

Solving quadratic equations by completing the square

Nikos Apostolakis

Spring 2017

Recall 1. Every positive real number a has two square roots, a positive one (also called *the principal square root*) and a negative one. We also know that negative numbers don't have square roots (at least not *real* ones) and that 0 has exactly one square root, itself.

We can rephrase the above in terms of equations and solutions.

Fact 1. *The equation*

$$x^2 = k$$

where a is a real number, has

- *exactly two real solution, $x = \sqrt{a}$ and $x = -\sqrt{a}$, if $k > 0$*
- *exactly one solution, $x = 0$, if $k = 0$*
- *no real solutions if $k < 0$.*

This fact can be used to solve some simple quadratic equations right a way.

Example 1. Solve the equation $x^2 = 4$.

Answer. According to Fact 1, this equation has two real solutions, $x = \sqrt{4}$ and $x = -\sqrt{4}$, that is $x = 2$ and $x = -2$. □

Example 2. Solve $x^2 = 5$.

Answer. Again according to Fact 1, we have two solutions: $x = \sqrt{5}$ and $x = -\sqrt{5}$. □

Example 3. Solve $x^2 = 0$.

Answer. According to Fact 1 there is only one solution: $x = 0$. □

Example 4. Solve $x^2 = -9$.

Answer. According to Fact 1 this equation has no real solutions. □

Remark 2. Quite often we get two opposite solutions to an equation. For instance in Example 1 we got the solutions $x = 2$ and $x = -2$ and in Example 2 we got the solutions $x = \sqrt{5}$ and $x = -\sqrt{5}$. In such situations we will abbreviate the two opposite numbers using the symbol “ \pm ”. Thus we will write “ $x = \pm 2$ ” to mean “ $x = 2$ or $x = -2$ ”.

In general we can solve this way any quadratic equation without a linear term, by transforming it to an equivalent equation of the form $x^2 = k$. Let's see a couple of examples:

Example 5. Solve $5x^2 = 3$.

Answer. We will first divide both sides by 5 in order to get the equation in the form of Fact 1:

$$\begin{aligned}5x^2 = 3 &\iff x^2 = \frac{3}{5} \\ &\iff x = \pm\sqrt{\frac{3}{5}} \\ &\iff x = \pm\frac{\sqrt{3}\sqrt{5}}{\sqrt{5}\sqrt{5}} \\ &\iff x = \pm\frac{\sqrt{15}}{5}\end{aligned}$$

□

Example 6. Solve $3x^2 - 5 = 7$

Answer. We have:

$$\begin{aligned}3x^2 - 5 = 7 &\iff 3x^2 = 12 \\ &\iff x^2 = 4 \\ &\iff x = \pm 2\end{aligned}$$

□

Fact 1 can be used to solve any equation whose LHS is a *complete square*. Here are a few examples:

Example 7. Solve: $(x - 3)^2 = 9$

Answer. According to Fact 1 we have:

$$\begin{aligned}(x - 3)^2 = 9 &\iff x - 3 = \pm\sqrt{9} \\ &\iff x - 3 = \pm 3 \\ &\iff x = 3 \pm 3 \\ &\iff x = 3 + 3 \text{ or } x = 3 - 3 \\ &\iff x = 6 \text{ or } x = 0\end{aligned}$$

□

Example 8. Solve: $(x + 2)^2 = 20$

Answer. We have:

$$\begin{aligned}(x + 2)^2 = 20 &\iff x + 2 = \pm\sqrt{20} \\ &\iff x + 2 = \pm\sqrt{4 \cdot 5} \\ &\iff x + 2 = \pm 2\sqrt{5} \\ &\iff x = -2 \pm 2\sqrt{5}\end{aligned}$$

□

Example 9. Solve: $3(x - 4)^2 + 7 = 61$

Answer. We have:

$$\begin{aligned}3(x - 4)^2 + 7 = 61 &\iff 3(x - 4)^2 = 54 \\ &\iff (x - 4)^2 = 18 \\ &\iff x - 4 = \pm\sqrt{18} \\ &\iff x - 4 = \pm 3\sqrt{2} \\ &\iff x = 4 \pm 3\sqrt{2}\end{aligned}$$

□

Example 10. Solve: $5(x + 6)^2 - 3 = 17$

Answer. We have:

$$\begin{aligned}5(x + 6)^2 - 3 = 17 &\iff 5(x + 6)^2 = 20 \\ &\iff (x + 6)^2 = 4 \\ &\iff x + 6 = \pm\sqrt{4} \\ &\iff x = -6 \pm 2 \\ &\iff x = -6 + 2 \text{ or } x = -6 - 2 \\ &\iff x = -4 \text{ or } x = -8\end{aligned}$$

□

Time to practice:

1. Solve: $9(x - 1)^2 = 36$

2. Solve $4(x + 8)^2 + 3 = 30$

Sometimes the LHS of the equation is a complete square in Simplified Expanded Form. In order to deal with these cases we should be able to recognize the SEF of the square of a binomial.

Recall 2. In a previous chapter we saw the following identity:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \quad (1)$$

In our case we will use the Identity (1) when the LHS is the square of a linear binomial. So let's state it in that form:

$$(ax \pm h)^2 = a^2x^2 \pm 2ahx + h^2 \quad (2)$$

Example 11. Solve $x^2 - 6x + 9 = 11$

Answer. The LHS is the expansion of $(x - 3)^2$. So the given equation is equivalent to

$$(x - 3)^2 = 11$$

So proceeding as before¹ we get that the solution is:

$$x = 3 \pm \sqrt{11}$$

□

Example 12. Solve: $4x^2 + 20x + 25 = 28$

Answer. The LHS is the expansion of $(2x + 5)^2$. So the equation is equivalent to

$$(2x + 5)^2 = 28$$

So we can solve:

$$\begin{aligned} (2x + 5)^2 = 28 &\iff 2x + 5 = \pm\sqrt{28} \\ &\iff 2x + 5 = \pm 2\sqrt{7} \\ &\iff 2x = -5 \pm 2\sqrt{7} \\ &\iff x = \frac{-5 \pm 2\sqrt{7}}{2} \end{aligned}$$

□

Now you do these:

1. Solve: $9x - 12x + 4 = 1$

¹How? Complete the missing steps!

2. Solve: $x^2 - 20x + 100 = 75$

It turns out that this idea can be used to solve *any* quadratic equation because of the following:

Fact 3. *Every quadratic equation can be written in terms of a complete square. In other words every quadratic equation is equivalent to an equation of the form:*

$$(x - h)^2 = k$$

where h and k are real numbers.

We start with a few examples that illustrate the general method:

Example 13. Solve: $x^2 - 4x = 7$

Answer. The idea is to transform the equation so that the LHS so that it is the expanded form of the square of a binomial. Notice that in the RHS of Equation (2) the quadratic term of the RHS is the square of the linear term of the binomial in the LHS and the linear term is twice the product of the linear term ax and the constant term $\pm h$ of the binomial in the LHS. In our equation the linear term is $-4x$ and the quadratic term is x^2 . The LHS thus is “part” of the expansion of $(x - 2)^2$. What is missing in order to have the complete expansion of $(x - 2)^2$ is the square of 2 that is 4. So in order to complete the square we add 4 to both sides of the equation².

So our equation is equivalent to:

$$x^2 - 4x + 4 = 11$$

Now the LHS is the expansion of $(x - 2)^2$. So our equation can be written as:

$$(x - 2)^2 = 11$$

Solving this as before³ we get: $x = 2 \pm \sqrt{11}$. □

Example 14. Solve: $x^2 + 6x - 27 = 0$

Answer. We first get the constant term “out of the way” by transferring it to the RHS:

$$x^2 + 6x = 27$$

The linear term of the LHS is 2 times $3x$. So the LHS is part of the expansion of $(x + 3)^2$. In order to complete the expansion we need to add the square of 3, i.e. 9. So we add 9 to both sides and we get:

$$x^2 + 6x + 9 = 36$$

²We need to add it to both sides in order to get an equivalent equation.

³Complete the missing steps!

Now the LHS is the expansion of $(x + 3)^2$. So we have the equation:

$$(x + 3)^2 = 36$$

Solving this equation⁴ gives $x = 3$ or $x = -9$. □

In general for a monic polynomial, to complete the square we need to add the square of half the coefficient of its linear term. We state this as a rule:

To complete $x + ph$ to a square we need to add the square of $\frac{p}{2}$, that is $\frac{p^2}{4}$.

Example 15. Solve $x^2 - 14x - 3 = 0$

Answer. We first transfer 3 to the RHS and then add the square of $\frac{14}{2}$ (i.e. 49) to both sides to complete the square. Then we solve:

$$\begin{aligned}x^2 - 14x - 3 = 0 &\iff x^2 - 14x = 3 \\ &\iff x^2 - 14x + 49 = 52 \\ &\iff (x - 7)^2 = 52 \\ &\iff x - 7 = \pm\sqrt{52} \\ &\iff x - 7 = \pm 2\sqrt{13} \\ &\iff x = 7 \pm 2\sqrt{13}\end{aligned}$$

□

Example 16. Solve $x^2 + 5x - 7 = 0$

Answer. We first transfer 7 to the RHS and then add the square of $\frac{5}{2}$ to both sides to complete the square. Then we solve:

$$\begin{aligned}x^2 + 5x - 7 = 0 &\iff x^2 + 5x = 7 \\ &\iff x^2 + 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2 \\ &\iff \left(x + \frac{5}{2}\right)^2 = 7 + \frac{25}{4} \\ &\iff \left(x + \frac{5}{2}\right)^2 = \frac{53}{4} \\ &\iff x + \frac{5}{2} = \pm\sqrt{\frac{53}{4}} \\ &\iff x + \frac{5}{2} = \pm\frac{\sqrt{53}}{2} \\ &\iff x = -\frac{5}{2} \pm \frac{\sqrt{53}}{2} \\ &\iff x = \frac{-5 \pm \sqrt{53}}{2}\end{aligned}$$

⁴Do it!

□

Even though the method is exactly the same the last example was more complicated because it involved operations with fraction. When $a \neq 1$ the calculations can get a bit more complicated.

Example 17. Solve $3x^2 - 7x - 5 = 0$

Answer. We first transfer the constant term to the RHS of the equation. Then we divide both sides by the leading coefficient 3. Then we add the square of half the linear term to both sides to complete the square and then we solve:

$$\begin{aligned}
 3x^2 - 7x - 5 = 0 &\iff 3x^2 - 7x = 5 \\
 &\iff \frac{3x^2 - 7x}{3} = \frac{5}{3} \\
 &\iff x^2 - \frac{7x}{3} = \frac{5}{3} \\
 &\iff x^2 - \frac{7x}{3} + \left(\frac{7}{6}\right)^2 = \frac{5}{3} + \left(\frac{7}{6}\right)^2 \\
 &\iff \left(x - \frac{7}{6}\right)^2 = \frac{5}{3} + \frac{49}{36} \\
 &\iff \left(x - \frac{7}{6}\right)^2 = \frac{60}{36} + \frac{49}{36} \\
 &\iff \left(x - \frac{7}{6}\right)^2 = \frac{109}{36} \\
 &\iff x - \frac{7}{6} = \pm\sqrt{\frac{109}{36}} \\
 &\iff x - \frac{7}{6} = \pm\frac{\sqrt{109}}{6} \\
 &\iff x = \frac{7}{6} + \pm\frac{\sqrt{109}}{6} \\
 &\iff x = \frac{7 \pm \sqrt{109}}{6}
 \end{aligned}$$

□

1 Exercises

1. Solve each of the following equations:

(a) $3x^2 - 7 = -4$

(b) $(x - 3)^2 = 49$

(c) $(3x - 1)^2 - 3 = 61$

(d) $(2x + 1)^2 = 5$

(e) $(x - 2)^2 + 7 = 3$

(f) $(3x + 5)^2 - 7 = -3$

$$(g) (x + 11)^2 - 2 = 10$$

$$(h) (3x - 7)^2 + 6 = 81$$

2. Solve each of the following equations by completing the square:

$$(a) x^2 - 10x = -19$$

$$(b) x^2 - 4x + 4 = 4$$

$$(c) x^2 - 6x + 9 = 11$$

$$(d) x^2 - 14x + 46 = 33$$

$$(e) x^2 + 10x + 25 = 13$$

$$(f) x^2 + 8x = 11$$

$$(g) x^2 - 22x + 121 = -5$$

$$(h) x^2 + 3x - 8 = 5x + 11$$

$$(i) x^2 + 6x - 12 = 4$$

$$(j) x^2 - 30x + 3 = 45$$

$$(k) x^2 - 3x - 30 = 3x + 10$$

$$(l) x^2 + 7x - 44 = 0$$

$$(m) x^2 + 3x - 6 = 11$$

$$(n) 3x^2 - x + 3 = 18$$

2 The general case and the quadratic formula

The last example (Example 17) of the previous section was rather complicated. The good news is that it was as complicated as it can get. Solving the general quadratic equation requires essentially the same calculations. So in this section we're going to solve the general quadratic equation $ax^2 + bx + c = 0$, so that we won't need to repeat these tedious calculations every time we see a quadratic equation. Here it goes: first we transfer c to the RHS, then we divide both sides by a ⁵, then we add the square of half of the linear term (that is the square of $\frac{b}{2a}$), we simplify and finally solve.

$$\begin{aligned}ax^2 + bx + c = 0 &\iff ax^2 + bx = -c \\ &\iff \frac{ax^2 + bx}{a} = -\frac{c}{a} \\ &\iff x^2 + \frac{bx}{a} = -\frac{c}{a} \\ &\iff x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\ &\iff \left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\ &\iff \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\ &\iff x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &\iff x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &\iff x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &\iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{aligned}$$

The last equation is the famous *quadratic formula*. Using this formula we can solve any quadratic equation by just substituting the coefficients in to the formula. The quantity $b^2 - 4ac$ that appears under the radical sign is important enough to deserve its own name: we call it the *discriminant* of the equation. So we can express the results of our calculations as follows:

Fact 4. *The solutions of $ax^2 + bx + c = 0$, with a, b, c real numbers and $a \neq 0$ is given by:*

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

where $D = b^2 - 4ac$ is the discriminant of the equation.

Let's see some examples:

Example 18. Solve: $x^2 + 4x - 21 = 0$

⁵Remember that $a \neq 0$.

Answer. We have $a = 1$, $b = 4$, $c = -21$. So the discriminant is

$$\begin{aligned} D &= 4^2 - 4 \cdot 1 \cdot (-21) \\ &= 16 + 84 \\ &= 100 \end{aligned}$$

The quadratic formula then gives:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{100}}{2 \cdot 1} \\ &= \frac{-4 \pm 10}{2} \\ &= \frac{-4 + 10}{2} \quad \text{or} \quad \frac{-4 - 10}{2} \\ &= \frac{6}{2} \quad \text{or} \quad \frac{-14}{2} \\ &= 3 \quad \text{or} \quad -7 \end{aligned}$$

□

Example 19. Solve: $x^2 - 2x + 2 = 0$

Answer. We have $a = 1$, $b = -2$, $c = 2$. So the discriminant is $D = (-2)^2 - 4 \cdot 1 \cdot 2 = 4 - 8 = -4$. So the quadratic formula gives:

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{-4}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 2i}{2} \\ &= 1 \pm 2i \end{aligned}$$

□

In the last example we got two complex solutions. This will happen every time the discriminant is negative since the quadratic formula involves taking the square root of D .

Example 20. Solve: $2x^2 + 7x - 15 = 0$

Answer. We have $a = 2$, $b = 7$, $c = -15$. So $D = 7^2 - 4 \cdot 2 \cdot (-15) = 49 + 120 = 169$. So the quadratic formula gives:

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{169}}{2 \cdot 2} \\ &= \frac{-7 \pm 13}{4} \\ &= \frac{-7 + 13}{4} \quad \text{or} \quad \frac{-7 - 13}{4} \\ &= \frac{6}{4} \quad \text{or} \quad \frac{-20}{4} \\ &= \frac{3}{2} \quad \text{or} \quad -5 \end{aligned}$$

□

Example 21. Solve $x^2 - 4x + 1 = 0$

Answer. We have $a = 1$, $b = -4$, $c = 1$. So $D = (-4)^2 - 4 \cdot 1 \cdot 1 = 16 - 4 = 12$. So the solutions are:

$$\begin{aligned}x &= \frac{-(-4) \pm \sqrt{12}}{2 \cdot 1} \\ &= \frac{4 \pm 2\sqrt{3}}{2} \\ &= 2 \pm \sqrt{3}\end{aligned}$$

□

3 Exercises

Solve each of the following equations using the quadratic formula.

1. $x^2 + 4x - 21 = 0$
2. $x^2 - 3x - 2 = 0$
3. $x^2 - 5 = 0$
4. $x^2 + 3x = 0$
5. $4x^2 - 3x + 7 = 0$
6. $12x^2 + 4x = 1$
7. $12x^2 + 13x - 4 = 0$
8. $5x^2 + 3x = 11$
9. $-2x^2 - 7x + 3 = 0$
10. $24x^2 - 5x + 5 = 3x + 20$
11. $8x^2 - 11x - 13 = 3x^2 - 15x - 8$