Exercise. Prove the "Parallelogram Rule for Addition" of vectors in $\mathbb{R}^2$. In other words:

Consider Figure 1 where O is the origin of the coordinate system, A has coordinates (a_1, a_2), B has coordinates (b_1, b_2) and $OACB$ is a parallelogram. Prove that C has coordinates $(a_1 + b_1, a_2 + b_2)$.

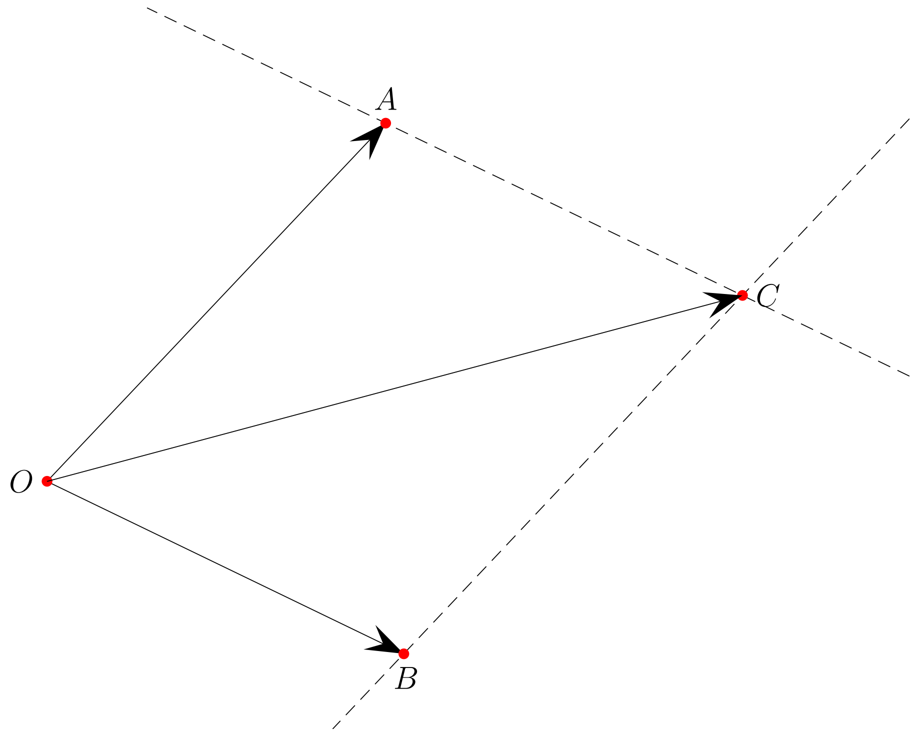

FIGURE 1. The parallelogram rule

Solution. We first note that points A and B have to be different than O, so at least one coordinate of each is different than 0. We also note that the points O, A, and B are not colinear–otherwise there is no parallelogram.

The line OA has equation $a_2x - a_1y = 0$ and the line OB has equation $b_2x - b_1y = 0$. Since B is not in the line OA we must have:

$$a_2b_1 - a_1b_2 \neq 0$$

The line AC is parallel to $0B$ and contains point B so it has equation $b_2x - b_1y = b_2a_1 - b_1a_2$. Similarly, the line BC has equation $a_2x - a_1y = a_2b_1 - a_1b_2$. So to find the coordinates of the point C we need to solve the system:

$$\begin{cases} b_2x - b_1y &= b_2a_1 - b_1a_2 \\ a_2x - a_1y &= a_2b_1 - a_1b_2 \end{cases}$$

The augmented matrix of this system is:

$$\begin{bmatrix} b_2 & -b_1 & b_2a_1 - b_1a_2 \\ a_2 & -a_1 & a_2b_1 - a_1b_2 \end{bmatrix}$$

If $b_2 \neq 0$ then adding to the second row $-\frac{a_2}{b_2}$ times the first we get:

$$\begin{bmatrix} b_2 & -b_1 & b_2a_1 - b_1a_2 \\ 0 & \dfrac{-a_1b_2 + b_1a_2}{b_2} & \dfrac{b_2\left(a_2b_1 - a_1b_2\right) - a_2\left(b_2a_1 - b_1a_2\right)}{b_2} \end{bmatrix}$$

Since $a_1b_2 - b_1a_2 \neq 0$, we can multiply the second row with $\frac{b_2}{a_1b_2 - b_1a_2}$ to get:

$$\begin{bmatrix} b_2 & b_1 & b_2a_1 - b_1a_2 \\ 0 & -1 & -b_2 - a_2 \end{bmatrix}$$

and adding to the first row b_1 times the second we obtain:

$$\begin{bmatrix} b_2 & 0 & b_2a_1 + b_1b_2 \\ 0 & -1 & -b_2 - a_2 \end{bmatrix}$$

Finally by dividing the first row by b_2 and the second by -1 we get the reduced echelon form:

$$\begin{bmatrix} 1 & 0 & a_1 + b_1 \\ 0 & 1 & b_2 + a_2 \end{bmatrix}$$

So the solution is $x = a_1 + b_1$, $y = a_2 + b_2$. If $b_2 = 0$ we have the matrix:

$$\begin{bmatrix} 0 & -b_1 & -b_1a_2 \\ a_2 & -a_1 & a_2b_1 \end{bmatrix}$$

which becomes:

$$\begin{bmatrix} a_2 & -a_1 & a_2b_1 \\ 0 & -b_1 & -b_1a_2 \end{bmatrix}$$

Now $b_1 \neq 0$ (since B is not the origin) so by dividing the second row by $-b_1$ we get

$$\begin{bmatrix} a_2 & -a_1 & a_2b_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

which gives the echelon form:

$$\begin{bmatrix} a_2 & 0 & a_2b_1 + a_2a_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

Since A is not in the line OB we have $a_2 \neq 0$ so dividing the first row by a_2 gives:

$$\begin{bmatrix} 1 & 0 & b_1 + a_1 \\ 0 & 1 & a_2 \end{bmatrix}$$

i.e. we have $x = a_1 + b_1$, $y = a_2$, again confirming the formula, since in this case $a_2 + b_2 = a_2$. $\square$