The Circle

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1 The equation of a circle

Definition 1. Given a point C in the plane and a positive number r, the *circle with center* C *and radius* r, is the locus of points that are at distance r from C.

Example 1. Find an equation for the circle with center $C(2,3)$ and radius 4.

Solution. A point $P(x,y)$ will be on the circle exactly when the distance from P to C is 4. If d is that distance then we have

$$\begin{aligned} d^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (x-2)^2 + (y-3)^2 \\ &= x^2 - 4x + 4 + y^2 - 6y + 9 \\ &= x^2 - 4x + y^2 - 6y + 13 \end{aligned}$$

So P is at distance 4 from C exactly when

$$x^2 - 4x + y^2 - 6y + 13 = 4^2$$

which is equivalent to

$$x^2 - 4x + y^2 - 6y = 3 \tag{1}$$

So (1) is an equation of the circle with center $(2,3)$ and radius 4. □

Example 2. Find the equation of the circle with center $C(-2,0)$ and radius $\sqrt{3}$.

Solution. We proceed as in the previous example: The square of the distance of a point $P(x,y)$ from C is:

$$\begin{aligned} d^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &= (x+2)^2 + (y-0)^2 \\ &= x^2 + 4x + 4 + y^2 \end{aligned}$$

So an equation for our circle is:

$$x^2 + 4x + 4 + y^2 = \left(\sqrt{3}\right)^2$$

Which becomes after simplification:

$$x^2 + 4x + y^2 = -1$$

□

The above procedure works in all cases. In general we have the following:

Fact 2. *An equation of the circle with center $C(h, k)$ and radius r is:*

$$(x - h)^2 + (y - k)^2 = r^2 \tag{2}$$

Justification. The left hand side of (2) is the square of the distance between a point $P(x, y)$ and the $C(h, k)$, while the right hand side is the square of the radius. □

Remark 3. Notice that (2) is not in simplified expanded form; so after using it to get the equation of a circle we'll have to expand and simplify.

Remark 4. The Equation (2), says that the square of the distance of the point with coordinates (h, k) from the point with coordinates (x, y) is r^2. So if we are given such an equation we know that it is an equation of a circle with center (h, k) and radius r. This remark will be relevant in the next section.

Example 3. Find the equation of the circle with radius 2 centered at $\left(\frac{1}{2}, -\frac{2}{3}\right)$.

Solution. According to Fact 2, an equation for the given circle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{2}{3}\right)^2 = 2^2$$

After expanding we get:

$$x^2 - x + \frac{1}{4} + y^2 + \frac{4}{3}y + \frac{4}{9} = 4$$

Next we clear denominators by multiplying both sides by 36 (the LCD):

$$36x^2 - 36x + 9 + 36y^2 + 48y + 16 = 144$$

Then we combine like terms:

$$36x^2 - 36x + 36y^2 + 48y + 25 = 144$$

And finally we transfer 25 to the right hand side to get:

$$36x^2 - 36x + 36y^2 + 48y = 119$$

□

Now let's practice:

Example 4. Find an equation for the circle of radius 5 centered at $(0, 0)$

Example 5. Find an equation for the circle with center $(-1, -3)$ and radius $\sqrt{2}$.

Example 6. A circle has radius 3 and center at $\left(0, \frac{3}{2}\right)$. Find an equation for this circle.

Example 7. Find an equation of the circle with radius $\frac{1}{3}$ and center $(0, -1)$.

If we know the center of the circle and one point *on* the circle then we can find the equation of the circle:

Example 8. A circle has center $(-1, 0)$ and passes through the point $(2, -1)$. Find an equation of the circle.

Solution. We know the center but we still need the radius of the circle. By definition the radius is the distance between any point of the circle and its center. So the square of the radius is:

$$\begin{aligned} r^2 &= (-1-2)^2 + (0+1)^2 \\ &= 10 \end{aligned}$$

So the equation of the circle is:

$$(x+1)^2 + y^2 = 10$$

After expanding and simplifying we get:

$$x^2 + 2x + y^2 = 9$$

□

As we will see in the next section knowing two points of a circle is not enough to determine its equation. However if the two points happen to be *diametrically opposite* (i.e. they are the endpoints of a diameter of the circle) then we can find the equation.

Example 9. The segment with endpoints $(-2, 3)$ and $(4, -1)$ is a diameter of a circle. What's the equation of the circle?

Solution. We need the center and the radius of the circle. The center will be the midpoint of the diameter. So the center is at

$$\left(\frac{-2+4}{2}, \frac{3+(-1)}{2}\right) = (1, 1)$$

The radius will be half the length of the diameter. The length of a segment is the distance between its endpoints. So the length of the diameter is

$$\sqrt{(-2-4)^2 + (3+1)^2} = \sqrt{52} = 2\sqrt{13}$$

It follows that the radius of the circle is $\sqrt{13}$.

Putting it all together we have that the equation of the circle is:

$$(x-1)^2 + (y-1)^2 = 13$$

After expanding and simplifying we get the equation:

$$x^2 - 2x + y^2 - 2y = 11$$

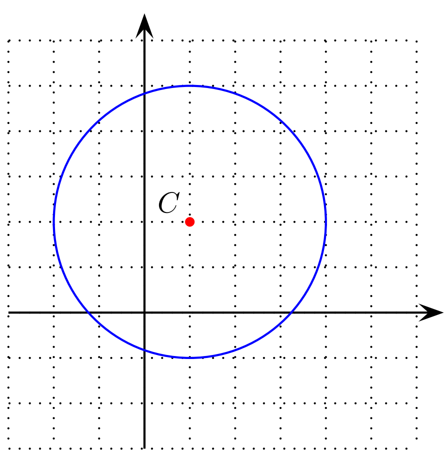
□

1.1 Exercises

1. Find the equation of the circle with the given center C and radius r:
   (a) $C = (0, 0)$, $r = 1$
   (b) $C = (-2, -3)$, $r = 6$
   (c) $C = (3, -4)$, $r = 3\sqrt{2}$
   (d) $C = \left(0, \frac{2}{3}\right)$, $r = \sqrt{5}$
   (e) $C = \left(\frac{1}{2}, -\frac{3}{2}\right)$, $r = \frac{5}{2}$

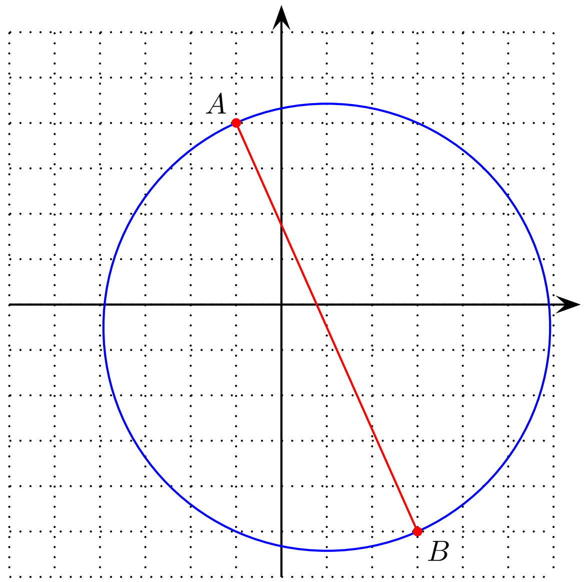
2. The point with coordinates $(4, -3)$ lies in a circle whose center is at the point with coordinates $(-3, 2)$. Find the equation of the circle.

3. Find the equation of the circle, given that C is its center.



4. The segment with endpoints $(-2, 5)$ and $(4, -3)$ is a diameter of a circle. Find an equation of that circle.

5. The segment AB is a diameter of the circle. Find an equation for the circle.



6. The segment with endpoints at $(1, 2)$ and $(-1, -1)$ is a radius of a circle. Additionally the point with coordinates $(2, 1)$ lies on that circle. Find the equation of the circle.

2 Finding the center and radius of a circle given its equation

In the previous section we saw how to find an equation of a circle if we know its center and its radius. In this section we will look at the reverse problem: if we know an equation of a circle how can we find its center and radius? The answer will be, "by completing the square".

Example 10. A circle has equation $x^2 - 4x + y^2 - 6y = -5$. Find its center and its radius.

Solution. If we have an equation in the form of Equation (2), we know that it is an equation of circle whose center has coordinates (h, k) and whose radius is r (see Remark 4 in the previous section). So we will transform the given equation to have that form, i.e. so that the LHS is the sum of the squares of two linear binomials one in x and one in y. To achieve this we will complete the square in x and in y.

To complete the terms involving x into a square we need to add 4 (the square of half of the coefficient of the linear term $-4x$) and to complete the terms involving y in to a square we need to add 9 (the square of of half of the coefficient of the linear term $-6y$). So:

$$\begin{aligned} x^2 - 4x + y^2 - 6y = -5 &\iff x^2 - 4x + 4 + y^2 - 6y + 9 = -5 + 4 + 9 \\ &\iff (x-2)^2 + (y-3)^2 = 8 \end{aligned}$$

So the circle has center with coordinates $(2, 3)$ and radius $\sqrt{8} = 2\sqrt{2}$. □

Example 11. A circle has equation $x^2 + 2x + y^2 = 8$. Find its center and radius.

Solution. As in the previous example, we will complete the squares in x and y. In this case y already appears as a complete square so we only need to complete the terms involving x.

$$\begin{aligned} x^2 + 2x + y^2 = 8 &\iff x^2 + 2x + 1 + y^2 = 8 + 1 \\ &\iff (x+1)^2 + y^2 = 9 \end{aligned}$$

So the circle has center $(1, 0)$ and radius 3. □

Example 12. Find the center and radius of the circle with equation $x^2 + 3x + y^2 - 4y = -1$.

Solution. As before, we complete the squares:

$$\begin{aligned} x^2 + 3x + y^2 - 4y = -1 &\iff x^2 + 3x + \left(\frac{3}{2}\right)^2 + y^2 - 4y + 4 = -1 + \left(\frac{3}{2}\right)^2 + 4 \\ &\iff \left(x + \frac{3}{2}\right)^2 + (y-2)^2 = 3 + \frac{9}{4} \\ &\iff \left(x + \frac{3}{2}\right)^2 + (y-2)^2 = \frac{21}{4} \end{aligned}$$

So the circle has center at the point $\left(-\frac{3}{2}, 2\right)$ and its radius is $\frac{\sqrt{21}}{2}$. □

2.1 Exercises

1. Find the center and radius of the circle with equation:
   (a) $x^2 + 4x + y^2 - 4y = -6$
   (b) $x^2 - 6x + y^2 - 2y = -3$
   (c) $x^2 + y^2 + 10y = 15$
   (d) $x^2 + 5x + y^2 - y = 3$
   (e) $4x^2 + 4y^2 = 3$

3 Circle through three points

Recall that a (straight) line is determined by two points: given two points in the plane there is a unique line that passes through them. This, of course, is not true for circles, given any two points there will be infinity many different circles that pass through these points. Indeed, let A and B be two points, if we take any point C in the perpendicular bisector of the segment AB, C will be equidistant from A and B. Therefore the circle with center C and radius determined by A will also pass through B. In Figure 1 we have five circles passing through two points. It turns out that a

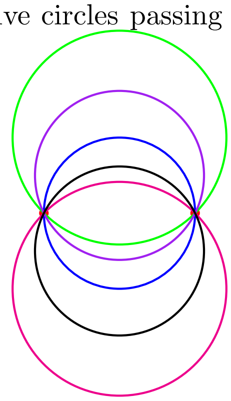


Figure 1: Many circles through two given points

circle is determined by three non-collinear points:

Fact 5. *Given any three non-co-linear points there is a unique circle passing through all three of them.*

Justification. Let X, Y and Z be three points that don't lie on the same line. In order to show that the three points are in the same circle, we need to show that there is a point C, the center of the circle, that is equidistant from all the three points. To show that we will show that there is a point which is equidistant from X and Y and, at the same time, equidistant from Y and Z. Now all points equidistant from X and Y lie in the perpendicular bisector of the segment XY, and all points equidistant from Y and Z lie in the perpendicular bisector of YZ. Since X, Y, Z are not collinear, the segments XY and YZ are not parallel (they intersect and are not part of the same line). It follows that their perpendicular bisectors are not parallel either, and therefore there is a point C that lies in both bisectors. Then the circle that has center C and passes through one of the points will pass through the other two as well. Therefore the three points lie on the same circle.

Furthermore, as we saw, the center of a circle that passes through all three points is determined. So there is only one such circle. □

Let's see how we can use the method suggested above in actual examples:

Example 13. Considered the points $X(0, 1)$, $Y(0, -1)$, and $Z = (1, 0)$

(a) Verify that X, Y, Z are not collinear.

(b) Find the equation of the circle determined by the three points X, Y, Z.

Solution. (a) The line that passes through X and Y is vertical (actually it's the y-axis $x = 0$) while the line that passes through Y and Z has slope $m = \dfrac{\Delta y}{\Delta x} = \dfrac{1-0}{0-(-1)} = 1$. So X, Y and Z are not collinear.

(b) To find the center C of the circle we need to find the intersection of the perpendicular bisector of XY and YZ.

The perpendicular bisector of a segment is the line that passes through the midpoint of the segment and is perpendicular to it. The segment XY is vertical and its midpoint of the segment XY is $(0,0)$. Thus the perpendicular bisector of XY is a horizontal line that passes through $(0,0)$; so it is the x-axis with equation $y=0$.

The midpoint of YZ is $M := \left(\frac{1}{2}, -\frac{1}{2}\right)$. We also saw in part (a) that its slope is 1. So the perpendicular bisector of YZ has slope -1 and passes through M. The equation of the bisector then will be of the form $y = -x + b$, where b is its y-intercept. Substituting the coordinates of M into this equation we get

$$-\frac{1}{2} = -\frac{1}{2} + b \Longleftrightarrow b = 0$$

Thus the equation of the perpendicular bisector of YZ is $y = -x$.

Therefore the center of the circle is the intersection of the lines $y = -x$ and $y = 0$. So the center of the circle is at the point $C(0,0)$.

The radius of the circle is the distance between the center C and one (anyone) of the points, say X. So the radius of the circle is 1.

So the circle determined by the three points X, Y, Z has center at $(0,0)$ and radius 1. So its equation is

$$x^2 + y^2 = 1$$

$\square$

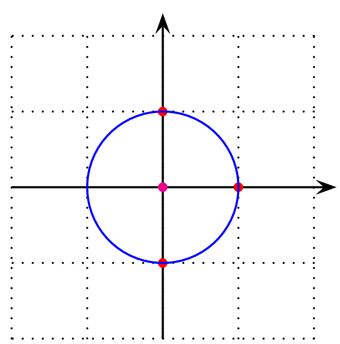


Figure 2: The circle of Example 13

Example 14. Find the equation of the circle determined by $X(0,0)$, $Y(3,-1)$, and $Z(-2,4)$

Solution. As in the previous example we find the intersection point of the perpendicular bisectors of two of the three segments defined by the three given points. Since the coordinates of X are the "easiest" we will work with XY and XZ.

The midpoint of XY is $\left(\frac{3}{2}, -\frac{1}{2}\right)$ and its slope is $\frac{-1-0}{3-0} = -\frac{1}{3}$. So its perpendicular bisector has slope 3. So its equation is $y = 3x + b$, where b is its y-intercept. Plugging the midpoint we get:

$$-\frac{1}{2} = 3 \cdot \frac{3}{2} + b \Longleftrightarrow b = -\frac{1}{2} - \frac{9}{2} \Longleftrightarrow b = -5$$

Thus the equation of the perpendicular bisector of XY is $y = 3x - 5$.

The midpoint of XZ is $(-1, 2)$ and its slope is $\frac{4}{-2} = -2$. So the slope of its perpendicular bisector is $\frac{1}{2}$. Its equation then will be $y = \frac{x}{2} + b$ where b is its y-intercept. Substituting the coordinates of the midpoint we get:

$$2 = \frac{-1}{2} + b \Longleftrightarrow b = \frac{5}{2}$$

Thus the equation of the perpendicular bisector of XZ is $y = \frac{x}{2} + \frac{5}{2}$.

To find the center of the circle we need to find the point of intersection of the two perpendicular bisectors. So we have to solve the system:

$$\begin{cases} y &= 3x - 5 \\ \\ y &= \frac{x}{2} + \frac{5}{2} \end{cases}$$

We proceed by substituting the value of y from the first equation into the second.

$$3x - 5 = \frac{x}{2} + \frac{5}{2} \Longleftrightarrow 6x - 10 = x + 5 \Longleftrightarrow 5x = 15 \Longleftrightarrow x = 3$$

Substituting the value of x into the first equation we find $y = 3 \cdot 3 - 5$, or $y = 4$.

Thus the center of the circle is $(3, 4)$. The radius of the circle will be the distance of the center to one of the points, say X:

$$r^2 = 3^2 + 4^2 = 25$$

Thus the equation of the circle in standard form is

$$(x - 3)^2 + (y - 4)^2 = 25$$

After expanding and simplifying we get the equation:

$$x^2 - 6x + y^2 - 8y = 0$$

□

The geometry of the above example is illustrated in Figure 3. The three points are shown with red dots and the two segments XY and XZ are shown as dashed red lines. The perpendicular bisectors are indicated by dashed magenta lines. Finally the center is the intersection of the two perpendicular bisectors, and shown as a magenta dot.

There is a more algebraic way we can use to find the circle determined by three points. It involves solving a system of three quadratic equations with three unknowns (the two coordinates of the center plus the radius). We illustrate this method with two examples:

Example 15. Find the equation of the circle passing through the points with coordinates $(1, 2)$, $(1, 4)$ and $(3, 2)$.

Solution. Let (h, k) be the coordinates of the center and r be the radius of the circle. Then the standard form of the equation of the circle is

$$(x - h)^2 + (y - k)^2 = r^2 \tag{3}$$

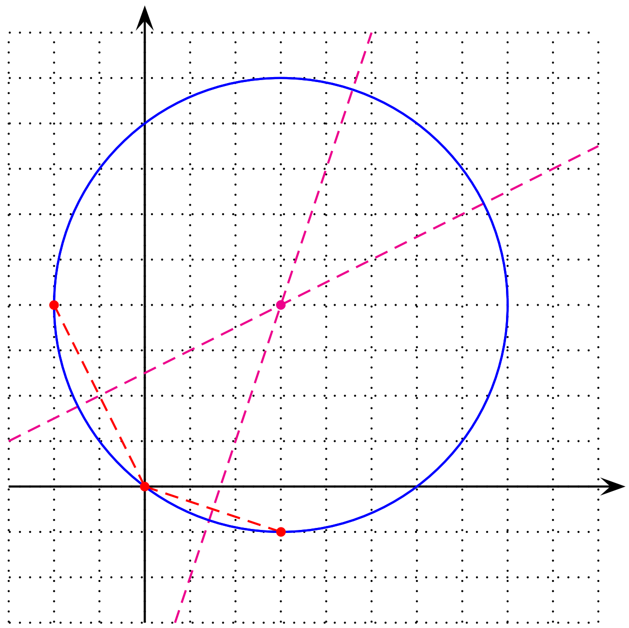


Figure 3: The circle of Example 14

Now we substitute the coordinates of each of the three points in turn to get three equations with three unknows (h, k, and r). We start with the point $(1, 2)$:

$$(1-h)^2 + (2-k)^2 = r^2 \iff h^2 - 2h + k^2 - 4k + 5 = r^2$$

Then we substitute the point $(1, 4)$:

$$(1-h)^2 + (4-k)^2 = r^2 \iff h^2 - 2h + k^2 - 8k + 17 = r^2$$

Finally we substitute the point $(3, 2)$:

$$(3-h)^2 + (2-k)^2 = r^2 \iff h^2 - 6h + k^2 - 4k + 13 = r^2$$

So we need to solve the following system:

$$\begin{cases} h^2 - 2h + k^2 - 4k + 5 &= r^2 \\ h^2 - 2h + k^2 - 8k + 17 &= r^2 \\ h^2 - 6h + k^2 - 4k + 13 &= r^2 \end{cases}$$

This is a system of quadratic equations, but notice that all the quadratic terms are the same in all three equations. This makes the system easy to solve. We start by subtracting the first equation from the second and the third equation. this gives us the system:

$$\begin{cases} -4k + 12 &= 0 \\ -4h + 8 &= 0 \end{cases}$$

So all the quadratic terms disappeared and we are left with a linear system of two unknowns, and we know how to solve those! In fact this particular system is extremely easy to solve: the solution is $h = 2$ and $k = 3$. Now that we know h and k we can substitute in one of the equations (anyone) to find r. So let's substitute to the first equation:

$$2^2 - 2 \cdot 2 + 3^2 - 4 \cdot 3 + 5 = r^2 \iff 2 = r^2$$

So the standard form of the equation of the circle is

$$(x-2)^2 + (y-3)^2 = 2$$

Which, upon expanding and simplifying gives:

$$x^2 - 4x + y^2 - 6y = -11$$

□

Example 16. The points $(2, 3)$, $(1, -4)$, and $(-1, 2)$ lie on a circle. Find the equation of the circle.

Solution. We proceed as in the previous example. Let (h, k) be the center and r the radius of the circle, then its equation in standard form is:

$$(x-h)^2 + (y-k)^2 = r^2$$

We now substitute each of the three points and (after expanding) get the system of equations:

$$\begin{cases} h^2 - 4h + k^2 - 6k + 13 &= r^2 \\ h^2 - 2h + k^2 + 8k + 17 &= r^2 \\ h^2 + 2h + k^2 - 4k + 5 &= r^2 \end{cases}$$

We subtract the first equation from the other two and get the system:

$$\begin{cases} 2h + 14k + 4 &= 0 \\ 6h + 2k - 8 &= 0 \end{cases}$$

To solve this we start by dividing each equation by 2 to get:

$$\begin{cases} h + 7k + 2 &= 0 \\ 3h + k - 4 &= 0 \end{cases}$$

Subtracting 3 times the first equation from the second we get:

$$-20k - 10 = 0 \iff k = -\frac{1}{2}$$

We then subtract 7 times the second equation from the first to get:

$$-20h + 14 = 0 \iff h = \frac{7}{10}$$

We then substitute the values of h and k in one of the three original equations to find r. We choose the third one:

$$\left(\frac{7}{10}\right)^2 + 2 \cdot \frac{7}{10} + \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 5 = r^2 \iff \frac{457}{50} = r^2$$

So the equation of the circle in standard form is:

$$\left(x - \frac{7}{10}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{457}{50}$$

After expanding we get

$$x^2 - \frac{7}{5}x + \frac{49}{100} + y^2 + y + \frac{1}{4} = \frac{457}{50}$$

Which after clearing denominators (by multiplying with the LCD 100) gives:

$$100x^2 - 140x + 49 + 100y^2 + 100y + 25 = 914$$

Which after simplifying becomes:

$$100x^2 - 140x + 49 + 100y^2 + 100y + 25 = 889$$

□

3.1 Exercises

1. Find the equation of the circle determined by the following three points:
   (a) $(-2, 0), (2, 0), (0, 2)$
   (b) $(1, 2), (2, 1), (1, 0)$
   (c) $(4, 1), (1, 2), (-1, 6)$
   (d) $(5, -1), (-8, 12), (-7, 7)$