BRONX COMMUNITY COLLEGE
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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

MATH 42 — Additional Review
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1. Find bases for the row space, the column space, and the null space f the following matrix A. Also find a basis for the orthogonal complement of the column space of A.

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

   Is A diagonalizable? If yes diagonalize it.

3. Prove that the set H of vectors in $\mathbb{R}^4$ of the form

$$\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}$$

   is a vector space. What's the dimension of H?

4. Let $\mathbb{M}_{3\times 3}$ be the vector space of real 3×3 matrices. As usual, for a matrix $A \in \mathbb{M}_{3\times 3}$ we write a_{ij} for the entry on the i row and j column. Which of the following subsets of $\mathbb{M}_{3\times 3}$ are subspaces? Justify your answers.
   (a) $\mathbb{G}_{3\times 3}$ the set of all invertible 3×3 matrices.
   (b) The set of all singular 3×3 matrices.
   (c) $\mathbb{D}_{3\times 3}$ the set of diagonal 3×3 matrices.
   (d) The set of 3×3 matrices whose second and third columns are equal.
   (e) The set of 3×3 matrices with the property that every entry of the second row is one more than the entry of the third row in the same column.
   (f) The set $\mathbb{A}_{3\times 3}$ of anti-symmetric 3×3 matrices, i.e. matrices A with the property that for all i, j with $1 \leq i, j \leq 3$, $a_{ij} = -a_{ji}$.

(g) The set of all 3×3 matrices that have trace 0[1].

For those subsets that are vector spaces find their dimension.

5. Recall the subspace $\mathbb{S}_{3\times 3}$ of symmetric 3×3 matrix from the take home exam. Define a function $T\colon \mathbb{M}_{3\times 3} \to \mathbb{S}_{3\times 3}$ by

$$T(A) = S, \quad \text{where } s_{i\,j} = a_{i\,j} + a_{j\,i}$$

   (a) Prove that $T(A)$ is indeed a symmetric matrix.

   (b) Prove that T is a linear transformation.

   (c) Find the kernel of T.

6. Fix a matrix M in $\mathbb{M}_{n\times n}$. Prove that the set of all matrices A in $\mathbb{M}_{n\times n}$ that commute with M, i.e. that satisfy the equation $AM = MA$, is a subspace of $\mathbb{M}_{n\times n}$.

7. Verify that the following 4×4 matrix U is orthogonal and use this fact to find U^{-1}.

$$U = \begin{bmatrix} 0 & 3 & 1 & 5 \\ 1 & 5 & 0 & -3 \\ -4 & 1 & 1 & -1 \\ -1 & 1 & -4 & 1 \end{bmatrix}$$

8. Prove that $\mathcal{B} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for $\mathbb{P}_2$, where:

$$\mathbf{p}_1 = 1, \quad \mathbf{p}_2 = 1 - x, \quad \mathbf{p}_3 = 1 - 2x + x^3.$$

   Find the coordinates of $\mathbf{p} = 1 + 2x + x^2$ with respect to the basis $\mathcal{B}$.

9. Let $T\colon \mathbb{P}_1 \to \mathbb{P}_2$ be defined by $T(\mathbf{p}(x)) = x\mathbf{p}(2x+1)$.

   (a) Prove that T is a linear transformation.

   (b) Find the matrix of T with respect to the standard bases of $\mathbb{P}_1$ and $\mathbb{P}_2$.

10. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (x - y, 2x + 4y)$$

   Find a basis $\mathcal{B}$ of $\mathbb{R}^2$ for which the matrix of T is diagonal and find the matrix of T with respect to that basis.

11. M is a 3×3 matrix with the property that $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 5, and $\begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ are eigenvectors with eigenvalue 2. Find the determinant of M.

[1] Recall that the *trace* of a matrix is the sum of its diagonal entries.