Review for the first exam

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Disclaimer The following is a set of questions to help you review what we have covered in class. If you know how to answer these questions then you should do well in the exam. However there is no guarantee that the questions in the actual exam will be perceived to be similar to these questions.

1. Calculate the following limits. If you think a certain limit doesn't exist state so and explain why.

   A. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 + x - 6}$. B. $\lim_{x\to -4} \frac{|x+4|}{x+4}$. C. $\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$ D. $\lim_{u\to 3} \frac{\sin 3x}{2x}$

   E. $\lim_{t\to 0} \frac{\sin 5t}{t}$ F. $\lim_{t\to 0} \frac{\sin 5t}{t}$ G. $\lim_{t\to 0} \frac{\tan^5 2t}{t^5}$ H. $\lim_{x\to 0} x \cot x$ I. $\lim_{x\to 0} x^2 \cos \frac{3}{x}$

   J. $\lim_{x\to 0} \frac{\sin x}{\tan x}$

2. Let

$$f(x) = \begin{cases} ax^2 - 3x + 4 & \text{if } x \leq 2 \\ x + 3a & \text{if } x > 2 \end{cases}$$

   Find the real number a so that $\lim_{x\to 2} f(x)$ exists.

3. Explain *in detail* why the function

$$f(x) = \frac{(2+x)^3 - 7}{1+x^2} - \sqrt{x^2+3} - \sin(\cos(3x))$$

   is continuous on $(-\infty, \infty)$.

4. Find the points that each of the following functions is discontinuous and identify the nature of the discontinuity:

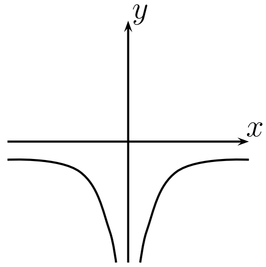
   (a) $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   (b) $g(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \end{cases}$

   (c) $g(x) = \begin{cases} \frac{3}{(x-5)^2} & \text{if } x \neq 5 \\ 5 & \text{if } x = 5 \end{cases}$

5. By examining the graphs calculate the required limits. If you think that a certain limit doesn't exist state so.

a)

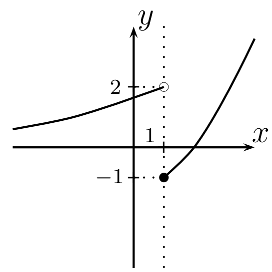


$\lim_{x\to 0^+} f(x) =$

$\lim_{x\to 0} f(x) =$

$\lim_{x\to 0^-} f(x) =$

b)

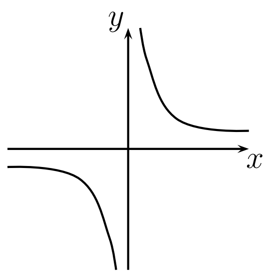


$\lim_{x\to 1^+} f(x) =$

$\lim_{x\to 1} f(x) =$

$\lim_{x\to 1^-} f(x) =$

c)

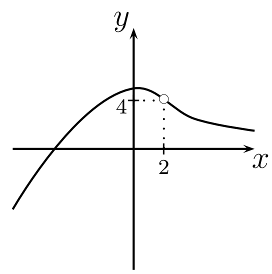


$\lim_{x\to 0^+} f(x) =$

$\lim_{x\to 0} f(x) =$

$\lim_{x\to 0^-} f(x) =$

c)



$\lim_{x\to 2^+} f(x) =$

$\lim_{x\to 2} f(x) =$

$\lim_{x\to 2^-} f(x) =$

6. Sketch the graph of a function f that satisfies all of the following conditions:

$$\lim_{x\to 0^+} f(x) = -2 \quad \lim_{x\to 0^-} f(x) = 1 \quad \text{f(0) = -1}$$
$$\lim_{x\to 2^-} = \infty \qquad \lim_{x\to 2^+} = -\infty$$

7. Give an example of a function that
   (a) has a jump discontinuity at $x = -5$.
   (b) has a removable singularity at $x = 0$.
   (c) has an infinite discontinuity at $x = 3$.
   (d) is continuous everywhere except at $x = 0$ and the discontinuity is not jump, removable or infinite.

8. Prove that each of the following equations has a solution in the given interval. State clearly what theorem you are using.
   (a) $x^4 - 2x^3 + 3x^2 - 2x - 6 = 0$ in $(-1, 1)$
   (b) $2^x = x^2$ in $(-1, 0)$
   (c) $\cos x = x$ in $\left(0, \dfrac{\pi}{2}\right)$

9. Consider the function $f(x) = \dfrac{1}{x-1}$; we have that $f(0) = -1$ and $f(2) = 1$. However even though 0 is between -1 and 1 there is no c in $(-1, 1)$ with $f(c) = 0$. Does this contradict the Intermediate Value Theorem? Why or why not?

10. Calculate each of the following derivatives using the definition of the derivative as the limit of the difference quotients:

    A. $\dfrac{d}{dx}(x^3 - 3x^2 + 5x - 2)$ B. $\left(\dfrac{1}{x^2}\right)'$ C. $\left(\dfrac{x+1}{x+2}\right)'$ D. $\dfrac{d}{dx}\left(\sqrt{2-3x}\right)$

11. Calculate y':

    A. $y = (x^3 + x)^5$ B. $y = \dfrac{-x^2 + x - 2}{\sqrt{x}}$ C. $y = \sqrt[3]{1 + \sec x}$ D. $y = \dfrac{1}{(x-4)^{42}}$

    E. $y = \cos(\sin 3x)$ F. $y = \dfrac{x^2+1}{\csc x}$ G. $y = \sin^3\left(\dfrac{x-1}{x^2+1}\right)$ H. $y = \pi \sin x \cos x \cot x$

    I. $y = \tan\sqrt{1-x}$ J. $x^4 y - 3xy^4 + x^2 y^2 = 3x + 4y$ K. $\sin xy = x^2 - y$

    L. $y = \dfrac{\sin x}{x}$ M. $y = \dfrac{(x+2)^2}{x^2+16}$ N. $y = \sqrt{x}\sin\sqrt{x}$ O. $y = \dfrac{1}{\sqrt[3]{1+\sqrt{x}}}$

12. Let $f(x) = |x^2 + 2x - 8|$.
    (a) Sketch the graph of $y = f(x)$.
    (b) Where does f fail to be differentiable?
    (c) Find $f'(x)$ when it exists.

13. Find an equation of the tangent to the curve at the given point:
    (a) $y = 4\sin^2 x$, at the point $(\frac{\pi}{6}, 1)$
    (b) $y = \dfrac{x^2-4}{x^2+4}$, at the point $(0, -1)$
    (c) $y = \sqrt{4 - 2\sin x}$, at the point $(0, 2)$
    (d) $x^3 + 3x^2 y - 2xy^2 - y^3 = 49$, at the point $(3, 2)$
    (e) $x^{2/3} + y^{2/3} = 4$, at the point $(-3\sqrt{3}, 1)$

14. At what points on the curve $y = \sin x - \cos x$, $0 \le x \le 2\pi$ is the tangent line horizontal?

15. Find the points on the ellipse $2x^2 + y^2 = 1$ where the tangent line has slope 1.

16. How many tangent lines to the curve $y = \dfrac{x}{1+x}$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

17. Give an example of a function f whose graph has a tangent line at $x = 0$ but $f'(0)$ does not exist.

18. A particle moves on a vertical line according to the law of motion

$$s(t) = 2t^3 - 9t^2 + 12t + 3, \qquad t \geq 0$$

where t is measured in seconds and s in meters.

(a) Find the velocity and the acceleration of the particle at time t.

(b) When is the particle moving upward and when is it moving downward?

(c) When is the particle speeding up and when is it slowing down?

(d) Find the total distance traveled by the particle during the first six seconds.

19. A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 3 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 4 m from the wall?

20. Boat A travels west at a 50 miles per hour and boat B travels north at 60 miles per hour. The two boats are going to collide in 3 hours. At what rate are the two boats approaching each other 1 hour before the collision?

21. The surface area of a cube is increasing at a rate of $2\,\text{cm}^2/\text{min}$. How fast is the volume of the cube increasing when the length of the edge is 20 cm?

22. Use appropriate linear approximations to estimate the following:

A. $\sqrt{9.04}$ B. $\sin 0.02$ C. $(1.03)^{-1/3}$ D. $\sqrt[3]{0.97}$