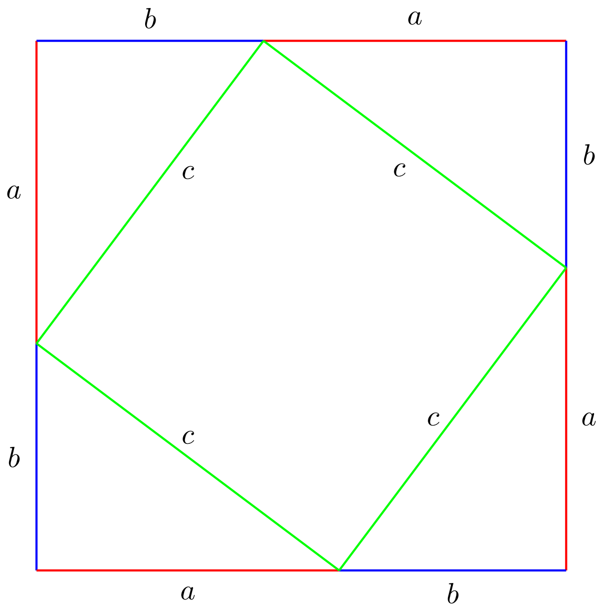


Figure 2: Figure for proving the Pythagorean theorem

13. **Extra Credit:** This is a puzzle that shows another way that one can use to prove the Pythagorean theorem. In Figure 3 a right triangle is shown with squares erected on each of its sides. Use a pair of scissors to cut the three squares and then cut the middle sized square into four pieces along the dashed lines. Your task then is to arrange these four pieces together with the smaller square to completely cover the larger square. Do you see why this is relevant to Pythagorean theorem?

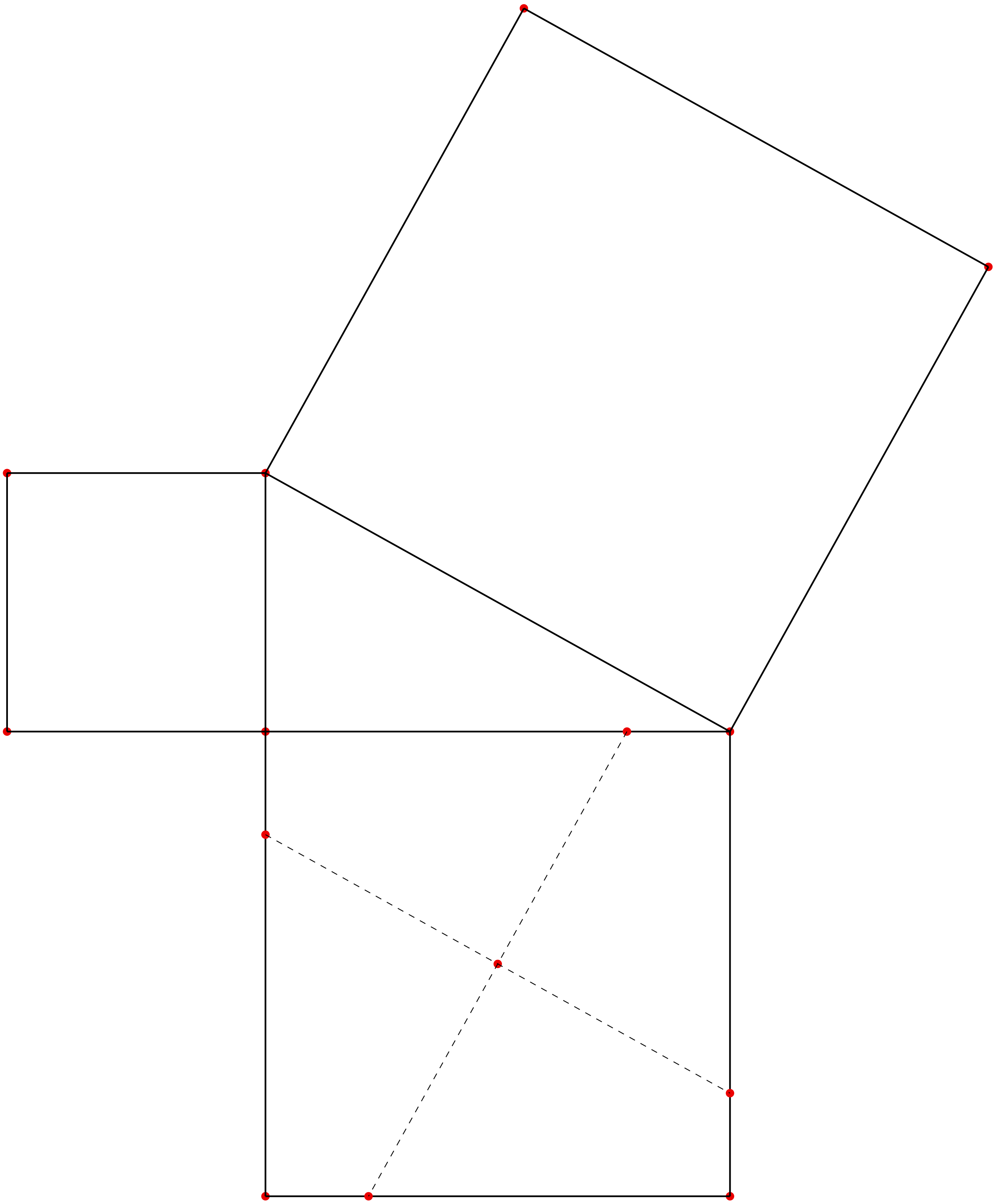


Figure 3: The Pythagorean puzzle

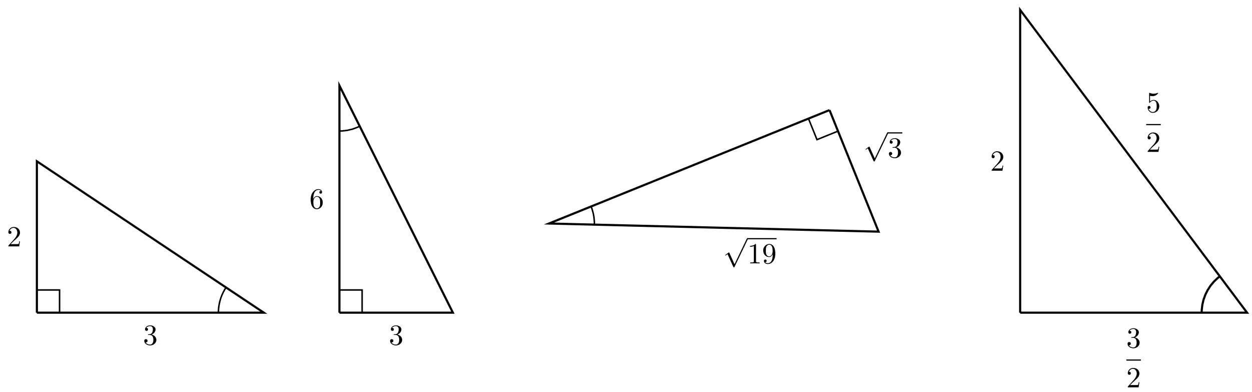
2 Useful Facts about Triangles

- Two triangles that have all three sides congruent are congruent.
- The sum of the measurements of the angles of any triangle is $180°$.
- Therefore a triangle can have *at most one* right angle.
- In a right triangle the sum of the measurement of the two acute angles is $90°$.
- If two triangles have all three angles congruent then they are *similar*. That means that one is a scaled version of the other, and so the ratios of their corresponding sides is the same.
- If two *right* triangles have one *acute* angle congruent then they are similar.
- In a triangle, the relation between two of its sides is the same as the relation between the angles opposite those sides. So if two angles are equal the sides opposite them are equal, and the larger angle is opposite of the larger side.
- Therefore, in a right triangle the hypotenuse (the side opposite to the right angle) is the largest side.

3 Trigonometric ratios

For the following questions you should give **exact answers**. Also you should **not** use your calculator.

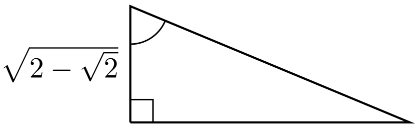
1. For each of the following triangles find the sine, the cosine and the tangent of the indicated angle.



2. In the triangle ABC we have, $A = 90°$, $a = 7$ and $b = 3$. Find $\cos B$ and $\tan C$.
3. In the triangle PQR we have $Q = 90°$, $p = \sqrt{5}$, $r = \sqrt{7}$. Find $\tan P$, $\sin P$ and $\cos P$.
4. In the triangle ABC we have $A = 60°$, $B = 90°$, $c = 2$. Find a and b.
5. The hypotenuse of a right triangle with an angle of $45°$ is $4\sqrt{2}$. What are the lengths of the legs?
6. In the triangle ABC we have $A = 90°$, $b = 5$, and $a = 10$. Find the measures of the two acute angles.

7. The legs of a right triangle have equal lengths. How many degrees is one acute angle in that triangle?

8. Given that the marked acute angle is $67.5°$ and that $\sin 22.5° = \dfrac{\sqrt{2-\sqrt{2}}}{2}$ find x.



4 Trigonometric ratios of special right triangles

Let's calculate the trigonometric rations of $45°$. So let ABC be right triangle where one of its angles is $45°$. Then the other angle is also $45°$ (why?) and therefore we have an *isosceles* right triangle. By scaling we can assume that the hypotenuse has length 2, so let x be the length of one (and hence both) of the legs. Refer to Figure 4

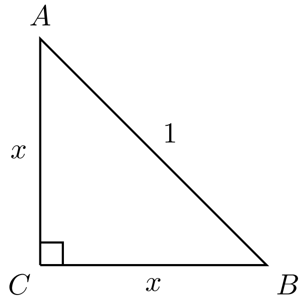


Figure 4: An isosceles right triangle

Using the Pythagorean Theorem we have:

$$\begin{aligned} x^2 + x^2 = 2^2 &\iff 2x^2 = 4 \\ &\iff x^2 = 2 \\ &\implies x = \sqrt{2} \end{aligned}$$

Now that we know all three sides of the triangle we can calculate the trigonometric ratios:

$$\begin{aligned} \sin 45° &= \sin B = \frac{\sqrt{2}}{2} \\ \cos 45° &= \cos B = \frac{\sqrt{2}}{2} \\ \tan 45° &= \tan B = 1 \end{aligned}$$

We can also calculate the trigonometric ratios of $30°$ and $60°$. Start with an equilateral triangle ABD where each side has length 2, and let C be the midpoint of AD. Draw the segment BC, and notice that the triangle ABC is a right triangle with right angle C, and that angle A is $60°$ while angle B is $30°$. Look at Figure 5.

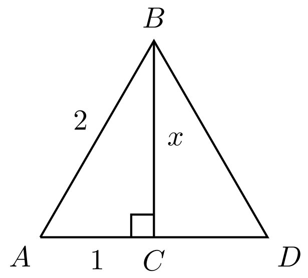


Figure 5: An equilateral triangle

Concentrating on the right triangle ABC, we know that AB has length 2, and AC has length 1. By the Pythagorean theorem we have that if x is the length of BC then:

$$1^2 + x^2 = 2^2 \implies 1 + x^2 = 4 \implies x^2 = 3 \implies x = \sqrt{3}$$

So we can now calculate:

$$\begin{aligned}
\sin 30^\circ &= \sin B = \frac{1}{2} \\
\cos 30^\circ &= \cos B = \frac{\sqrt{3}}{2} \\
\tan 30^\circ &= \tan B = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}
\end{aligned}$$

and,

$$\begin{aligned}
\sin 60^\circ &= \sin A = \frac{\sqrt{3}}{2} \\
\cos 60^\circ &= \cos A = \frac{1}{2} \\
\tan 60^\circ &= \tan A = \frac{\sqrt{3}}{1} = \sqrt{3}
\end{aligned}$$

In sum, we have the following table of *exact* values:

θ	$\sin\theta$	$\cos\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$