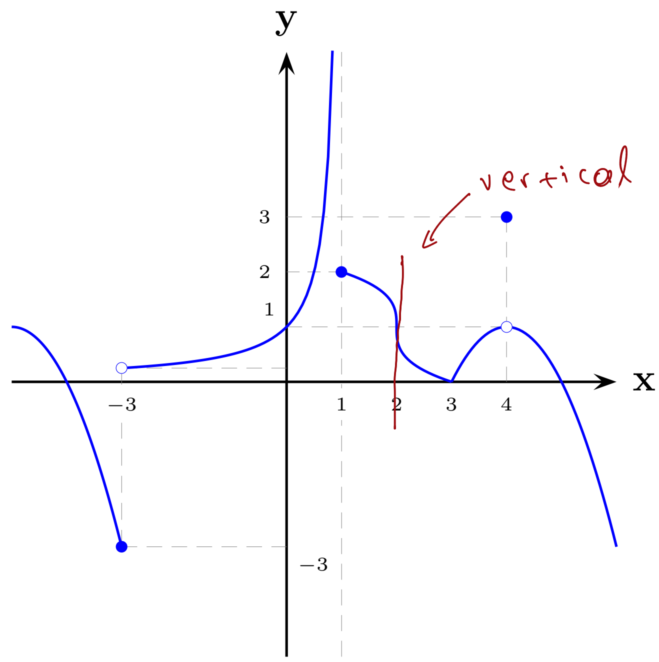
Solutions to the First Exam

Name: Key

1. The graph of $y = f(x)$ is shown bellow:



(a) (50 points) At which points is f discontinuous? What is the nature of discontinuity at each of these points?

f is discontinuous at:

- $x=-3$. We have $\lim_{x\to -3^-} f(x) = -3$ and $\lim_{x\to -3^+} f(x) = \frac{1}{4}$. Since these are different we have jump discontinuity.
- $x=1$, because $\lim_{x\to 1^-} f(x) = +\infty$. We have infinite discontinuity.
- $x=4$, because $\lim_{x\to 4} f(x)=1$, while $f(4)=3$. Since $\lim_{x\to 4} f(x)$ exists we have removable discontinuity.

(b) (50 points) At which points does f fail to be differentiable? Give reasons.

f is not differentiable at

- $x=-3, 1, 4$ because it's not continuous at those points.
- $x=2$, because $y=f(x)$ has a vertical tangent there.
- $x=3$ because there is no tangent line there ($y=f(x)$ has a cusp there.

2. Find the following limits. Your answer should be a real number, $+\infty$, $-\infty$, or *Does Not Exist*.

(a) (25 points) $\lim_{x\to -5} \dfrac{x^2-2x-35}{x+5}$

$x^2-2x-35 = (x-7)(x+5)$

So near -5, for $x \neq -5$ we have $\dfrac{x^2-2x-35}{x+5} = \dfrac{(x-7)\cancel{(x+5)}}{\cancel{x+5}} = x-7$

Thus $\lim_{x\to -5} \dfrac{x^2-2x-35}{x+5} = \lim_{x\to -5}(x-7) = -5-7 = \boxed{-12}$

(b) (25 points) $\lim_{x\to 0} \dfrac{\sin 5x}{3x}$

$\dfrac{\sin 5x}{3x} = \dfrac{5}{3}\,\dfrac{\sin 5x}{5x}$. Let $u = 5x$. Then $x \to 0 \Rightarrow u \to 0$

and we have $\lim_{x\to 0}\dfrac{\sin 5x}{5x} = \lim_{u\to 0}\dfrac{\sin u}{u} = 1$

So $\lim_{x\to 0}\dfrac{\sin 5x}{3x} = \lim_{x\to 0}\dfrac{5}{3}\,\dfrac{\sin 5x}{5x} = \dfrac{5}{3}\lim_{x\to 0}\dfrac{\sin 5x}{5x} = \dfrac{5}{3}\cdot 1 = \boxed{\dfrac{5}{3}}$

(c) (25 points) $\lim_{x\to -7} \dfrac{|x+7|}{x+7}$

$-7 \to x$

$x+7$ $\quad - \quad | \quad +$

$$|x+7| = \begin{cases} x+7, & \text{if } x \geq -7 \\ -x-7, & \text{if } x < -7 \end{cases}$$

So:

$$\frac{|x+7|}{x+7} = \begin{cases} 1, & \text{if } x > -7 \\ -1, & \text{if } x < -7 \end{cases}$$

Thus $\lim_{x\to -7^+}\dfrac{|x+7|}{x+7} = \lim_{x\to -7^+}(+1) = 1$

$\lim_{x\to -7^-}\dfrac{|x+7|}{x+7} = \lim_{x\to -7^-}(-1) = -1$

Since these are different we have $\lim_{x\to -7}\dfrac{|x+7|}{x+7}$ Does not exist

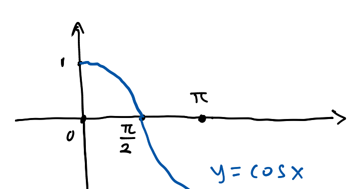
(d) (25 points) $\lim_{x\to \frac{\pi}{2}^+} \tan x$

We have $\tan x = \dfrac{\sin x}{\cos x}$. Now $\lim_{x\to\frac{\pi}{2}^+} \sin x = 0$

while $\lim_{x\to\frac{\pi}{2}^+} \cos x = 0$. Near $\frac{\pi}{2}$, for $x > \frac{\pi}{2}$ we have $\cos x < 0$

So $\lim_{x\to\frac{\pi}{2}} \cos x = 0^-$

Thus $\lim_{x\to\frac{\pi}{2}^+} \tan x = \lim_{x\to\frac{\pi}{2}^+}\dfrac{\sin x}{\cos x} = \dfrac{1}{0^-} = -\infty$



when $x > \frac{\pi}{2}$ $\cos x < 0$

3. (100 points) Prove that the equation $5x^3 - 7x^2 + 8x - 1 = 0$ has a solution in the interval $(0, 1)$. Name any theorems you're using.

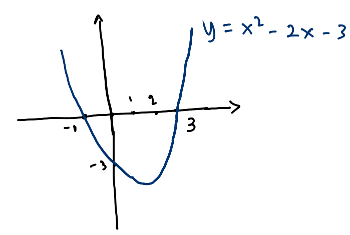
Let $f(x) = 5x^3 - 7x^2 + 8x - 1$. f is a polynomial function and hence continuous.

We have $f(0) = -1$, while $f(1) = 5 - 7 + 8 - 1 = 5$. Since 0 is between $f(0)$ and $f(1)$ by the Intermediate Value Theorem we have that there is a c in the interval $(0,1)$ with $f(c) = 0$. That c is a solution of the equation $5x^3 - 7x^2 + 8x + 1 = 0$.

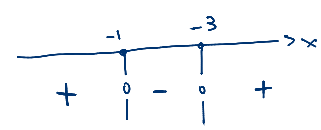
4. Let $f(x) = |x^2 - 2x - 3|$.

(a) (50 points) Sketch a graph of $y = f(x)$.

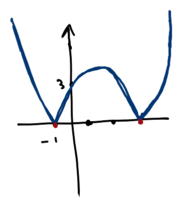
We first graph the parabola $y = x^2 - 2x - 3$



So we have:



So the graph of $y = |x^2 - 2x - 3|$ is



(b) (50 points) At what points f fails to be differentible?

From the graph we see that f is not differentiable at $x = -1$ and $x = 3$.

(c) (50 points) Find $f'(x)$ where it exists.

From the table of signs we have:

$$f(x) = \begin{cases} x^2 - 2x - 3, & \text{if } x \leq -1 \text{ or } x \geq 3 \\ -x^2 + 2x + 3, & \text{if } -1 < x < 3 \end{cases}$$

So:

$$f'(x) = \begin{cases} 2x - 2, & \text{if } x < -1 \text{ or } x > 3 \\ -2x + 2, & \text{if } -1 < x < 3 \end{cases}$$

5. (100 points) Calculate $\dfrac{d}{dx}\left(x^2 - 5x\right)$ using the definition of the derivative as a limit of the difference quotient.

Let $f(x) = x^2 - 5x$. Then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{(x+h)^2 - 5(x+h) - (x^2 - 5x)}{h} = \dfrac{\cancel{x^2} + 2xh + h^2 - \cancel{5x} - 5h - \cancel{x^2} + \cancel{5x}}{h} = \dfrac{2x\cancel{h} + h^2 - 5\cancel{h}}{\cancel{h}} = 2x - 5 + h$

So $\dfrac{d}{dx}(x^2 - 5x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} (2x - 5 + h) = 2x - 5.$

6. Calculate the following derivatives. Simplify your answer as much as possible:

(a) (25 points) $\left(\dfrac{x}{x-1}\right)'$

$$= \frac{(x)'(x-1) - x(x-1)'}{(x-1)^2}$$

$$= \frac{1 \cdot (x-1) - x \cdot (1)}{(x-1)^2}$$

$$= \frac{x - 1 - x}{(x-1)^2}$$

$$= -\frac{1}{(x-1)^2}$$

(b) (35 points) $\left(\sqrt{x}\cos\sqrt{x}\,\right)'$ $= (\sqrt{x})' \cos\sqrt{x} + \sqrt{x}\,(\cos\sqrt{x})' = \dfrac{1}{2\sqrt{x}}\cos\sqrt{x} + \sqrt{x}\left(-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}\right)$

Let $u = \sqrt{x}$, then $u' = \dfrac{1}{2\sqrt{x}}$

so $(\cos\sqrt{x})' = (\cos u)'$

$= -\sin u \cdot u'$

$= -\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

$$= \boxed{\frac{\cos\sqrt{x}}{2\sqrt{x}} - \frac{\sin\sqrt{x}}{2}}$$

(c) (40 points) $\left(\sqrt[5]{x\tan x}\right)'$

Let $u = x\tan x$. Then $u' = (x)'\tan x + x(\tan x)'$

$= \tan x + x\sec^2 x$

So $\left(\sqrt[5]{x\tan x}\right)' = \left(\sqrt[5]{u}\right)'$

$= \left(u^{1/5}\right)'$

$= \dfrac{1}{5}u^{-4/5}\,u'$

$$= \boxed{\frac{\tan x + x\sec^2 x}{5\sqrt[5]{x^4\tan^4 x}}}$$

7. Consider the curve:

$$y^3 + x^3 = 2xy^2 + x - 1$$

(a) (100 points) Find an equation for y', at the points that this equation can be solved to express y as a function of x,

Differentiate both sides with respect to x: $3y^2y' + 3x^2 = 2y^2 + 2x \cdot 2yy' + 1$

$$\Leftrightarrow 3y^2y' + 3x^2 = 2y^2 + 4xyy' + 1$$

$$\Leftrightarrow 3y^2y' - 4xyy' = -3x^2 + 2y^2 + 1$$

$$\Leftrightarrow (3y^2 - 4xy)y' = -3x^2 + 2y^2 + 1$$

$$\Leftrightarrow \boxed{y' = \frac{-3x^2 + 2y^2 + 1}{3y^2 - 4xy}}$$

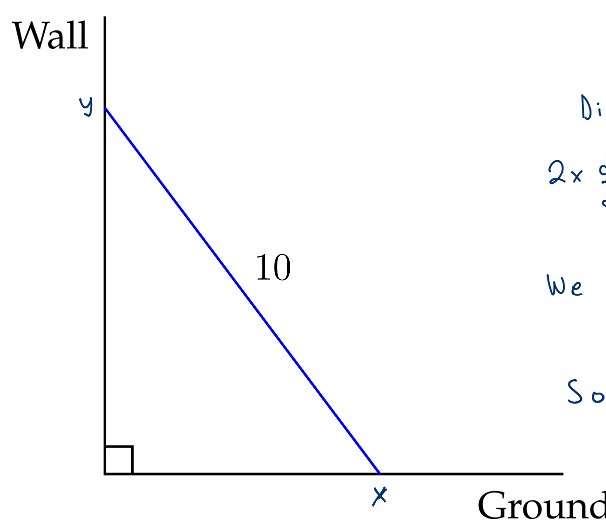
(b) (50 points) Find the equation of the line tangent to the curve at the point $(-2, -1)$

The slope of the tangent line is

$$y'\Big|_{\substack{x=-2\\y=-1}} = \frac{-3(-2)^2 + 2(-1)^2 + 1}{3(-1)^2 - 4(-2)(-1)} = \frac{-12+2+1}{3-8} = \frac{9}{5}$$

So tangent line is $\boxed{y + 1 = \frac{9}{5}(x+2)}$

8. (150 points) A ladder 10 ft long rests against a vertical wall as in the figure bellow.



$x^2 + y^2 = 100$

Differentiating both sides w.r.t. t:

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$.

We are given that $\frac{dx}{dt} = 2$ ft/s

So $\frac{dy}{dt} = -\frac{2x}{y}$

The bottom of the ladder slides away from the wall at a rate of 2 ft/s. How fast is the top of the ladder sliding down when the bottom of the ladder is 8 ft away from the wall?

When $x = 8$ we have

$8^2 + y^2 = 100 \Rightarrow y^2 = 36$

$\Rightarrow y = 6$

So when $x = 8$ we have $\frac{dy}{dt} = -\frac{2 \cdot 8}{6} = -\frac{8}{3}$ ft/s. So the top of the ladder slides down at the rate of $\boxed{8/3 \text{ ft/s.}}$

9. A particle moves on a horizontal line according to the law of motion

$$s(t) = t^3 - 6t^2 + 9t + 5, \qquad t \geq 0$$

where t is measured in seconds and s in meters.

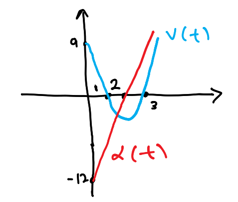
(a) (50 points) Find the velocity and acceleration of the particle as functions of time.

Velocity: $v(t) = \frac{ds}{dt} = \boxed{3t^2 - 12t + 9}$

Acceleration: $a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = \boxed{6t - 12}$

Table of signs for $v(t)$ and $a(t)$

$v(t) = 3(t-3)(t-1)$



t	0		1		2		3	
$v(t)$		+	0	−		−	0	+
$a(t)$		−		−	0	+		+

(b) (25 points) When is the particle moving forward and when is it moving backwards?

From the table of signs we see that the particle

– moves forward for $t < 1$ and for $t > 3$

– Moves backwards for $1 < t < 3$

(c) (25 points) When is the particle speeding up and when is it slowing down?

The particle is speeding up when the $v(t)$ and $a(t)$ have the same sign.

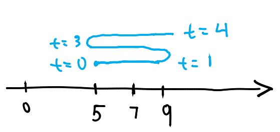
This happens for $1 < t < 2$ and for $t > 3$

The particle slows down when $v(t)$ and $a(t)$ have opposite sign.

This happens for $t < 1$ and for $1 < t < 2$

(d) (25 points) Find the total distance traveled by the particle in the first four seconds.

Schematically the trajectory is:



$s(0) = 5$ $\quad s(2) = 7$ $\quad s(4) = 9$

$s(1) = 9$ $\quad s(3) = 5$

So the total distance is

$9\,m + 9\,m + 9\,m = 27\,m.$

10. Let $f(x) = \sqrt[3]{x-2}$.

(a) (50 points) Find the linearization of f at $a = 10$

The linearization is given by $F(x) = f(10) + f'(10)(x-10)$.

$f(10) = \sqrt[3]{10-2} = \sqrt[3]{8} = 2$

$$f'(x) = \left((x-2)^{1/3} \right)'$$

$$= \frac{1}{3}(x-2)^{-2/3}$$

$$= \frac{1}{3\sqrt[3]{(x-2)^2}}$$

So $f'(10) = \frac{1}{3\sqrt[3]{(10-2)^2}}$

$$= \frac{1}{3\sqrt[3]{64}}$$

$$= \frac{1}{12}$$

So the linearization is: $\boxed{F(x) = 2 + \frac{x-10}{12}}$

(b) (25 points) Use this linearization to estimate $\sqrt[3]{7.98}$.

$\sqrt[3]{7.98} = f(9.98)$ We use the linearization:

$$F(10.2) = 2 + \frac{9.98-10}{12}$$

$$= 2 - \frac{.02}{12}$$

$$\approx 1.9833$$

So $\boxed{\sqrt[3]{7.98} \approx 1.9833}$