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1 Complex numbers

Recall from Elementary Algebra, that negative numbers can't possibly have square roots. Indeed if a is a real number then a^2 is always positive or zero. Nevertheless we can investigate what happens if we introduce a new kind of numbers, called *imaginary* numbers with the property that when we square them we get negative numbers. So we introduced the *imaginary unit*, a number i that has the property

$$i^2 = -1$$

and we assume that it has as many properties of the real numbers as possible. So for example, $-i$ should also make sense and we should have that also $(-i)^2 = -1$, so that -1 has two "square roots", $\pm i$. Similarly to the case of real numbers we make the convention that $\sqrt{-1}$ stands for i.

Since i has the algebraic properties of real numbers, now we have that all negative numbers have square roots, for example:

$$(3i)^2 = 3^2 i^2 = 9(-1) = -9, \qquad \left(i\sqrt{2}\right)^2 = i^2\left(\sqrt{2}\right)^2 = -1 \cdot 2 = -2$$

In general:

> If a is a positive real number then
> $$\sqrt{-a} = i\sqrt{a}$$

Example 1. Simplify: $\sqrt{-45}$

Answer. $\sqrt{-45} = i\sqrt{45} = 3i\sqrt{5}$ □

If we add or subtract two imaginary numbers we get a new imaginary number:

Example 2. $3i + 5i = 8i, \quad 7i - 12i = 5i$

When we multiply two imaginary numbers we get a real number:

Example 3. $(2i)(3i) = 6i^2 = -6, \quad (-5i)(i\sqrt{3}) = -i^2 5\sqrt{3} = 5\sqrt{3}$

If we add a real number and an imaginary number we get a *complex number*. For example $2 + 3i$ is a complex number. We can add an multiply complex numbers and the result will be a complex number.

Example 4. Here are a few examples to refresh your memory:

1. $(1 - 2i)(3 + 5i) = 3 + 5i - 6i - 10i^2 = 3 - i + 10 = 13 - i$
2. $(2 - 3i)(2 + 3i) = 4 - 9i^2 = 4 + 9 = 13$
3. $(2 - 3i)^2 = 4 + (3i)^2 - 12i = 4 - 9 - 12i = -5 - 12i$

To divide two complex numbers we use the technique of *rationalizing the denominator* that we have already seen, after all i is just a shorthand for $\sqrt{-1}$.

> The *conjugate* of the complex number $a + b\,i$ is the complex number $a - b\,i$

Example 5. The conjugate of $5 + 2i$ is $5 - 2i$, the conjugate of $4 - 3i$ is $4 + 3i$.

Now notice what happens when we multiply tow conjugate numbers:

$$(a + b\,i)(a - b\,i) = a^2 - b^2 i^2 = a^2 + b^2$$

> To divide two complex numbers, we multiply the numerator and denominator by the conjugate of the denominator.

Example 6. Divide: $\dfrac{2-4i}{1+i}$

Answer.

$$\begin{aligned}\frac{2-4i}{1+i} &= \frac{2-4i}{1+i}\,\frac{1-i}{1-i}\\ &= \frac{2-2i-4i-4}{1+1}\\ &= \frac{-2-6i}{2}\\ &= -1-3i\end{aligned}$$

□

Example 7. Perform the division: $\dfrac{2+3i}{3-4i}$

Answer.

$$\begin{aligned}\frac{2+3i}{3-4i} &= \frac{2+3i}{3-4i}\,\frac{3+4i}{3+4i}\\ &= \frac{6+8i+9i-12}{9+16}\\ &= \frac{-6+17i}{25}\\ &= -\frac{6}{25}+\frac{17}{25}i\end{aligned}$$

□

Powers of i. We know that $i^2 = -1$. What about higher powers? Let's see:

$$\begin{aligned}&i^0 = 1\\ &i^1 = i\\ &i^2 = -1\\ &i^3 = i^2 \cdot i = -i\\ &i^4 = i^3 \cdot i = -i \cdot i = 1\\ &i^5 = i^4 \cdot i = 1 \cdot i = i\\ &i^6 = i^5 \cdot i = i \cdot i = -1\\ &i^7 = i^6 \cdot i = -1 \cdot i = -i\\ &i^8 = i^7 \cdot i = -i \cdot i = 1\\ &\cdots\cdots\end{aligned}$$

After doing this for a while you will probably realize that there is a repeating pattern: the results cycle through $1, i, -1, -i$. This happens because $i^4 = 1$, once once we reach the fourth power it's as if we start from the beginning, so we go through i, then -1, then $-i$, and then back to 1, where the whole cycle will begin again. For multiples of 4 we will get 1, if the remainder of the division by 4 is 1, we'll get i, if the reminder is 2 will get -1, and if the remainder is 3 we'll get $-i$.

Example 8. $i^{15} = -i$ because 15 divided by 4 leaves remainder 3. $i^{22} = -1$ because the remainder of 22 divided by 4 is 2.

The pattern of the powers of i continues for negative exponents as well. The pattern, starting with exponent 0 is $1, i, -1, -i$ and it repeats for ever. In this pattern before 1 comes $-i$, so since the exponent -1 comes before the exponent 0 we should have $i^{-1} = -i$; since exponent -2 comes before exponent -1 and in the pattern -1 comes before $-i$ we should have $i^{-2} = -1$, and so on.

Here is the calculations that show why this is so.

$$i^{-1} = \frac{1}{i^1}$$

Now since $1+3=4$ and $i^4 = 1$ if we multiply both numerator and denominator with i^3 we get a denominator of i^4 which equals 1. So:

$$\begin{aligned} i^{-1} &= \frac{1}{i^1} \\ &= \frac{1}{i^1}\,\frac{i^3}{i^3} \\ &= \frac{i^3}{i^4} \\ &= \frac{i^3}{1} \\ &= i^3 \\ &= -i \end{aligned}$$

For i^{-2} we have:

$$\begin{aligned} i^{-2} &= \frac{1}{i^2} \\ &= \frac{1}{i^2}\,\frac{i^2}{i^2} \\ &= \frac{i^2}{i^4} \\ &= \frac{i^2}{1} \\ &= i^2 \\ &= -1 \end{aligned}$$

And for example for i^{-35} we have, since $35 = 4 \cdot 8 + 3$:

$$\begin{aligned} i^{-35} &= \frac{1}{i^{35}} \\ &= \frac{1}{i^3} \\ &= \frac{1}{i^3}\,\frac{i}{i} \\ &= \frac{i}{i^4} \\ &= \frac{i}{1} \\ &= i \end{aligned}$$

2 Exercises

1. Perform the following operations. Give your answers in the form $a + bi$ where a and b are real numbers.
   (a) $\sqrt{-16}$
   (b) $\sqrt{-12} + \sqrt{-20}$
   (c) $\sqrt{-3}(1 - \sqrt{-75})$
   (d) i^{29}
   (e) $-i^{10}$
   (f) $2i(7 - 4i)$
   (g) $\left(\frac{2}{3} + 3i\right) + \left(6 - \frac{3i}{5}\right)$
   (h) $(-3 + 5i) - (5 - 3i)$
   (i) $(4 + 3i)(4 - 3i)$
   (j) $\left(\frac{1}{3} - \frac{3i}{2}\right)\left(\frac{1}{3} + \frac{3i}{2}\right)$
   (k) $(2 - 3i)(-5 + 4i)$
   (l) $(6 - 5i)^2$
   (m) $(3 + 5i)^2$
   (n) $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^2$
   (o) $(2 - i)^3$
   (p) $\left(\frac{-1 + i\sqrt{3}}{2}\right)^3$
   (q) $\left(\sqrt{2} + i\sqrt{2}\right)^4$
   (r) $\frac{7}{2 - 3i}$
   (s) $\frac{-5 + 10i}{-2 + i}$
   (t) $\frac{2 - i}{-i}$
   (u) $\frac{(3 - 4i)(-1 + 2i)}{2 - i}$
   (v) $\frac{-1 - i}{2 - 3i}$

2. Evaluate each of the following expressions when $z = -2 + 3i$:
   (a) $z^2 + 4z$
   (b) $3z^3 - 138$
   (c) $z^3 - z^2 - 7z - 65$
   (d) $\frac{3 + 2i}{iz}$
   (e) $-\frac{3z^2}{12i + 5}$

3 Quadratic Equations

Recall that the solutions to quadratic equations with real coefficients are in general complex numbers. Indeed from the quadratic formula

$$x = \frac{-b \pm \sqrt{D}}{2a}, \text{ where } D = b^2 - 4ac$$

we see that if the discriminant D is a negative number we get square roots of negative numbers.

Example 9. Solve $x^2 - 4x + 13 = 0$

Answer. We have $D = (-4)^2 - 4 * 1 * 13 = 16 - 52 = -36$, so the solutions will be complex numbers. Indeed:

$$x = \frac{-(-4) \pm \sqrt{-36}}{2 \cdot 1} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

□

4 Exercises

Solve the following quadratic equations;

1. $x^2 - 12x = -61$
2. $9x^2 - 6x + 2 = 0$
3. $4x^2 + 9 = 0$
4. $x^2 = 4x + 7 = 0$