# BRONX COMMUNITY COLLEGE
## of the City University of New York

## DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 42 — Take Home Exam
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1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 & 2 \\ 2 & 4 & -4 & 6 & 10 \\ 3 & 6 & -6 & 9 & 13 \end{bmatrix}$$

   Find a basis, and thus the dimension, for each of the following:

   (a) The null space of $A$.

   (b) The column space of $A$.

   (c) The row space of $A$.

2. Recall that $\mathbb{P}_4$ stands for the vector space of polynomials of degree at most 4. Consider the following vectors in $\mathbb{P}_4$:

$$\mathbf{p}_0(x) = 1, \quad \mathbf{p}_1(x) = x, \quad \mathbf{p}_2(x) = 2x^2 - 1, \quad \mathbf{p}_3(x) = 4x^3 - 3x, \quad \mathbf{p}_4(x) = 8x^4 - 8x^2 + 1$$

   (a) Prove that $\mathcal{C} = \{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ is a basis of $\mathbb{P}_4$.

   (b) Find the change of coordinate matrix that changes coordinates from the *standard basis* of $\mathbb{P}_4$ to $\mathcal{C}$.

   (c) Let $\mathbf{p}(x) = -8x^4 + 4x^3 + 6x^2 - 2x - 1$. Find the coordinates of $\mathbf{p}$ in the basis $\mathcal{C}$, i.e. find $[\,\mathbf{p}\,]_{\mathcal{C}}$.

3. Find the eigenvalues and a basis for each eigenspace for the following matrix:

$$A = \begin{bmatrix} 19 & -9 & -6 \\ 25 & -11 & -9 \\ 17 & -9 & -4 \end{bmatrix}$$

4. (a) Let $A$ and $B$ be similar matrices. Prove that for every natural number $n$, $A^n$ and $B^n$ are also similar.

   (b) Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of the matrix $A$. Prove that for each natural number $n$, the eigenvalues of $A^n$ are $\lambda_1^n, \ldots, \lambda_k^n$.

   (c) Let

$$A = \begin{bmatrix} 17 & 22 & 3 \\ -1 & 4 & -33 \\ -7 & -14 & 15 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ -1 & -2 & 3 \end{bmatrix}$$

   Calculate $PAP^{-1}$ and use the result together with parts (a) and (b) to find the eigenvalues of $A^5$.

5. Let $\mathbb{M}_{3\times3}$ be the set of all $3 \times 3$ matrices.

    (a) Prove that $\mathbb{M}_{3\times3}$ with the usual addition and scalar multiplication of matrices is a vector space.

    (b) What is the dimension of $\mathbb{M}_{3\times3}$?

    (c) A $3 \times 3$ matrix $A = [a_{ij}]$ is called *symmetric* if for all $i, j$, $a_{ij} = a_{ji}$. Prove that $\mathbb{S}_{3\times3}$, the set of $3 \times 3$ symmetric matrices, is a subspace of $\mathbb{M}_{3\times3}$.

    (d) Find the dimension of $\mathbb{S}_{3\times3}$.

6. Let $\mathbf{v} \in \mathbb{R}^3$ and consider the function: $T_\mathbf{v} \colon \mathbb{M}_{3\times3} \to \mathbb{R}^3$ be defined by $T(X) = X\mathbf{v}$.

    (a) Prove that $T$ is a linear transformation.

    (b) If $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, find $ker T_\mathbf{v}$.

7. Let $V$ be a vector space. Define the $V^*$, the *dual* of $V$ to be the set of linear transformations from $V$ to $\mathbb{R}$, i.e.

$$V^* = \{f \colon V \to \mathbb{R} : f \text{ is linear}\}$$

    (a) Prove that $V^*$, with the usual addition and scalar multiplication, is a vector space.

    (b) What is the standard matrix of an element $f \in (\mathbb{R}^n)^*$ ?

    (c) Prove that $\dim(\mathbb{R}^n)^* = n$.
    **Hint.** Use part (b) to find a basis of $(\mathbb{R}^n)^*$.

8. **Extra Credit:** Generalize exercise 5 for $n \times n$ matrices, that is define $\mathbb{M}_{n\times n}$ and $\mathbb{S}_{n\times n}$ and consider the same questions.