MTH 05 Lecture Notes

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1.1 Introduction to fractions

Growing up, the first numbers we encounter are *whole numbers*: $0, 1, 2, \ldots$. These numbers (with the possible exception of zero) are as concrete as a number can be. While I may not be able to picture a "three," I have a clear idea of what three books mean, three dollars are, three fingers, etc. The whole numbers, together with their opposites ("negative whole numbers") are called *integers*.

As soon as we start dividing whole numbers, though, we encounter the problem that the quotient of two whole numbers may not be a whole number.

A *fraction* is a symbolic way of writing a quotient, which is the result of dividing two numbers. In this way, the operation of division is "built into" the notion of a fraction.

For example, $\frac{1}{2}$ is a symbol representing the number that results by performing the operation $1 \div 2$.

Some things to notice right away: A fraction is *one symbol* consisting of two numbers separated by a bar (the bar representing the operation of division). One number is written above the bar—it is called the *numerator*. The other number, written below the bar, is called the *denominator*. The two numbers play different roles. After all, division is not commutative: $1 \div 2$ does not give the same result as $2 \div 1$. So $\frac{1}{2}$ is not the same as $\frac{2}{1}$.

For the first few chapters of this book, most of the numbers we will encounter will be *rational numbers*. A rational number is the result of dividing two integers. Said differently, a rational number is a number which can be written as a fraction whose numerator and denominator are both integers. Remember that since division by zero causes very fundamental problems, the denominator of a fraction representing a rational number must not be zero.

1.2 Decimal representation

Fractions are not the only way to represent the result of a division. Using long division, the quotient of two numbers can be expressed in decimal notation. Here is a simple example:

$$2 \overline{\smash{)}1.0} \quad = 0.5$$

Changing from fraction to decimal notation

To change a fraction to a number in decimal form, perform long division of the numerator by the denominator.

Be sure to practice this, as there are several different ways of performing long division depending on in which country you went to school!

One of the unfortunate features of decimal representations of numbers is that they may not terminate nicely, as the previous simple example did. For example, $1 \div 3 = \frac{1}{3} = 0.\overline{3} = 0.33333\ldots$.

A basic fact of rational numbers, however, is that their decimal representation must either terminate or repeat.

1.2.1 Exercises

Convert the following fractions to decimals.

1. $\frac{3}{4}$

2. $\frac{5}{11}$

3. $\frac{2}{7}$

Throughout the text, starred exercises are those that might be slightly more challenging, or explore a topic in greater depth.

4. (*) Write the number $0.\overline{123}$ in fraction notation.

   (Hint: Represent the number $0.\overline{123}$ by the letter N. Since N has three digits repeating, multiply N by 1000. What is $1000N$? Using these values, subtract $1000N - N = 999N$. Then divide by 999 to "solve for N.")

5. (*) Use the hint in the previous exercise to write the following repeating decimals using fraction notation.

   (a) $0.\overline{123456}$
   (b) $3.\overline{14}$

1.3 Other conventions: Mixed numbers

You may recall that fractions whose numerator is smaller (in magnitude) than the denominator is called a *proper fraction*. For this reason, a fraction whose numerator is greater than or equal to the denominator is sometimes called an *improper fraction*. However, there is nothing improper about improper fractions at all—we will work with them routinely. In fact, in most circumstances, it is better to work with improper factions than their alternative.

However, there is another common way of expressing improper fractions. These are what are called *mixed numbers*. A mixed number has an integer part and a proper fraction part.

To convert from an improper fraction to a mixed number, perform the indicated division. The quotient will be the integer part of the mixed number. The remainder will be the numerator of the proper fraction part; the denominator is the same as the denominator of the original improper fraction.

Example 1.3.1. *Convert* $\frac{22}{7}$ *to a mixed number.*

Answer. $22 \div 7 = 3\ R\ 1$. *So*

$$\frac{22}{7} = 3\frac{1}{7}.$$

The answer is $3\frac{1}{7}$.

Notice that $3\frac{1}{7}$ actually means $3+\frac{1}{7}$. This is an unfortunate notation, since normally the absence of a symbol for an operation between two numbers implies multiplication. But at this point, the notation is a historical fact of life.

To convert from a (positive) mixed number to an improper fraction, multiply the integer part by the denominator of the fraction part and add the numerator of the fraction part; the result will be the numerator of the improper fraction. The denominator of the improper fraction is the same as the denominator of the fraction part of the mixed number.

Example 1.3.2. *Convert* $5\frac{2}{3}$ *to an improper fraction.*

Answer. *To obtain the new numerator, first multiply the denominator by the integer part:* $3 \times 5 = 15$. *Then add the numerator of the fraction part:* $15 + 2 = 17$.

The denominator is the same as the denominator of the fraction part, in this case 3.

The answer is $\frac{17}{3}$.

We will not insist that improper fractions be converted to mixed numbers! In most cases, we will not work with mixed numbers at all.

1.3.1 Exercises

Change the following improper fractions to mixed numbers.

1. $\frac{19}{5}$

2. $\frac{100}{3}$

Change the following mixed numbers to improper fractions.

3. $4\frac{1}{8}$

4. $2\frac{3}{10}$

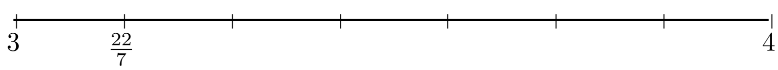
1.4 Graphical representation of fractions

You may remember from your youth seeing your math teacher drawing pictures of pizzas on the board to illustrate a way to represent fractions. Throughout this text, the most convenient way to represent numbers , including fractions, will be using a *number line*. In this representation, every number will correspond to a geometrical point.

The key features of a number line are: (1) it extends infinitely in both directions; (2) one direction is designated the positive direction (to the right) while the other is the negative direction (to the left); (3) there is one distinguished point on the line, representing the number 0; and (4) there is a "unit distance" with length one, which allows us to mark off all the points representing integers.

Proper fractions (at least the positive ones) are then represented by points lying between those labeled 0 and 1. The denominator tells us how many units to subdivide the segment between the points labeled 0 and 1; the numerator tells us how many of these sub-units to count from 0.

Improper fractions are handled similarly. For these fractions, it is more convenient to represent the number as a mixed number. Instead of dividing the segment between the points representing 0 and 1, we divide the segment starting at the point representing the integer part of the number and the next point representing an integer away from the point representing 0. For example, thinking of the improper fraction $\frac{22}{7}$ as the mixed number $3\frac{1}{7}$, we can represent this number with a point $\frac{1}{7}$ of a unit between 3 and 4:



1.5 Equivalent fractions and fractions in simplest form

One of the most important features of fractions is that two different-looking fractions might represent the same number. When you think about it, this shouldn't be a big surprise. After all, there are many different division problems that give the same result. For example, $9 \div 3$ is the same as $6 \div 2$—both are 3. That's one way to see that the fractions $\dfrac{9}{3}$ and $\dfrac{6}{2}$ are different ways of symbolizing the same number.

Two fractions are called *equivalent* if they represent the same number.

The basic principle we will need to keep in mind is the following: **Multiplying or dividing both the numerator and the denominator of a fraction by the same non-zero number gives an equivalent fraction**. In fact, this procedure amounts to multiplying a number by 1, which of course does not change the number.

There are two main reasons that we will be interested in equivalent fractions: writing fractions with common denominators, and writing fractions in simplest form.

1.5.1 Writing fractions with common denominators

As we will see below, there are many situations when we would like to write two fractions in an equivalent way so that they have the same denominator.

Example 1.5.1. *Write the fractions* $\frac{3}{10}$ *and* $\frac{7}{15}$ *with a common denominator.*

Answer. *There are two major steps in writing two fractions using a common denominator.*

Step 1. *Find a common denominator.*

We will find the least common multiple of the two denominators 10 *and* 15. *That is, we will find the smallest whole number which is multiple of both 10 and 15.*

Multiples of 10: $10, 20, 30, 40, \ldots$

Multiples of 15: $15, 30, 45, 60, \ldots$

$LCM(10, 15) = 30$.

Step 2. *For each of the fractions, decide what number is needed to multiply the original denominator in order to obtain the new, common denominator. Then multiply both the numerator and the denominator of the fractions by this number to obtain the equivalent fraction with the common denominator.*

For the fraction $\frac{3}{10}$, *what number do we need to multiply the original denominator* 10 *by to obtain the new common denominator* 30*?* $30 \div 10 = 3$. *So:*

$$\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30}.$$

Likewise, for the fraction $\frac{7}{15}$, *what number do we need to multiply* 15 *by to obtain* 30*?* $30 \div 15 = 2$. *So:*

$$\frac{7}{15} = \frac{7 \times 2}{15 \times 2} = \frac{14}{30}.$$

The answer is $\frac{9}{30}$ *and* $\frac{14}{30}$.

1.5.2 Writing fractions in simplest form

A fraction is said to be in *simplest form* when the numerator and denominator have no factors in common (except 1).

For example, the fraction $\frac{8}{52}$ is not in simplest form, since 4 is a factor of 8 (since $8 = 4 \times 2$) and 4 is also a factor of 52 (since $52 = 4 \times 13$). When dealing with large numbers as numerators and denominators, it is sometimes helpful to see their *prime factorizations*. We will not emphasis that here; most of the time we will be able to see common factors without much trouble.

Example 1.5.2. *Write the fraction* $\frac{8}{52}$ *in simplest form.*

Answer. *We saw above that the numerator and the denominator have a common factor of* 4. *To write the fraction in simplest form, we will apply the opposite procedure we used above in writing fractions with a common denominator: we will* ***divide both the numerator and denominator by the common factor****.*

$$\frac{8}{52} = \frac{8 \div 4}{52 \div 4} = \frac{2}{13}.$$

Notice that 2 *and* 13 *have no factors in common, so* $\frac{2}{13}$ *is in simplest formand equivalent to the original fraction* $\frac{8}{52}$.

The answer is $\frac{2}{13}$.

To repeat, multiplying or dividing both the numerator and denominator by the same nonzero number results in an equivalent fraction.

1.5.3 Exercises

Write the following fractions in simplest form.

1. $\frac{6}{8}$
2. $\frac{20}{25}$
3. $\frac{18}{6}$
4. $\frac{118}{177}$
5. $\frac{14}{10}$

1.6 Operations with fractions: Multiplying and dividing

Now that we have reviewed the basic properties of fractions, we will review the rules for performing arithmetic operations using this symbolism. We will begin with multiplying and dividing. That might seem strange if you think that adding and subtracting are "easier" operations to work with, but since fraction notation is based on the operation of division, it should not be too hard to believe that multiplying and diving are much more suited to the notation than adding and subtracting.

1.6.1 Multiplying fractions

Multiplying whole numbers has a clear relationship to the operation of addition. For example, 2×3 means the same as "three added together two times," or $3+3$. However, if we want to extend the operation of multiplication to fractions, negative numbers, and other more exotic numbers, we have to make sure that certain basic properties are preserved, like the *commutative* and *associative* properties (which, in the case of whole number multiplication, are just easy consequences of the corresponding properties for addition). In addition, multiplication and addition must be related by the *distributive* property.

We will not review these properties here. We only mention them to indicate that the rules for multiplying (and dividing) fractions are not arbitrary, but are carefully constructed so that all our basic operations interact in the same way we expect them to do based on our experience with whole numbers.

Multiplying fractions

The product of two fractions is a new fraction whose numerator is the product of the two numerators and whose denominator is the product of the two denominators.

Example 1.6.1. *Multiply:* $\frac{2}{3} \times \frac{4}{5}$.

Answer.

$$\frac{2}{3} \times \frac{4}{5}$$
$$\frac{2 \times 4}{3 \times 5}$$
$$\frac{8}{15}.$$

The answer is $\frac{8}{15}$.

Note that the answer is in simplest form. That doesn't always happen, as the next example shows.

Example 1.6.2. *Multiply:* $\frac{1}{4} \times \frac{2}{3}$.

Answer. *The two fractions we begin with are in simplest form. Multiplying,*

$$\frac{1}{4} \times \frac{2}{3}$$
$$\frac{1 \times 2}{4 \times 3}$$
$$\frac{2}{12} \quad \textit{(Note that the numerator and denominator have a common factor of } 2\textit{)}$$
$$\frac{2^{\div 2}}{12^{\div 2}}$$
$$\frac{1}{6}.$$

The answer, in simplest form, is $\frac{1}{6}$.

We use one more example as a reminder that whole numbers can be thought of as fractions, most easily as "itself divided by 1".

Example 1.6.3. *Multiply:* $15 \times \frac{4}{7}$.

Answer.

$$15 \times \frac{4}{7}$$
$$\frac{15}{1} \times \frac{4}{7}$$
$$\frac{15 \times 4}{1 \times 7}$$
$$\frac{60}{7}.$$

The final answer, $\frac{60}{7}$*, is in simplest form. According to our convention, we will not bother to change it to a mixed number. (If you wanted to, it would be* $8\ \frac{4}{7}$*.)*

1.6.2 Dividing fractions

To divide fractions, we will take advantage of the fact that division is the "opposite of," or more precisely the *inverse operation* of multiplication. In order to take advantage of this fact, we recall the idea of the *reciprocal* of a number.

Two numbers are *reciprocals* if their product is 1.

In practice, to find the reciprocal of a number written as a fraction, we interchange the numerator and the denominator.

Example 1.6.4. *The reciprocal of* $\frac{2}{3}$ *is* $\frac{3}{2}$.

After all,

$$\frac{2}{3} \cdot \frac{3}{2} = \frac{6}{6} = 1.$$

Example 1.6.5. *The reciprocal of* 4 *(*$= \frac{4}{1}$*) is* $\frac{1}{4}$.

After all,

$$\frac{4}{1} \cdot \frac{1}{4} = \frac{4}{4} = 1.$$

Dividing fractions

The quotient of two fractions is the same as the product of the first fraction by the reciprocal of the second fraction.

So, for example, we will rewrite $\dfrac{3}{4} \div \dfrac{2}{3}$ as $\dfrac{3}{4} \cdot \dfrac{3}{2}$.

Example 1.6.6. *Divide:* $\dfrac{7}{8} \div \dfrac{3}{4}$.

Answer.

$$\begin{aligned} &\frac{7}{8} \div \frac{3}{4} \\ &\frac{7}{8} \cdot \frac{4}{3} && \textit{Rewriting as a product} \\ &\frac{7 \times 4}{8 \times 3} \\ &\frac{28}{24} \\ &\frac{28^{\div 4}}{24^{\div 4}} \\ &\frac{7}{6}. \end{aligned}$$

The answer is $\dfrac{7}{6}$.

Example 1.6.7. *Divide:* $15 \div \dfrac{2}{3}$.

Answer.

$$
\begin{aligned}
&15 \div \frac{2}{3} \\
&\frac{15}{1} \div \frac{2}{3} && \textit{Rewriting the whole number as a fraction} \\
&\frac{15}{1} \cdot \frac{3}{2} && \textit{Rewriting as a product} \\
&\frac{15 \times 3}{1 \times 2} \\
&\frac{45}{2}.
\end{aligned}
$$

The answer is $\frac{45}{2}$.

Example 1.6.8. *Divide:* $\frac{9}{2} \div 15$.

Answer.

$$
\begin{aligned}
&\frac{9}{2} \div \frac{15}{1} \\
&\frac{9}{2} \cdot \frac{1}{15} && \textit{Rewriting as a product} \\
&\frac{9 \times 1}{2 \times 15} \\
&\frac{9}{30} \\
&\frac{9^{\div 3}}{30^{\div 3}} && \textit{Reducing} \\
&\frac{3}{10}.
\end{aligned}
$$

The answer is $\frac{3}{10}$.

Keep in find that division is also be indicated by the fraction bar, as the following example illustrates.

Example 1.6.9. *Divide:* $\frac{\frac{5}{4}}{12}$.

Answer.

$$
\begin{aligned}
&\frac{5}{4} \div \frac{12}{1} \\
&\frac{5}{4} \cdot \frac{1}{12} && \textit{Rewriting as a product} \\
&\frac{5 \times 1}{4 \times 12} \\
&\frac{5}{48}.
\end{aligned}
$$

The answer is $\frac{5}{48}$.

1.6.3 Exercises

Perform the indicated operation.

1. $\frac{3}{4} \cdot \frac{2}{5}$

2. $\frac{2}{3} \div \frac{4}{5}$

3. $\frac{17}{2} \cdot \frac{4}{3}$

4. $15 \cdot \frac{1}{5}$

5. $\frac{3}{8} \div 18$

6. $\frac{\frac{7}{12}}{\frac{14}{3}}$

1.7 Operations with fractions: Adding and subtracting

Unlike multiplication and division, addition and subtraction does not follow a rule that sounds like "do the operation to the top and bottom separately." Instead, they follow a rule of a different pattern, one which we will see many times ahead. Adding and subtracting fractions requires that the fractions be "of the same kind." For example, we will have a way to understand adding sixths to sixths, but adding sixths to fifths will require that we express them as fractions "of the same kind.".

Adding and subtracting fractions

To add or subtract two fractions:

1. Write the two fractions, using equivalent fractions if necessary, with a common denominator.
2. Add (or subtract) the numerators, while keeping the same (common) denominator.

Example 1.7.1. *Add:* $\frac{3}{4} + \frac{3}{8}$.

Answer. *The two fractions are not written with a common denominator. The least common denominator is* 8*.*

$$\begin{aligned} &\frac{3}{4} + \frac{3}{8} \\ &\frac{3^{\times 2}}{4^{\times 2}} + \frac{3^{\times 1}}{8^{\times 1}} \quad \textit{Rewrite using common denominator} \\ &\frac{6}{8} + \frac{3}{8} \\ &\frac{6+3}{8} \\ &\frac{9}{8}. \end{aligned}$$

The answer is $\frac{9}{8}$.

Example 1.7.2. *Subtract:* $2 - \frac{4}{7}$.

Answer. *We will rewrite the whole number* 2 *as* $\frac{2}{1}$. *The least common denominator for the two fractions is* 7*.*

$$2 - \frac{4}{7}$$
$$\frac{2}{1} - \frac{4}{7} \quad \textit{Rewrite the whole number as a fraction}$$
$$\frac{2^{\times 7}}{1^{\times 7}} - \frac{4^{\times 1}}{7^{\times 1}} \quad \textit{Rewrite using common denominator}$$
$$\frac{14}{7} - \frac{4}{7}$$
$$\frac{14-4}{7}$$
$$\frac{10}{7}.$$

The answer is $\frac{10}{7}$.

Example 1.7.3. *Subtract:* $\frac{5}{6} - \frac{1}{8}$.

Answer. *The least common denominator of the two fractions is* 24.

$$\frac{5}{6} - \frac{1}{8}$$
$$\frac{5^{\times 4}}{6^{\times 4}} - \frac{1^{\times 3}}{8^{\times 3}} \quad \textit{Rewrite using common denominator}$$
$$\frac{20}{24} - \frac{3}{24}$$
$$\frac{20-3}{24}$$
$$\frac{17}{24}.$$

The answer is $\frac{17}{24}$.

1.7.1 Exercises

Perform the indicated operation.

1. $\frac{8}{7} + \frac{4}{5}$

2. $\frac{5}{6} - \frac{2}{3}$

3. $\frac{4}{5} + \frac{3}{10}$

4. $\dfrac{5}{12} - \dfrac{1}{3}$

5. $5 - \dfrac{2}{3}$

6. $1\dfrac{2}{3} - \dfrac{3}{4}$.

1.8 Chapter summary

- A fraction is a way of representing the result of dividing two numbers.
- A fraction can be changed to the corresponding decimal representation by performing long division.
- The only time we will use mixed numbers will be when we want to represent a fraction graphically.
- Multiplying or dividing BOTH the numerator AND the denominator of a fraction by the same nonzero number will result in an equivalent fraction.
- Dividing two fractions is performed as multiplication of the first fraction by the reciprocal of the second fraction.
- Adding and subtracting fractions (unlike multiplying and dividing) require that both fractions have the same (common) denominator.
- Final answers involving fractions should always be expressed in simplest form, but may be improper fractions.

Chapter 2

Signed numbers

Vocabulary

- Magnitude
- Absolute value
- Positive
- Negative
- Opposite of a number
- Base (of an exponential expression)

2.1 Introduction

When a child learns to count, numbers only go in one direction—they "get bigger." In fact, for thousands of years, civilizations rose and fell using only positive numbers. After all, numbers first arise in answer to the question, "How many?" How many what? How many things—the things being things which might be picked up, looked at, put on a shelf, etc.

Negative numbers are more complicated. It doesn't make very much sense to say, "I have -5 books." Historically, negative numbers arose to take into account losses or debt, and was undoubtedly connected to the emergence of money or coin. In this way, there is a meaning to the sentence, "I have -5 dollars"—it simply means that I owe 5 dollars, instead of having 5 dollars. In this context, "to owe" is the opposite of "to have," and the negative numbers will be the opposite of the the more familiar positive numbers, in a way that will be made precise below.

For now, let's say that every positive number has an opposite, and this opposite (of a positive number) will be a negative number. Zero will be special, in that it is neither positive nor negative; it is "neutral," and we will say that zero "is its own opposite."

Our goal in this chapter will be to describe how to perform the basic arithmetic operations—adding, subtracting, multiplying, dividing—with these negative numbers.

As soon as we allow negative numbers, we will need to take into account two aspects of every nonzero number: its *sign*, which can be either positive or negative, and its *magnitude*, which is a positive numerical value. A positive number is indicated by a "+" symbol along with a magnitude. For example, the symbol $+5$ will represent the number whose sign is positive and whose magnitude is 5. A negative number is indicated by a "$-$" symbol along with a magnitude. For example, the symbol -5 will represent the number whose sign is negative and whose magnitude is 5. Notice that *the magnitude of a nonzero number is always positive.* Operations with signed numbers will have to take both of these aspects into account.

(The magnitude of a number is represented symbolically by means of the *absolute value* symbol $|\ \cdot\ |$. For example, we can summarize the preceding paragraph with $|5| = 5$ and $|-5| = 5$.)

We have already seen that zero is a special number when it comes to signs. In fact, the very idea of "opposite" that we have used to motivate the negative numbers will be defined relative to the number zero. Along with the fact that the number 0 will be neither positive nor negative, we will say that 0 has magnitude zero.

Warning: Do not confuse the meaning of symbols for the sign of a number with the meaning of the symbols for addition and subtraction. It is an unfortunate fact of history that the symbols are in fact the same, but the meanings are very different, as we will see below.

Convention: When a sign is not indicated for a number, it will be assumed to be positive. For example, the symbol "5" will have the same meaning as "+5."

2.2 Graphical representation and comparison of signed numbers

While there are several ways to understand negative numbers, the graphical representation of numbers on a number line is particularly helpful. Recall that a number line has three essential components: it extends infinitely (from left to right), it has a special point representing zero, and it has a specified unit length. In this representation, positive numbers will be those numbers represented by points *to the right of zero*, while negative numbers will be those numbers represented by points *to the left of zero.*

When we represent a signed number on a number line, the number's sign will tell us on which side of zero it will be represented, while its magnitude will tell us the distance (in terms of the specified unit length) from the representative point to the point representing zero. Thinking of the magnitude as the "distance from zero" corresponds to the convention that magnitudes of nonzero numbers, like distances, are always positive quantities.

The number line representation of signed numbers also gives us an easy was to visualize comparisons of signed numbers. By comparison, we mean either "less than," "equal to," or "greater than." Symbolically, these three possible comparisons are written as $<$ ("is less than"), $=$ ("is equal to") and $>$ ("is greater than").

Comparing positive numbers corresponds to our standard notions of quantity. So for example, $15 < 27$. Comparing positive numbers in decimal or fraction notation is only a little more challenging, in that we first need to see them as like quantities before comparing. So $0.043 > 0.0099$ (since $0.043 = 0.0430$ and $430 > 99$) and $\frac{3}{11} < \frac{2}{7}$ (since $\frac{3}{11} = \frac{21}{77}$, $\frac{2}{7} = \frac{22}{77}$, and $21 < 22$). But which is bigger, -10 or -15?

Using the number line representation and comparison of positive numbers as our guide, we will translate "less than" as "to the left of," and "greater than" as "to the right of." In this way, $-10 > -15$ since the point representing -10 is to the right of the point representing -15 on the number line.

This reasoning can be summarized in the following guide for comparing signed numbers. Note that the signs and the magnitudes are both important in comparing two signed numbers.

- The lesser of *two positive numbers* is the positive number with the lesser magnitude.
- The lesser of *one positive and one negative number* is the negative number.
- The lesser of *two negative numbers* is the negative number with the greater magnitude (the "most negative" number).

2.3 Operations with signed numbers: Addition and subtraction

How much money do you have at the end of the following situations? Think of debt as being represented by negative numbers and money you have as positive numbers.

- You have \$100. Your partner hands you \$250.
- You have \$100. Your partner hands you an \$80 phone bill.
- You have an \$80 phone bill. Your partner hands you \$250.
- You have an \$80 phone bill. Your partner hands you a \$100 electric bill.

In all four scenarios, you have something and your partner adds to what you have. But the way you treat the four cases is different.

The goal in this section is to arrive at rules for adding and subtracting signed numbers. Because we now have to keep track of two aspects of each number—its sign and its magnitude—the rules will be more complicated than the rules for adding and subtracting positive numbers that we learned in grade school.

2.3.1 Adding signed numbers

Before listing the rules for addition, let's give another illustration using the graphical representation of numbers on a number line. If you had to draw a picture, using the number line model of numbers, of the familiar equation "$2 + 3 = 5$," perhaps the best way to do it would be as follows:

First, draw an arrow starting at 0 and stretching for 2 units in the positive direction—to the right. Then, draw another arrow starting where the first arrow ended (at the point representing 2) and stretching for 3 units, also in the positive direction. The sum is represented by the point where the second arrow ends: at the point representing 5. See Figure 2.1.

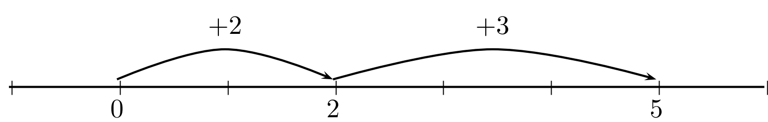


Figure 2.1: Adding 3 to 2.

The advantage of this graphical representation of addition is that it is very easily adapted to take into account negative numbers. We will simply draw negative numbers using arrows pointing in the negative direction–arrows pointing to the left. For example, Figure 2.2 is a number line representation of the sum $4 + (-5)$.

The result indicates that $4 + (-5) = -1$.

If you take a few minutes to draw a few more of these number line pictures, a few things should become clear. First, the pattern depends on whether the two numbers being added have the *same sign* or *different sign*. Depending on that, we will end up wither adding or subtracting the magnitudes.

Here is a summary of the conclusions of this discussion.

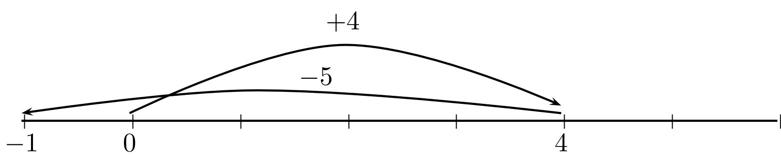


Figure 2.2: Adding −5 to 4.

Rules for adding signed numbers

- *To add signed numbers with the* ***same sign***:
  - The sum has the *same sign* as the sign of the two original numbers;
  - The sum has magnitude which is the *sum* of the magnitudes of the two original numbers.
- *To add signed numbers with* ***different signs***:
  - The sum has the *sign of the number with the larger magnitude* of the two orginal numbers;
  - The sum has magnitude which is the *difference* of the magnitudes of the two original numbers (subtracting the smaller magnitude from the larger).

Here are some examples.

Example 2.3.1. *Add:* $(-12) + (-15)$.

Notice that we are adding two numbers with the same sign—both are negative. This tells us two things:

- *The sum will have the same sign—it will also be negative.*
- *The magnitude will be the sum of the magnitudes:* $12+15 = 27$. *(Remember: magnitudes are always positive quantities!)*

So $(-12) + (-15) = -27$.
The answer is -27.

The parentheses in the preceding example are grouping symbols, meant to indicate that the symbol representing the sign of the numbers "goes with" the number, and are "separate from" the symbol representing addition. It is in mathematical "bad taste" to write expressions like $-12 + -15$.

Example 2.3.2. *Add:* $(-4) + 12$.

This time we are adding numbers with different signs. Notice the $+$ *sign represents addition, not a sign. However, reading the phrase as "negative four plus twelve," we see that the number* 12*, since it does not have a sign specified, is in fact positive.*

- *The sum will have the sign of the number with the bigger magnitude, which is positive. (After all,* $+12$ *has magnitude* 12*, and* -4 *has magnitude* 4*.)*
- *The sum will be the difference of the larger magnitude and the smaller magnitude:* $12 - 4 = 8$.

So $(-4) + 12 = 8$.
The answer is 8.

We illustrate one more example using fractions.

Example 2.3.3. *Add:* $-\dfrac{4}{5} + \dfrac{3}{10}$.

As usual when adding fractions, we will need to rewrite the problem using equivalent fractions with a common denominator. This will also allow us to compare the magnitudes of the fractions.

We are adding two numbers with opposite signs. The magnitude of $-\frac{4}{5}$ *is* $\frac{4}{5}\left(=\frac{8}{10}\right)$*. The magnitude of* $\frac{3}{10}$ *is* $\frac{3}{10}$*. The sign of the final answer will be negative, since the number with greater magnitude is negative.*

Subtracting the magnitudes:

$$\begin{aligned}
&\frac{4}{5} - \frac{3}{10} \\
&\frac{4^{\times 2}}{5^{\times 2}} - \frac{3^{\times 1}}{10^{\times 1}} && \textit{Writing with common denominator } 10 \\
&\frac{8}{10} - \frac{3}{10} \\
&\frac{8-3}{10} \\
&\frac{5}{10} \\
&\frac{1}{2} && \textit{Reducing.}
\end{aligned}$$

Hence: $-\dfrac{4}{5} + \dfrac{3}{10} = -\dfrac{1}{2}$.
The answer is $-\dfrac{1}{2}$.

What should be clear from the above rules and examples is that addition of signed numbers is a two-step process, corresponding to the two "parts" of signed numbers. We first establish the sign of the result. We then determine the magnitude of the results by either adding or subtracting the magnitudes, depending on the signs of the original two numbers.

Warning! Notice that these rules do NOT say that "a negative and a negative gives a positive," which is a famous distortion of a correct rule (see below). In fact, the sum of negative numbers is negative, not positive!

2.3.2 Subtracting signed numbers

We will not make a new list of rules for subtracting signed numbers. Instead, we will illustrate a procedure to rewrite subtraction problems as addition problems, and then rely on the same rules for adding signed numbers that we outlined in the previous section.

When subtraction is first presented in grade school, it is usually described as the operation of "taking away." Unfortunately, while this analogy still holds for negative numbers, it can be more confusing. Is it obvious to you that taking away debt has the same effect as giving you cash?

Instead, we revert to the number line representations that gave us a clue to our rules of adding signed numbers above. What would the number line picture look like for the subtraction problem $6 - 2 = 4$?

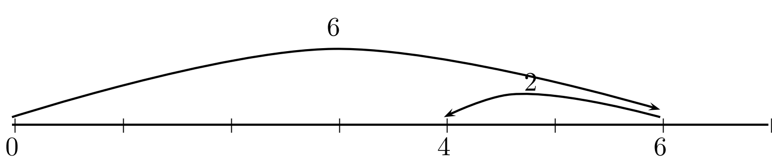


Figure 2.3: Subtracting 2 from 6.

Subtracting (positive!) 2 is represented with an arrow going in the opposite direction ("taking away from") the positive direction. See Figure 2.3.

Now compare Figure 2.3 to the number line representation of the addition problem $6+(-2)$ in Figure 2.4. The two pictures are identical! We have labelled them slightly differently to take into account that the second diagram depicts addition, where the first diagram depicts subtraction.

These two number line diagrams reflect a very important mathematical fact:

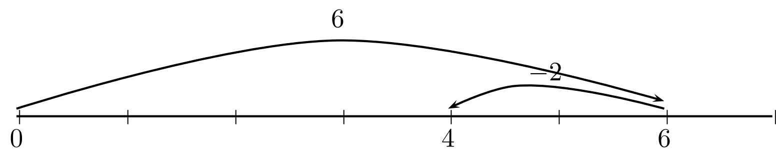


Figure 2.4: Adding −2 to 6.

Subtraction is the same as "adding the opposite."

To be more precise, we need to understand what "the opposite" means. In the context of addition, the opposite of a number is the number with the same magnitude, but opposite sign. So the opposite of 3 is −3, whereas the opposite of −8 is 8. To be even more precise, we will say that *two numbers are opposites if their sum is* 0. Notice that $3 + (-3) = 0$ and $(-8) + 8 = 0$.

Notation: Many times, it will be convenient to think of a negative sign as representing "the opposite of." So the symbol −5 can be thought of interchangeably as "negative five" and "the opposite of (positive) five." More importantly, the symbol $-(-12)$ should be understood as "the opposite of negative twelve"—which is of course positive twelve!

We are now going to outline a procedure for subtracting signed numbers. Keep in mind that in any subtraction problem, the order of the numbers matter! For example, $7-5$ is not the same as $5-7$. (In mathematical terms, subtraction is not *commutative*.)

Procedure for subtracting signed numbers

1. Rewrite the subtraction problem as the sum of the first number and the *opposite of the second number*;
2. Then apply the rules for adding signed numbers in the preceding section.

Important note: In applying this procedure for subtracting signed numbers, it is important to emphasize the difference in meaning between the symbol for subtraction and the symbol for "negative" (or "opposite of"). Most times, reading the problem out loud will indicate whether the $-$ symbol represents subtraction ("minus") or opposite ("negative").

Exercise 2.3.4. *Subtract:* $-10 - 12$.

Answer. *The problem reads: "Negative ten minus twelve." The numbers that are being subtracted are* -10 *and (positive)* 12. *Rewriting the subtraction as "Negative ten plus negative twelve,"*

$$\begin{aligned} -10 - 12 \\ -10 + (-12) \\ -22. \end{aligned}$$

The answer is -22.

Notice that this procedure involves two separate changes: the operation changes from subtraction to addition, and the sign of the second number changes. The first number does not change.

Exercise 2.3.5. *Subtract:* $(-15) - (-8)$.

Answer. *The problem reads: "Negative fifteen minus negative eight." The numbers being subtracted are* -15 *and* -8. *Rewriting,*

$$\begin{aligned} -15 - (-8) \\ -15 + (+8) \\ -7. \end{aligned}$$

The answer is -7.

Here is one more example, this time with fractions:

Exercise 2.3.6. *Subtract:* $\frac{1}{4} - \frac{5}{6}$.

Answer. *The problem reads: "(Positive) one-fourth minus (positive) five-sixths." Rewriting:*

$$\begin{aligned} &\frac{1}{4} - \frac{5}{6} \\ &\frac{1}{4} + \left(-\frac{5}{6}\right) \\ &\frac{3}{12} + \left(-\frac{10}{12}\right) \quad \textit{(rewriting with least common denominator)} \\ &-\frac{7}{12}. \end{aligned}$$

We will comment more on the placement of the negative sign in the next section, after discussing dividing signed numbers. Here, we obtained the last step by noting that the number with larger magnitude ($-\frac{10}{12}$) was negative, so the sum must be negative.

The answer is $-\dfrac{7}{12}$.

2.3.3 Exercises

Perform the indicated operations.

1. $(-4) + (-3)$
2. $(-5) + (-2)$
3. $3.25 + (-1.8)$
4. $(-12) + (-2)$
5. $10 + (-10)$
6. $\left(-\dfrac{2}{5}\right) + \left(-\dfrac{1}{4}\right)$
7. $\left(\dfrac{1}{8}\right) + \left(-\dfrac{1}{12}\right)$
8. $(-5) - (-2)$
9. $5 - 12$
10. $-4 - 4$
11. $(-4.03) - (-2.1)$
12. $\left(-\dfrac{3}{4}\right) - \left(\dfrac{1}{4}\right)$
13. $\left(\dfrac{1}{3}\right) - \left(-\dfrac{1}{4}\right)$
14. $\left(-\dfrac{4}{7}\right) - \left(-\dfrac{5}{14}\right)$
15. $\dfrac{3}{4} - \dfrac{7}{8}$

2.4 Operations with signed numbers: Multiplication and division

Multiplication of whole numbers emerged as an abbreviated form of addition. So, for example, "6×3" means 3 added to itself 6 times. But to extend this simple understanding to all kinds of numbers, we require that the commutative, associative, and distributive properties continue to hold.

It is not too hard, for example, to extend the definition of multiplication to all integers. First, multiplying a positive number by a negative number, we can reason as in the following example: We can think of $6 \times (-3)$ as -3 added to itself 6 times:

$$(-3) + (-3) + (-3) + (-3) + (-3) + (-3) = -18.$$

If the negative number appears first, this is slightly more difficult: How can we think of $(-2) \times 4$ as 4 added to itself -2 times? However, if we insist that multiplication of signed numbers should still be commutative, then $(-2) \times 4$ must be the same as $4 \times (-2)$, which is

$$(-2) + (-2) + (-2) + (-2) = -8.$$

What should be clear from both of these examples is that *the product of a negative number and a positive number is a negative number.*

What about multiplying a negative number by a negative number? First, you should convince yourself that the opposite of a positive number is negative, and the opposite of a negative number is positive. Second, remember that two numbers are opposites if their sum is 0.

Let's consider the example $(-2) \times (-3)$. We will show that this number is the opposite of $2 \times (-3)$. After all,

$$\begin{aligned} (-2) \times (-3) + 2 \times (-3) &= (2 + (-2)) \times (-3) \quad \text{by the distributive property} \\ &= 0 \times (-3) \\ &= 0. \end{aligned}$$

This calculation shows that $(-2) \times (-3)$ is the opposite of $2 \times (-3)$. But we saw above that $2 \times (-3) = -6$, so $(-2) \times (-3)$ is the opposite of -6. In conclusion, assuming that the distributive property is to hold, $(-2) \times (-3) = +6$.

This example is meant to illustrate a famous but often little-understood property of multiplication of signed numbers: *the product of two negative numbers is a positive number.*

Finally, a word about division of signed numbers. We saw (in the context of fractions) that division is the same as multiplication by the reciprocal. Keeping in mind that two numbers are reciprocals if their product is (positive!) 1, it should not be too hard to see that the reciprocal of a negative number is also negative, and the reciprocal of a positive number is also positive. So for the purpose of signed numbers, division will follow exactly the same rules as multiplication.

This brief discussion leads to the following rules for multiplying signed numbers:

Rules for multiplying and dividing signed numbers

- The product (or quotient) of two numbers with the *same sign* is positive.
- The product (or quotient) of two numbers with *different signs* is negative.

In both case, the magnitude of the product (or quotient) is just the product (or quotient) of the magnitudes of the two numbers.

Notice how nice these rules are compared to the rules for adding signed numbers! We never have to worry about which number has larger magnitude, and we always perform exactly the operation indicated in the problem.

Let's look at a couple of examples. We'll take the opportunity to review operations with decimals also.

Example 2.4.1. *Multiply:* $(-0.004) \times (-2.68)$.

Answer. *Putting aside that the two numbers are written in decimal form, we have the product of two numbers with the same sign—they are both negative. The answer will be positive.*

We will calculate the magnitude separately, by multiplying $(0.004) \times (2.68)$. *Recall that keeping track of the fact that there are a total of five decimal places to the right of the units place (three from the first number, two from the second), we will first multiply* $4 \times 268 = 1072$. *Now we will ensure that the final answer shows five decimal places:* 0.01072.

The answer is $+0.01072$.

Example 2.4.2. *Divide:* $(15.3) \div (-0.03)$.

We are dividing two numbers with different signs. The result will be negative.

We will divide the magnitudes $(15.3) \div (0.03)$. Recall that we will do this as a long division problem, making sure that the divisor (the *second* number in the divison problem) is a whole number. The divisor here has two decimal places; we move the decimal place for both numbers two places to the right (which amounts to multiplying both numbers by 100): $1530 \div 3 = 510$.

In conclusion, $(15.3) \div (-0.03) = -510$.

The answer is -5100.

2.4.1 Exercises

Perform the indicated operations.

1. $(3)(-12)$
2. $(-1)(-25)$
3. $6 \div (-2)$
4. $(-3.2) \div (-0.04)$
5. $\frac{6}{7} \cdot \left(-\frac{2}{3}\right)$
6. $\left(-1\frac{3}{4}\right) \div \left(-1\frac{1}{3}\right)$

2.5 Operations with signed numbers: Exponents and square roots

Whole number exponents indicate repeated multiplication, in the same way that whole number multiplication represent repeated addition.

The *base* of an exponential expression is the quantity immediately to the left of the exponent.

Example 2.5.1. *The base of* $(-4)^{10}$ *is* -4*. (The parentheses immediately to the left of the exponent tells us that the number inside the parentheses is the base.)*

Example 2.5.2. *The base of* -2^5 *is* 2*. Unlike the previous example, the quantity immediately to the left of the exponent is* 2*.*

Since we will only be concerned (for now!) with whole number exponents, the rules for exponentials with negative base follow immediately from the rules for multiplying signed numbers. In particular, you should notice the following fact:

The sign of exponentials with a negative base

A negative base raised to an even exponent is positive. A negative base raised to an odd exponent is negative

We will also from time to time encounter square roots, written with the symbol $\sqrt{\ }$

.

Remember that the square root of a number is the (non-negative) number which, when raised to the second power ("squared"), gives the original number. So, for example, $\sqrt{16} = 4$ since $(4)^2 = 16$.

For now, we won't go into the different issues that arise from the operation of taking square roots. In fact, for now, we will only consider square roots of *perfect squares*: whole numbers which are the second power of another whole number. The first few perfect squares are

$$1, 4, 9, 16, 25, \ldots$$

If you haven't worked with perfect squares for a while, you should take a moment to make a list of the first 12 or 15 of them (by squaring the numbers from 1 to 15, for example).

There is one important fact that we cannot ignore when we are talking about negative numbers and square roots in the same section. We have already seen that any number, positive or negative, when squared, will result in a positive number. After all, "squaring" a number is raising the number to an even exponent (of 2). For that reason, no real number[1] can be the square root of a negative number. Said differently, **the square root of a negative number cannot be a real number.**

Warning: $-\sqrt{9}$ does not mean the same as $\sqrt{-9}$. $-\sqrt{9}$ means "the opposite of the square root of 9," which is "the opposite of 3," or -3. $\sqrt{-9}$ is the square root of -9, which is not a real number, as we just saw.

2.5.1 Exercises

Perform the indicated operations.

1. $(-3)^4$
2. $(-2)^3$
3. $-(-2)^5$
4. $-\sqrt{81}$
5. $\sqrt{-36}$

2.6 Chapter summary

- Every nonzero number has two "parts:" a magnitude and a sign.
- The number 0 is neither positive nor negative, and has magnitude zero.

[1]For our purposes, the real numbers are all those that can be represented as points on the number line in the manner we have described above. In particular, they can be ordered, and there are points "infinitesimally close" to any other point on the number line. Later, we will see "numbers" that are not real numbers.

- Negative numbers are represented on a number line to the left of 0, while positive numbers are represented to the right of 0. In both cases, the magnitude of the number is represented by the distance from the point representing it to the origin.

- The rules for adding two signed numbers depend on whether the numbers have the same sign or different signs. If the numbers have the same sign, the magnitudes are added and the sign is the same as that of the two numbers. If the numbers have different signs, the magnitudes are subtracted and the sign is the sign of the number with the larger magnitude.

- The rules for multiplying or dividing two signed numbers depend on whether the numbers have the same sign or different signs. If the numbers have the same sign, the result is positive. If the numbers have different signs, the result is negative. The magnitude of the product (or quotient) is the product (or quotient) of the magnitudes.

- A negative base raised to an even power will be positive. A negative base raised to an odd power will be negative.

- The square root of a negative number cannot be a real number.

Chapter 3

Introduction to algebra

Vocabulary

- The order of operations
- Parentheses (and other grouping symbols)
- Variables
- Constants
- Algebraic expression
- Evaluate
- Function notation

3.1 The order of operations

In the previous chapters, we reviewed the basic arithmetic operations in the context of fractions and signed numbers. In a typical problem that we are going to encounter, however, there will be more than just one operation to perform. For that reason, the order in which we perform the operations is important. Usually, if we do operations in a different order, we will obtain different answers. For example, consider the expression with two operations (both division):

$$8 \div 4 \div 2.$$

If we perform the first division first, we obtain $2 \div 2 = 1$. If we perform the second division first, we obtain $8 \div 2 = 4$. The answers are different. Which is correct?

The order of operations is largely a matter of convention that has developed in history, and in fact more a question of typography than of mathematics. That being said, it is for the most part agreed upon as standard. We describe the order of operations here as a rule.

The order of operations

When an expression involves more than one operation, the operations are performed according to the following order:

1. **Operations inside grouping symbols**, from inside to outside. Parentheses are the most common grouping symbols, but there are many others.
2. **Exponents and roots**. When more than one exponent or root occur in the same expression, it is standard to evaluate them from left to right.
3. **Multiplication and division**. When there are more than one multiplication or division in the same expression, they must be performed in order from left to right.
4. **Addition and subtraction**. When there are more than one addition or subtraction in the same expression, they must be performed in order from left to right.

Referring to the example at the beginning of the section, the expression $8 \div 4 \div 2$ involves two division operations. Since both are at the same "level" in the order of operations, the divisions should be performed left-to-right:

$$8 \div 4 \div 2 = 2 \div 2 = 1.$$

If we had wanted to write an expression where the second division was performed first, we would need to use grouping symbols: $8 \div (4 \div 2)$.

It is worth noting at this point that both the associative properties of addition and of multiplication, as well as the distributive property of multiplication over addition, are all properties that concern the order of operations. For example, the associative property of addition says that repeated addition actually does not need to be performed from left to right, but can be performed in any order at all. The same holds, of course, for multiplicattion. This is *not* true for subtraction or division, however! Neither subtraction nor division is commutative or associative.

Important note: The unfortunate slogan "PEMDAS," sometimes taught to help memorize the order of operations, might imply that the first operation performed is "parentheses." This short-hand terminology is misleading, first and foremost because *parentheses do not represent an operation.* In particular, we draw special attention to a fact that often causes confusion:

> **Parentheses never mean multiplication.**
>
> Parentheses are grouping symbols, indicating that what is inside the parentheses should be considered as a single expression.
>
> **However, when no operation symbol is indicated between two expressions, the assumed operation will be multiplication.**

The expression 2(3) involves multiplication ("two times three") for the same reason that that we will see that $2x$ involves multiplication ("two times x"). In both cases, the multiplication is indicated by the fact that there is no symbol between the two factors (of 2 and 3 in the first case, 2 and x in the second.) Of course, 23 is simply the number twenty-three, so the parentheses in the expression 2(3) serve to separate the three, as its own "group," from the two.

Keep in mind also that parentheses are not the only grouping symbols. In addition to various shaped parentheses, like brackets $[,]$ and braces $\{,\}$ some commonly encountered grouping symbols include the bar used to write a fraction $\frac{\cdots}{\cdots}$ (where the numerator and the denominator are both considered as two separate groups) and the bar over an expression in a radical symbol $\sqrt{\cdots}$ (where everything "inside" the radical sign under the bar is considered as one group.)

The following examples illustrate the order of operations, especially involving grouping symbols.

Example 3.1.1. *Perform the indicated operations:* $3 - 2(-4 + 11)$.

Answer. *There are are three operations: An addition, a subtraction and a multiplication. Since the addition is grouped with parentheses, it will be performed first. Of the remaining two, multiplication takes priority over subtraction. Hence we will perform the operations as indicated below:*

$$3 \overset{\textcircled{3}}{-} 2 \overset{\textcircled{2}}{\cdot} (-4 \overset{\textcircled{1}}{+} 11).$$

$$\begin{aligned} &3 - 2(-4 + 11) \\ 3 - 2(7) &\quad \textcircled{1} \\ 3 - 14 &\quad \textcircled{2} \\ 3 + (-14) &\quad \textit{changing subtraction to "adding the opposite"} \\ -11 &\quad \textcircled{3} \end{aligned}$$

The answer is -11.

Example 3.1.2. *Perform the indicated operations:* $\dfrac{(-2) - (-6)}{1 - 5}$.

Answer. *There are three operations: two subtractions (one each in the numerator and denominator) and one division (indicated by the fraction bar). Since the fraction groups the numerator and the denominator separately, we perform the subtractions first, followed by the division, as indicated here:*

$$\frac{(-2) \overset{\textcircled{1}}{-} (-6)}{1 \underset{\textcircled{2}}{-} 5} \textcircled{3}.$$

$$\begin{aligned} &\frac{(-2)-(-6)}{1-5} \\ &\frac{(-2)+6}{1+(-5)} && \text{Changing both subtractions to "adding the opposite"} \\ &\frac{4}{1+(-5)} && \textcircled{1} \\ &\frac{4}{-4} && \textcircled{2} \\ &-1 && \textcircled{3} \end{aligned}$$

The answer is -1.

Example 3.1.3. *Perform the indicated operations:* $\sqrt{(-5)^2 - 4(1)(-6)}$.

Answer. *This time there are five operations: one exponent, two multiplications, a subtraction and a square root. Since the square root symbol groups everything inside, we perform those (four) operations first, and the square root last. Within the group, the "usual" order of operations apply: first, the exponent, followed by the two multiplications, followed by the subtraction. The order is indicated here:*

$$\textcircled{5}\sqrt{(-5)^{\overset{\textcircled{1}}{2}} \overset{\textcircled{4}}{-} 4 \overset{\textcircled{2}}{\cdot} (1) \overset{\textcircled{3}}{\cdot} (-6)}.$$

$$\begin{aligned} &\sqrt{(-5)^2 - 4(1)(-6)} \\ &\sqrt{25 - 4(1)(-6)} && \textcircled{1} \\ &\sqrt{25 - 4(-6)} && \textcircled{2} \\ &\sqrt{25 - (-24)} && \textcircled{3} \\ &\sqrt{25 + 24} && \text{Changing subtraction to "adding the opposite"} \\ &\sqrt{49} && \textcircled{4} \\ &7 && \textcircled{5} \end{aligned}$$

The answer is 7.

We close with the reminder that the commutative and associative properties give great flexibility with the order of operations involving addition and multiplication. For example, consider the expression

$$22 - 75 + (-18) - 52 - (-16) + 48 + (-12).$$

According to the order of operations, the six operations would be performed from left to right. However, changing all subtractions to "adding the opposite" and using the commutative property (to change the order) and the associative property (to re-group positive and negative terms together), it is much easier to perform the operations as

$$(22 + 16 + 48) + \left[(-75) + (-18) + (-52) + (-12)\right].$$

3.1.1 Exercises

For each of the problems below: (1) Count the number of operations; (2) list the operations in order; and (3) perform the operations.

1. $3 - 5(4 - 1)$
2. $\sqrt{(-3)^2 + (-4)^2}$
3. $\dfrac{(-3) - (-1)}{-2 - (-1)}$
4. $-(-3)^2 + (2)(1)$
5. $3\left(-\dfrac{1}{4}\right)^2 - 2\left(-\dfrac{1}{4}\right) + 1$
6. $\sqrt{(-1)^2 - 4(2)(-3)}$
7. $\left(\dfrac{1}{2}\right)\left(\dfrac{2}{3}\right) - \left(\dfrac{1}{4}\right)(-2)$
8. $\dfrac{-(-1) + \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$

3.2 Algebraic expressions

All numbers are symbols. The number "5" is a symbol that indicates quantity, answering the question "how many." More abstractly, the number "5" represents the quality that all collections of five objects have in common.

There are times when it is convenient to introduce other symbols that represent numbers. For example, you might read in an astronomy book that light travels at the speed of nearly $300,000$ kilometers per second. This is actually a rule that tells you that if you know how long a light ray has been traveling, then you actually also know how far it has traveled. In 1 second, the light ray has traveled $300,000$ kilometers (more than half the distance from the earth to the moon). In 2 seconds, a light ray will travel $300,000 \times 2 = 600,000$ kilometers. In 750 seconds (12.5 minutes) , a light ray travels $300,000 \times 750 = 225,000,000$ kilometers, which is the average distance from the earth to the planet Mars.

You might summarize this rule as follows: If t represents the number of seconds that a light ray travels, then the distance it will travel is $300,000 \times t$, or simply $300000t$. (Remember the convention: When no operation is indicated, there is an assumed multiplication!)

For our purposes, we will call a *variable* any symbol (we will always use letters) which is meant to represent a number whose value is not specified. A variable generally indicates that the value of the number is either unknown or, as in the example above, changing with time. Because of this, we will sometimes call numbers *constants* to distinguish them from variables.

From this point of view, algebra will be the study of expressions formed by combining both numbers and variables by using the standard operations of addition, subtraction, multiplication, division, (numerical) exponents, and roots.[1]

The main feature which distinguishes algebra from arithmetic, then, is the use of variables.

Convention: By far the most popular symbol to represent an unknown quantity[2] is the letter x. Because of this, we will avoid using the symbol $\times$ to represent multiplication from now on.

3.3 Evaluating algebraic expressions

Given an algebraic expression, there is not much that we can do with it apart from identifying the variables and the operations involved in the expression.

Example 3.3.1. *Consider the algebraic expression* $3x^2-5x+4$. *This expression involves one variable* x, *and five operations: an exponent, two multiplications, a subtraction, and an addition. (Locate them!)*

[1]There are other operations, like logarithms for example, which from this point of view are technically not algebraic, even though they are often treated in algebra courses.

[2]It is in this spirit that Malcolm X undoubtedly chose his name, to indicate that his true family name was unknown as a legacy of slavery.

If, on the other hand we are given values for *all* variables appearing in a given algebraic expression, we can evaluate[3] the expression for those given values. We do this by *substituting* the given values at every instance of the variable.

In the context of specifying values for a variable, we will use a short-hand notation with the equal sign "=". For example, we will say, "Evaluate an expression when $x = 1$." This means, "Evaluate the expression when x has the value 1." The use of the equal sign in this context is unfortunate, because its meaning is very different from the way we will use it for the rest of the book. However, like many unfortunate things, it is standard, and we will use it in this section.

Example 3.3.2. *Evaluate* $3x^2 - 5x + 4$ *when* $x = -2$.

Answer. *The variable x appears in the expression twice. We will substitute the value -2 in both instances. Then we will proceed according to the order of operations. We can indicate this order schematically as follows:*

$$3 \overset{\textcircled{2}}{\cdot} x^{\overset{\textcircled{1}}{2}} \overset{\textcircled{4}}{-} 5 \overset{\textcircled{3}}{\cdot} x \overset{\textcircled{5}}{+} 4.$$

In other words, the exponential is evaluated first, then the first multiplication, etc.

$$\begin{aligned}
3(-2)^2 - 5(-2) + 4 &\quad \textit{Substituting } -2 \textit{ for } x \\
3(4) - 5(-2) + 4 &\quad \textcircled{1} \\
12 - 5(-2) + 4 &\quad \textcircled{2} \\
12 - (-10) + 4 &\quad \textcircled{3} \\
12 + 10 + 4 &\quad \textit{Changing subtraction to ``adding the opposite''} \\
22 + 4 &\quad \textcircled{4} \\
26. &\quad \textcircled{5}
\end{aligned}$$

Notice the use of parentheses when we substitute a value for the variable. We can think of the variable as a placeholder, for which we insert the given value. In other words, we can think of the expression as being $3(\cdots)^2 - 5(\cdots) + 4$, and we will substitute the given value into the parentheses.

Example 3.3.3. *Evaluate* $\dfrac{y_2 - y_1}{x_2 - x_1}$ *when* $x_1 = 4$, $x_2 = -6$, $y_1 = -3$ *and* $y_2 = -18$.

Answer. *Notice that this algebraic expression has four variables, each of which appear once. (Be careful! These variable have* subscripts*, which should not be*

[3] The word *evaluate* means "find the value of"

mistaken for exponents or any other kind of operation. The subscripts belong with the symbol for the variable.)

There are three operations involved in this expression. We will first perform the subtraction on top, then the subtraction on bottom, and finally the division.

$$\frac{y_2 \overset{①}{-} y_1}{x_2 \underset{②}{-} x_1} ③.$$

$\dfrac{(-18)-(-3)}{(-6)-(4)}$ *Substituting*

$\dfrac{(-18)+(3)}{(-6)+(-4)}$ *Changing both subtractions to "adding the opposite"*

$\dfrac{-15}{(-6)+(-4)}$ ①

$\dfrac{-15}{-10}$ ②

$\dfrac{3}{2}$ ③.

The answer is 3/2. Notice that the final answer, as the quotient of two negative numbers, is positive. Also, in performing the division, we write the fraction in reduced form since the answer is not an integer.

3.3.1 Function notation

A function is a mathematical concept that is meant to express a relationship between two or more quantities. Students who will go on to study calculus will be expected to become familiar with these mathematical objects. The concept of a function is the main concept that is introduced in "precalculus" classes, and is central for any understanding of higher mathematics and many applications.

Very roughly, we can think of a function as a rule that takes one quantity and assigns to it another quantity. For our purposes, the quantities involved can be understood to be numbers.

Here, we are only going to discuss the most basic notation associated to functions. Functions will be given a "name," which we will denote with a letter, usually f but also g, h, etc. If the variable x represents a (numerical) quantity, the symbol $f(x)$ will represent the quantity that the function f assigns to the value x. *The notation $f(x)$ should be understood as a single symbol which represents a value.* In particular, *there is no multiplication implied in the notation $f(x)$.*

Many times, a function will be defined by means of an algebraic expression. For example, we might encounter a function described as

$$f(x) = 3x^2 - 5x + 4.$$

This just means that for any value of x, the function f assigns to x the value of $3x^2 - 5x + 4$ (evaluated with a given value for x). Notice the only variable appearing in the expression $3x^2 - 5x + 4$ is x. The notation $f(x)$ is meant to indicated that the function named f depends only of the value of the variable x.

When a function is defined algebraically using function notation, $f(x)$ can also be thought of as the "value" of f for the variable x. Since x is presumed unknown, the "value" of f is also unknown, but it is given in terms of x by the given algebraic expression. If the value of x is known, the value of f can be also be found by substituting the given value of the variable into the algebraic expression defining f. The notation $f(1)$, for example, means "the value of f when $x = 1$." Likewise, $f(-3)$ means "the value of f when $x = -3$.

Example 3.3.4. *Find $f(-1)$ if $f(x) = x^2 - 3x + 1$.*

Answer. *In this example, $f(x)$ is represented by the expression $x^2 - 3x + 1$. So $f(-1)$ is simply the value of $x^2 - 3x + 1$ when $x = -1$. We proceed exactly like we did in the previous section.*

$$\begin{aligned}
f(-1) &= (-1)^2 - 3(-1) + 1 && \textit{Substituting} \\
&= (1) - 3(-1) + 1 && \textit{Exponent first} \\
&= 1 - (-3) + 1 && \textit{Multiplication second} \\
&= 1 + 3 + 1 && \textit{Changing subtraction to "adding the opposite"} \\
&= 4 + 1 && \textit{The first addition (formerly subtraction)} \\
&= 5. && \textit{The second addition}
\end{aligned}$$

The answer is 5.

3.3.2 Exercises

1. Evaluate $-3x^2 + 7x - 5$ when $x = -5$.

2. Evaluate $\sqrt{x^2 + y^2}$ when $x = -5$ and $y = 12$.

3. Evaluate $b^2 - 4ac$ when $a = 1$, $b = -5$, $c = 6$.

4. Evaluate $\dfrac{I}{RT}$ when $I = 150$, $R = 0.04$, $T = 2$.

5. Evaluate $2y^2 - 9y - 1$ when $y = -\dfrac{2}{3}$.

6. Evaluate $\dfrac{y_2 - y_1}{x_2 - x_1}$ when
   (a) $x_1 = 0$, $y_1 = -4$, $x_2 = 2$, $y_2 = 0$
   (b) $x_1 = -2$, $y_1 = 4$, $x_2 = 4$, $y_2 = -8$
   (c) $x_1 = 1$, $y_1 = -1$, $x_2 = 4$, $y_2 = 5$
   (d) $x_1 = -1/3$, $y_1 = 0$, $x_2 = 0$, $y_2 = -5$
   (e) $x_1 = 3$, $y_1 = 5$, $x_2 = -2$, $y_2 = -8$

7. Evaluate $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ when:
   (a) $a = 1$, $b = -1$, $c = -6$
   (b) $a = 2$, $b = -7$, $c = 3$
   (c) $a = 1$, $b = 2$, $c = -8$
   (d) $a = 1/2$, $b = -1/3$, $c = -1/6$
   (e) $a = 4$, $b = 16$, $c = 15$

8. Evaluate $\dfrac{9}{5}C + 32$ when:
   (a) $C = -40$
   (b) $C = -15$
   (c) $C = 10$
   (d) $C = 25$
   (e) $C = 32$

9. For the function $f(x) = x^2 - 4x + 3$, evaluate
   (a) $f(-3)$
   (b) $f(1/2)$
   (c) $f(2)$

10. For the function $f(x) = 3x^3 + 2x^2 - 5x - 12$, evaluate
   (a) $f(0)$
   (b) $f(-1)$
   (c) $f(2)$

3.4 Translating algebraic expressions

Sometimes, we want to translate between an algebraic expression, represented symbolically with variables, numbers, and operations, and a verbal expression with the same meaning. This is most useful when it comes to so-called "word problems" involving applications of algebra.

The first thing to identify in translating an algebraic expression are the variable or unknown quantities, which will be represented with a letter. When there are more than one unknown quantity, or when an unknown quantity occurs more than once in the expression, it is important to distinguish between using the same variable or different variables.

Example 3.4.1. *Translate the following expression into words:* $3x^2 + 2x + 4$.

Answer. *Notice that the expression involves only one variable, which we will translate as "an unknown number." Also notice that the expression involves five operations.*

Here is one possible translation:

"Three times the square of an unknown number, increased by twice the same number, increased again by four."

Can you identify the operations in the sentence above?

Subtraction sometimes causes confusion in translation, because of the importance that the order plays. For example, if someone ask you, "What is four less than ten?", or tells you to "subtract four from ten," you will perform the operation $10 - 4$. The same operation, however, could be expressed by, "Ten decreased by four."

Example 3.4.2. *Translate into an algebraic expression: Five less than three times a number.*

Answer. *We right away identify the unknown quantity, in this case "a number," and represent it by the variable x.*

The phrase "less than" is one of the phrases for subtraction that reverses the normal left-to-right order. Hence, one translation would be:

$$3x - 5.$$

Another issue to be aware of in translating is the presence of implied grouping.

Example 3.4.3. *Translate: "Twice the sum of a number and five, decreased by the difference of three times another number and two."*

Answer. *Notice right away that there are two unknown quantities, "a number" and "another number." We will call them s and t.*

This sentence is more complicated than the others because there is implied grouping. You might see this by noticing that the phrase has the following structure:

"Twice SOMETHING decreased by SOMETHING ELSE."

The SOMETHING and the SOMETHING ELSE are groups, each involving alegbraic expressions themsleves. However, right away, we can expect the answer to have the form

$$2(\qquad) - (\qquad).$$

What is the SOMETHING? "The sum of a number and five," translated as $s+5$*.*

What is the SOMETHING ELSE? "The difference of three times another number and two," translated as $3t-2$*.*

So one translation for the expression would be:

$$2(s+5) - (3t-2).$$

Let's summarize a few common issues to watch out for in translating between algebra and words:

- Make sure to represent the same unknown quantity with the same variable name, no matter how many times it appears in the expression, while different unknown quantities should be represented by different variables.
- Subtraction has several verbalizations that reverse the usual left-to-right order. "Subtract something from something else," or "Something less than something else," both reverse the usual left-to-right order, while "Something decreased by something else," or, "The difference of something and something else," maintain the usual left-to-right order.
- Grouped operations are often implied, for example, by phrases like "the quantity of." But they are also expressed in phrases like, "Three times the difference..."

3.4.1 Exercises

Translate the following phrases into algebraic expressions:

1. Twice the sum of an unknown quantity and 8.
2. Seven less than half of a number.
3. One-fourth of the difference of some number and 12.

Translate the following algebraic expressions into words:

4. $7y - 30$
5. $3(x+2)$
6. $(x-5)^2 + 3x$
7. $\dfrac{2x-1}{3}$
8. $\dfrac{7x-3}{x-3}$

3.5 Chapter summary

- In expressions involving several operations, the operations are performed according to the order of operations.
- Parentheses and other grouping symbols indicate the expressions inside the symbols are to be treated as one group.
- Parentheses and other grouping symbols prioritize the operations within groups.
- Algebraic expressions consist of variables and numbers combined using the operations of addition, subtraction, multiplication, division, (numerical) exponents and roots.
- Algebraic expressions can be evaluated if values are given for all of the variables involved. In this case, the values are substituted for the variables and the operations are performed according to the order of operations.
- It is often necessary to translate back and forth between algebraic expressions and their English-language translation.

Part II

Linear equations and inequalities

Chapter 4

Linear equations and inequalities in one variable

Vocabulary

- Algebraic statement
- Equation
- Inequality
- Strict inequality
- Conditional statement
- Solution
- Ordered pair
- Solve
- Linear equation
- Coefficient
- Equivalent equations
- Addition principle
- Multiplication principle
- Like terms
- Coefficient
- Identity
- Contradiction
- Literal equation

4.1 Algebraic statements and solutions

In the last chapter, we considered algebraic expressions: expressions formed by combining numbers and variables with the operations of addition, subtraction, multiplication, division, exponents and roots. However, the only thing that we did with them was to evaluate them for given values of the variables involved. As soon as we took one step into the world of algebra, we quickly went right back into the world of arithmetic, evaluating expressions by performing operations with numbers.

In this chapter, we will begin see some important differences between the world of algebra and that of arithmetic.

A *mathematical statement* is a comparison of two expressions by means of the relations of equality (denoted with the symbol =), greater than (with the symbol $>$), less than (with the symbol $<$), and the compound relations "greater than or equal to" (denoted with the symbol $\geq$) and "less than or equal to" (with the symbol $\leq$). They can be broadly categorized as *equations* (those involving equality) and *inequalities* (all the rest). The basic inequalities $<$ and $>$ are called *strict* inequalities, while the compound inequalities $\leq$ and $\geq$ are sometimes called non-strict inequalities.

The first thing to notice about mathematical statements, unlike mathematical expressions, is that *a statement may be true or false.* For example, $1+2=3$ is an example of a *true* equation, while $1+1=5$ is an example of a *false* equation. Both examples are equations, but one is true and one is false.

In arithmetic, whether a statement is true or false can be completely decided by performing all operations, then comparing the resulting numbers. In algebra, however, this is not usually true.

An algebraic statement is a mathematical statement involving algebraic expressions. A typical algebraic equation or inequality may be true or false, depending on the values of the variables involved. Such a statement is called *conditional*. By contrast, there are no conditional statements in arithmetic: every statement is simply true or false, since there are no unknown quantities.

Example 4.1.1. *Consider the algebraic equation* $x+1=4$.

- *The equation* $x+1=4$ *is true when the value of* x *is* 3.
- *The equation* $x+1=4$ *is false when the value of* x *is* -2.

These two statements show that the equation $x+1=4$ *is a conditional equation, since whether it is true or false depends on the value of the variable* x.

The fact that the typical algebraic statement is conditional prompts the following definition. We will come back to this definition over and over again throughout the text.

A solution of an algebraic statement

A *solution* of an algebraic equation or inequality is a value for each of the variables which, when substituted into the statement, make the statement true.

Using this language, we can reformulate the results of Example 4.1.1: 3 is a solution of the equation $x+1=4$, but -2 is not a solution of $x+1=4$.

Example 4.1.2. *Determine whether* -2 *is a solution of the equation*

$$x^2+5x+6=0.$$

Answer. *We will substitute* -2 *for* x*:*

$$\begin{aligned}(-2)^2 + 5(-2) + 6 &= 0\\ 4 + 5(-2) + 6 &= 0\\ 4 + (-10) + 6 &= 0\\ -6 + 6 &= 0\\ 0 &= 0.\end{aligned}$$

The equation is true when x *is* -2*. So* -2 *is a solution of the equation* $x^2 + 5x + 6 = 0$*.*

Example 4.1.3. *Determine whether* -3 *is a solution of the equation*

$$x^2 + 5x + 6 = 0.$$

Answer. *We will substitute* -3 *for* x*:*

$$\begin{aligned}(-3)^2 + 5(-3) + 6 &= 0\\ 9 + 5(-3) + 6 &= 0\\ 9 + (-15) + 6 &= 0\\ -6 + 6 &= 0\\ 0 &= 0.\end{aligned}$$

The equation is true when x *is* -3*. So* -3 *is a solution of the equation* $x^2 + 5x + 6 = 0$*.*

Notice that in the previous two examples, we found two *different* solutions to the same equation.

In the case of statements with more than one variable, we need to specify a value for *each* variable in order to specify a solution. Many times, we will encounter equations or inequalities in two variables x and y.

Convention: When specifying a solution of an equation or inequality in two variables x and y, we will use ordered pair notation. An *ordered pair* is a pair of two numbers grouped with parentheses. For example, a typical ordered pair might be $(4, 7)$. In this case, the first number will always represent a value for x, while the second value will always represent a value for y. A full treatment of two-variable statements is found in the next chapter. However, we can already get a feeling for them in the context of deciding whether a given ordered pair is a solution or not.

Example 4.1.4. *Determine whether* $(1, -2)$ *is a solution of the equation*

$$3x + 4y = 11.$$

Answer. *The ordered pair* $(1, -2)$ *indicates that the value of* x *is* 1 *and the value of* y *is* -2*. Substituting:*

$$\begin{aligned}3(1) + 4(-2) &= 11\\ 3 + (-8) &= 11\\ -5 &= 11.\end{aligned}$$

The equation is false. So $(1,-2)$ *is NOT a solution of the equation*

$$3x + 4y = 11.$$

Example 4.1.5. *Determine whether* $(5,-1)$ *is a solution of the equation* $3x + 4y = 11$.

Answer. *The ordered pair* $(5,-1)$ *represents the case that the value of* x *is* 5 *and the value of* y *is* -1. *Substituting:*

$$\begin{aligned} 3(5) + 4(-1) &= 11 \\ 15 + (-4) &= 11 \\ 11 &= 11. \end{aligned}$$

The equation is true. So $(5,-1)$ *IS a solution of the equation* $3x + 4y = 11$.

Notice that in the previous two examples (with the same equation), we have found ONE solution of the equation, namely $(5,-1)$. The one solution is made up of two numbers—the values for x and for y both need to be specified for a single solution.

Example 4.1.6. *Determine whether* $-1/2$ *is a solution of the inequality*

$$5x + 1 \geq -1.$$

Answer. *The inequality has only one variable, so a solution is simply a number. We substitute the given candidate:*

$$\begin{aligned} 5\left(-\frac{1}{2}\right) + 1 &\geq -1 \\ \left(\frac{-5}{2}\right) + 1 &\geq -1 \\ \left(\frac{-5}{2}\right) + \frac{2}{2} &\geq -1 \\ -\frac{3}{2} &\geq -1. \end{aligned}$$

The inequality is false. So $-1/2$ *is NOT a solution of the inequality* $5x+1 \geq -1$.

We close this section with one of the most important definitions in this text:

Solving an algebraic statement

To **solve** an algebraic equation or inequality means to **find all solutions**.

Every word in this definition is important:

1. **Solutions**: To solve an equation or inequality, we need to keep in mind what a typical solution looks like. Is it a number, as is the case for statements with one variable? Is it an ordered pair, as is the case for statements with two variables?

2. **All**: To solve an equation or inequality, we need to know *how many solutions* the statement has. Does it have one solution? Two? Infinitely many? None?

3. **Find**: Rather than guess solutions and then check whether in fact they are solutions or not, we would like to have a procedure that produces (or "finds") the solutions for us.

4.1.1 Exercises

1. Is -2 a solution of $4x + 3 = -5$?

2. Is 0 a solution of $3(x + 2) - 2 = x - 4(x - 1)$?

3. Is 11/3 a solution of $5(x - 2) = 2x + 1$?

4. Is 3 a solution of $3x - 2 = -x - 4$?

5. Is $-1/4$ a solution of $x + 2 = 3x - 2(x - 1)$?

6. Is -1 a solution of $3x + 2 < -2$?

7. Is 1/3 a solution of $3x + 2 < -2$?

8. Is 3 a solution of $2x^2 - 5x = 3$?

9. Is $-1/2$ a solution of $2x^2 - 5x = 3$?

10. Is -3 a solution of $2x^2 - 5x = 3$?

11. Is $-1/2$ a solution of $4x^3 + 12x^2 - x = 3$?

12. Is $(2, 3)$ a solution of $x - 3y = -7$?

13. Is $(-7, 0)$ a solution of $x - 3y = -7$?

14. Is $(1, 1)$ a solution of $x - 3y \geq -7$?

15. Is $(-4, 1)$ a solution of $x - 3y \geq -7$?

4.2 Solving linear equations in one variable

In this section, our goal will be to develop a method that will "find" all the solutions for certain equations in one variable.

Let's start off with a very, very simple example.

Example 4.2.1 (A very, very simple example)**.** *Solve:* $x = -2$.

Answer. *One solution of the equation is obvious:* -2. *Just to make sure, substituting,* $-2 = -2$ *is a true statement.*

On the other hand, it is just as obvious that any value of x *which is not* -2 *will NOT be a solution of the equation.*

In other words, the equation has only one solution.

The solution is -2.

If only it were so easy to find the solutions to every equation! In fact, our first goal of producing a method to find all solutions to a given equation will be to convert it to an *equivalent equation* that has the same easy form as our very, very easy example. We say that two equations are **equivalent** if they have exactly the same solutions.

In order to do this, let's first specify what made our very, very easy example so easy:

- One side of the equation is a number. All operations involving numbers have been performed.
- The other side of the equation is an algebraic expression containing just one variable and no operations. (One sometimes says: "x is by itself on one side of the equation.")

A consequence of the second item is that the equation is *linear*. **A linear equation is an equation where the only operations performed on a variable are addition, subtraction, and multiplication by a constant (called a coefficient).**[1] So, for example, the equation $3x + 4y = 11$ is a linear equation (with two variables). The equation $x^2 + 5x + 6 = 0$ is not linear, because the x^2 term involves a variable raised to a power different from 1.

The remainder of this section will be devoted to the following sentence:

Every linear equation (almost[2]) in one variable, let's say x, is equivalent to an easy equation of the form $x =$ ___. (The blank will be a number.)

There is an important consequence of this fact: **(Almost[2]) Every linear equation in one variable has exactly one solution.**

[1] We will see a more concise way of stating this definition in Chapter 5.

[2] See section 4.2.3 below.

4.2.1 The rules of the game

Given any linear equation in one variable x, no matter how complicated, we will develop a method to find an equivalent equation of the form $x = \_\_\_$. Then we will be able to read off the solution to the easy equation; it will also be the solution of the (maybe more complicated) original equation also.

The method will be based on the following two properties of equations:

- **The addition principle**: Adding (or subtracting) the same quantity to both sides of an equation will produce an equivalent equation.
- **The multiplication principle**: Multiplying (or dividing) the same *nonzero* quantity to both sides of an equation will produce an equivalent equation.

These properties can be summarized in the loose slogan: "Doing the same thing to both sides of an equation does not change the solutions of the equation." As usual with loose slogans, though, we should be aware of the fine print. For example, multiplying both sides of an equation (which may be false) by 0 will yield the equation $0 = 0$, which is always true!

4.2.2 Applying the rules: Solving linear equations

We will use the properties in the previous section to attempt to start with any linear equation in one variable (say x) and obtain an equivalent equation of the form

$$x = \_\_\_.$$

Since the equations have the same solution, this solution will be obvious by considering the simple equation.

Recall that one of the special features of the easy equation $x = \_\_\_$ is that the expression with the variable has no operations involved. **Our guiding strategy for solving linear equations will be to identify the operations involved in the expression involving the variable, and then to "undo" them, one by one, by using the addition and multiplication principles**.

Example 4.2.2. *Solve:* $x - 41 = 36$.

Answer. *Notice that the left-hand side of the equation (which involves the variable x) involves just one operation: subtraction. To "undo" the operation of* subtracting 41*, we will do the opposite:* add 41 *to both sides:*

$$\begin{aligned} x - 41 + 41 &= 36 + 41 && \textit{Addition principle} \\ x + 0 &= 77 && \textit{Performing addition on both sides} \\ x &= 77. \end{aligned}$$

In other words, the equation $x - 41 = 36$ is equivalent to the (easy!) equation $x = 77$.

The solution is 77.

It is common to write the addition principle "vertically." The preceding example would be written:

$$\begin{array}{rcrcr} x & - & 41 & = & 36 \\ & + & 41 & \vdots & +41 \\ \hline x & & & = & 77. \end{array}$$

Notice the way that the terms are carefully lined up. The dots $\vdots$ under the equal sign are meant to remind us that we must add the same quantity on both sides.

Example 4.2.3. *Solve:* $5x + 18 = 12$.

Answer. *This time, the left hand side (involving the variable x) involves two operations: addition and multiplication. We will "undo" them in the order* opposite *the order of operations. First, we will undo the addition, then the multiplication.*

$$\begin{array}{rcrcr} 5x & + & 18 & = & 12 \\ & - & 18 & \vdots & -18 \\ \hline 5x & & & = & -6 \\ \frac{5x}{5} & & & = & \frac{-6}{5} \\ x & & & = & -\frac{6}{5}. \end{array}$$

The solution is $-6/5$.

Sometimes, we will encounter statements with variables on *both* sides of the equation or inequality. In this case, we will need to take an extra step to make sure that the equivalent equation has variables on only one side of the statement, like our easy equation. This can be done using the same addition principle that we have been using so far. But we need to make one thing that we have been using behind the scenes more explicit.

In a linear equation in one variable, **like terms** are identified according to whether they involve the variable or not. In other words, terms involving the variable will be like terms, and terms not involving the variable will be also be called like terms.

The most important feature of like terms for now is that **like terms can be added** (or "combined"). When we add like terms involving variables, we add the *coefficients* of the terms, but leave the variable part the same. For example, in the expression $5x + 2 + 3x + 9$, the terms $5x$ and $3x$ are like terms, and $5x + 3x = 8x$. Likewise, 2 and 9 are like terms, and $2 + 9 = 11$. So $5x + 2 + 3x + 9 = 8x + 11$.

When we use the addition principle, we will make a habit of writing like terms in the same column.

Example 4.2.4. *Solve:* $3x - 9 = 8x + 7$.

Answer (First method)**.** *This equation has variables on both sides of the equation. As mentioned, our first job will be to write an equivalent equation which only has a variable on one side of the equation. Which side? It doesn't really matter, as we will illustrate here.*

One popular line of thinking goes like this: Our easy equation looks like $x =$ ___*, so let's try to write the variable on the left hand side, just like our easy model.*

In that case, we need to eliminate the $8x$ *from the right-hand side. We do that by adding the opposite, which is* $-8x$*, to both sides.*

$$\begin{array}{ccccccc} 3x & - & 9 & = & 8x & + & 12 \\ -8x & & & \vdots & -8x & & \\ \hline -5x & - & 9 & = & & & 12 \\ & + & 9 & \vdots & & + & 9 \\ \hline -5x & & & = & & & 21 \\ \frac{-5x}{-5} & & & = & & & \frac{21}{-5} \\ x & & & = & & & -\frac{21}{5}. \end{array}$$

The solution is $-21/5$*.*

Answer (Second method)**.** *Let's see what would have happened if we had written the equivalent equation with the variable on the* right hand side*. In that case, we would like to eliminate the* $3x$ *term from the left-hand side.*

$$\begin{array}{ccccccc} 3x & - & 9 & = & 8x & + & 12 \\ -3x & & & \vdots & -3x & & \\ \hline & - & 9 & = & 5x & + & 12 \\ & - & 12 & \vdots & & - & 12 \\ \hline & & -21 & = & 5x & & \\ & & \frac{-21}{5} & = & \frac{5x}{5} & & \\ & & -\frac{21}{5} & = & x & & . \end{array}$$

Even though this equation is not exactly in the form $x =$ ___*, it's just as easy—the variable is "by itself" on one side of the equation.*

The solution is $-21/5$*.*

Looking at the two methods above, the second method has the advantage that the coefficient of the variable term is positive after using the addition principle to obtain an equation with the variable on one side only. That is because in the original equation, the variable term with the larger coefficient was on the right. From now on, we will follow the custom of writing our equivalent equation with the variable on the side where the original coefficient of the variable term was larger.

The only thing that can make a linear equation in one variable more complicated than the examples we have seen above is if there are more operations on one or both sides of the equation. For example, we may have to perform multiplication (using the distributive law if necessary), or we may have to combine like terms one one or both sides separately. The following example illustrates this situation. Notice that before we begin to apply the addition or multiplication principles ("doing the same thing to both sides") we will perform operations on *each side separately.*

Example 4.2.5. *Solve:* $4(x-3)+2x=7x-9$.

Answer. *Notice that on the left hand side, the* 4 *will distribute over both terms in the parentheses. In addition, there is more than one term on the left-hand side with a variable; we will have to combine like terms.*

$$\begin{aligned} 4(x-3)+2x &= 7x-9 \\ 4(x)+4(-3)+2x &= 7x-9 \\ 4x-12+2x &= 7x-9 \\ 6x-12 &= 7x-9. \end{aligned}$$

Now that both sides are simplified, we can apply the addition and multiplication principles as above. Notice that since there are variables on both sides of the equation, and the higher coefficient of x is on the right-hand side, we will write our equivalent equation with the variable on the right.

$$\begin{array}{cccccccc} 6x & - & 12 & = & 7x & - & 9 \\ -6x & & & \vdots & -6x & & \\ \hline & - & 12 & = & x & - & 9 \\ & + & 9 & \vdots & & + & 9 \\ \hline & & -3 & = & x & & . \end{array}$$

The solution is -3.

Summary: Solving linear equations in one variable x

1. Simplify each side of the equation separately:
   - Perform any multiplication, using the distributive property if necessary;
   - Combine like terms.
2. Use the addition property to form an equivalent equation with all variables on one side of the equation;
3. Use the addition property to form an equivalent equation with all constant terms on the side of the equation not containing the variable term;
4. Use the multiplication property to form an equivalent equation of the form $x = \_\_\_$.

4.2.3 Some unusual cases: Linear equations in one variable that do not have exactly one solution

We mentioned above that for *almost* every linear equation in one variable x, there is an equivalent equation of the form $x = \_\_\_$, and so *almost* every linear equation has exactly one solution.

In this section, we will illustrate what can go wrong.

Example 4.2.6. *Solve:* $2(x-3)+4 = 2x-2$.

Answer. *We will apply the general method as usual, simplifying first and then writing an equivalent equation with variables on the left-hand side.*

$$2(x-3)+4 = 2x-2$$
$$2x-6+4 = 2x-2$$

$$\begin{array}{rcrcrcr} 2x & - & 2 & = & 2x & - & 2 \\ -2x & & & \vdots & -2x & & \\ \hline & - & 2 & = & & - & 2. \end{array}$$

Wait! What happened to the variable? Since the coefficients of the variable were the same on both sides, eliminating from one side (using the addition principle) actually eliminated the variables from both sides.

Actually, that's not bad at all. The equivalent equation, $-2 = -2$*, is an arithmetic equation which is always true*—no matter the value for x. *(You may try substituting several different values for* x *into the original equation* $2(x-3)+4 = 2x-2$ *to confirm that they will all give a true statement.)*

All real numbers *are solutions for this equation. In particular, the equation has infinitely many solutions, not just one. An equation which is true for all values of the variables involved is called an* ***identity****.*

Here is another example where a similar problem occurs.

Example 4.2.7. *Solve:* $x + (2x - 9) = 3(x + 1)$.

Answer.

$$x + (2x - 9) = 3(x + 1)$$
$$x + 2x - 9 = 3x + 3$$

$$\begin{array}{rrrcrrr} 3x & - & 9 & = & 3x & + & 3 \\ -3x & & & \vdots & -3x & & \\ \hline & - & 9 & = & & & 3. \end{array}$$

Again, eliminating the variable from one side had the effect of eliminating the variable entirely from the equation. But this time, the original equation is equivalent to an equation which is false*—no matter the value of* x*.*

The equation has no solution*. An equation which has no solution is called a* ***contradiction****.*

We can summarize the results about solutions to linear equations in the following general fact (which is not *almost always* true, but is always true!):

For every linear equation in one variable, one and only one of the following statements must hold:

- The equation has exactly one solution; OR
- It is an identity, and every number is a solution; OR
- It is a contradiction, and it no solution.

This has what might be a startling conclusion: If a linear equation in one variable has two different solutions, then it must have infinitely many.

4.2.4 Another use of the multiplication principle: Equations involving fractions

In this section we illustrate how to use the multiplication principle in order to "clean up" equations involving fractions. While this is an extra step, compared to the four-step process for solving linear equations in two variables above, it does reduce the need to perform arithmetic with fractions that requires extra care and attention to detail.

The extra step that we are going to introduce when we attempt to solve an equation involving fractions is to *multiply both sides of the equation by a common denominator of all fractions appearing in the equation.* By the distributive property, this amounts to multiplying every term on both sides of the equation by the common denominator. Since by definition, every denominator will be a factor of the common denominator, the multiplication will have the effect of ensuring that every term will only involve integers.

Example 4.2.8. *Solve:* $\frac{2x}{3} + \frac{1}{4} = 4$.

Answer. *The least common denominator of the two fractions involved is* 12. *We will multiply both sides of the equation by* 12*:*

$$\begin{aligned} (12)\left(\frac{2x}{3} + \frac{1}{4}\right) &= (12)(4) \\ (12)\left(\frac{2x}{3}\right) + (12)\left(\frac{1}{4}\right) &= 48 \\ \left(\frac{12}{1}\right)\left(\frac{2x}{3}\right) + \left(\frac{12}{1}\right)\left(\frac{1}{4}\right) &= 48 \\ \frac{24x}{3} + \frac{12}{4} &= 48 \\ 8x + 3 &= 48. \end{aligned}$$

Notice that even though we aren't done yet, the new equivalent equation $8x + 3 = 48$ *is much simpler to work with than the original equation.*

Now that both sides are simplified, we can apply the addition and multiplication principles as above.

$$\begin{array}{cccccc} 8x & + & 3 & = & 48 \\ & - & 3 & \vdots & -3 \\ \hline 8x & & & = & 45 \\ \frac{8x}{8} & & & = & \frac{45}{8} \\ x & & & = & \frac{45}{8}. \end{array}$$

The solution is 45/8.

In the previous example, we wrote out all the steps involved in applying the distributive law after multiplying both sides by a common denominator. In the future, we will omit the step showing the multiplication of both sides, and apply the distributive law directly by *multiplying every term on both sides of the equation by a common denominator.*

Example 4.2.9. *Solve:* $\frac{x-2}{3} + \frac{1}{2} = \frac{x}{6}$.

Answer. *The least common denominator of all three fractions in* 6. *Pay careful attention to the fact that the first fraction on the left-hand side has a numerator involving a group with two terms.*

$$
\begin{aligned}
(6)\left(\frac{x-2}{3}\right)+(6)\left(\frac{1}{2}\right) &= (6)\left(\frac{1}{6}\right)\\
\frac{6(x-2)}{3}+\frac{6}{2} &= \frac{6}{6}\\
2(x-2)+3 &= 1.
\end{aligned}
$$

Now we proceed as usual:

$$
\begin{array}{cccccc}
2(x-2) & & + & 3 & = & 1\\
2x & -\ 4 & + & 3 & = & 1\\
2x & & - & 1 & = & 1\\
 & & + & 1 & \vdots & +1\\
\hline
2x & & & & = & 2\\
\frac{2x}{2} & & & & = & \frac{2}{2}\\
x & & & & = & 1.
\end{array}
$$

The solution is 1.

4.2.5 Some word problems

In this section we will apply the techniques of solving linear equations in one variable to some mathematical word problems. In doing so, we will pay special attention to the problem of translating between English and algebra. The technique outlined here emphasizes the translation aspect by using words (like "dictionary") which are essential ingredients in translating from one language to another.

For each word problem, we will follow the following four-step process:

A four-step strategy to approach word problems

1. **Create a "dictionary" for the problem**. The dictionary consists of a list of all unknown quantities, each expressed both in English and as an algebraic expression.

2. **Write an algebraic equation that represents the problem**. This involves using the dictionary from the previous step.

3. **Solve the equation**. This is just using the technique we have been discussing in this chapter. It is the one step where the techniques of algebra are used.

4. **Answer the question**. At this point, the solution of the equation (from the previous step), along with the dictionary, should allow us to answer the question asked in the original problem.

Let's see how this four-step technique works in a few examples.

Example 4.2.10. *The sum of three consecutive integers is* -237*. Find the integers.*

Answer. ***Step 1: Create a dictionary***. *Notice that there are three unknown quantities: the first integer, the second integer, and the third integer. So our dictionary will need three entries, one for each.*

We can always call one of the unknown quantities x*. Since our techniques so far have been with equations in one variable, we will try to write the other two unknown quantities in terms of* x*. For example, in this problem, we know the integers are* consecutive*. So if we call the first integer* x*, then the next two consecutive integers will be* $x+1$ *and* $x+2$*. We will write the dictionary as:*

Dictionary

First integer	x
Second integer	$x+1$
Third integer	$x+2$

Step 2: Write an equation. *Many times, looking for an equation in the original problem amounts to finding the word "is." In this problem, we see an equation in the sentence, "The sum of three integers IS* -237*." The word "sum" tell us that we will be adding—what? The three integers, which we translate using our dictionary. In other word, we can write:*

$$x + (x + 1) + (x + 2) = -237.$$

(Note that we have introduced parentheses in order to clearly see the three unknown quantities.)

Step 3: Solve the equation.

$$\begin{aligned} x + (x+1) + (x+2) &= -237 & \\ 3x + 3 &= -237 & \textit{Combining like terms} \\ 3x &= -240 & \textit{Adding } -3 \textit{ to both sides} \\ x &= -80. & \textit{Dividing both sides by } 3. \end{aligned}$$

The solution is -80.

Step 4: Answer the question. *Notice that solving the equation in Step 3 does not completely answer the question. In particular, the problem asked to find all three integers. The first integer, represented by* x, *is* -80. *The second, represented by* $x+1$, *is* $-80+1$, *or* -79. *The third integer, represented by* $x+2$, *is* $-80+2$, *or* -78.

The three integers are -80, -79, *and* -78.

Example 4.2.11. *The length of a rectangle is* 3 *more than twice the width. Find the dimensions of the rectangle if the perimeter is* 75 *inches. (Use the fact that the perimeter is given by the formula* $P = 2L + 2W$*.)*

Answer. ***Step 1: Create a dictionary***. *In this problem, we have two unknown quantities: the length and the width. As usual, we will call one of them* x, *and then try to write the other in terms of* x. *Notice that in this problem, one of the unknowns (the length) is expressed in terms of the other (the width): "The length is ... the width." In cases like this, it is much easier to call the unknown appearing second (in this case, the width) as* x. *That way, we can translate the expression for the first directly:* 3 *more than twice the width will be written as* $2x + 3$. *So the dictionary will be:*

Dictionary

Width	x
Length	$2x+3$

Step 2: Write an equation. *In this case, the equation will come from the formula for the perimeter, along with the dictionary:*

$$2(x) + 2(2x+3) = 75.$$

Step 3: Solve the equation.

$$\begin{aligned} 2(x) + 2(2x+3) &= 75 & \\ 2x + 4x + 6 &= 75 & \textit{Multiplying} \\ 6x + 6 &= 75 & \textit{Combining like terms} \\ 6x &= 69 & \textit{Adding } -6 \textit{ to both sides} \\ x &= \frac{69}{6} & \textit{Dividing both sides by } 6 \\ x &= \frac{23}{2} & \textit{Reducing to lowest terms} \end{aligned}$$

The solution is $23/2$.

Step 4: Answer the question. *The variable* x *represented the width of the rectangle, so the width is* $23/2 = 11.5$ *inches. The length was given by* $2x + 3$. *Substituting* $23/2$ *for* x, *we obtain* $2(23/2)+3 = 23+3 = 26$ *so the length is* 26.

4.2.6 Exercises

Solve each of the following linear equations.

1. $2x - 5 = -1$

2. $3(x - 2) = 4$

3. $2x + 1 = 7x + 6$

4. $3(2x - 1) - 3 = x - 5$

5. $3(x - 2) + 4(2x - 1) = 11x - 17$

6. $2(x - 1) + 3(2x - 3) = x + 4$

7. $2(x - 3) = 3(x - 2) - x$

8. $5(3x + 2) + 3(x - 7) = 6x - 11$

9. $\frac{2}{3}x - \frac{1}{2} = \frac{3}{4}$

10. $\frac{x-1}{4} - \frac{1}{2} = \frac{x}{6}$

11. $\frac{3}{4}x - 3 = \frac{1}{2}$

12. $\frac{2x-1}{3} - \frac{1}{2} = \frac{x}{6}$

For each of the following, set up an equation representing the problem. Then solve the equation to answer the question.

11. The sum of three consecutive integers is 93. Find the integers.

12. The sum of two consecutive odd integers is 264. Find the integers.

13. One number is three less than seven times another number. If the sum of the two numbers is 15, find the two numbers.

14. Joshua brings home $ 1,080 per week in net pay. If his deductions amount to 28 % of his gross pay, what is his weekly gross pay?

15. Suppose that for three consecutive odd integers, the sum of the first two and twice the third is 6. Find the integers.

16. One number is two less than three times a number. If the sum of the two numbers is 15, find the two numbers.

4.3 A detour: "Solving" literal equations

The method outlined above in Section 4.2 gives an effective procedure, or algorithm[3], to solve any linear equation in one variable. It turns out that this algorithm is also effective in a more general symbolic setting.

A *literal equation*, or *formula*, is an equation relating two or more variables. For example,

$$F = \frac{9}{5}C + 32, \qquad I = Prt, \qquad h^2 = a^2 + b^2, \qquad y = x^2$$

are four different literal equations. The first relates the two variables F and C (representing the temperature in degrees Fahrenheit and Celsius). The second relates the four variables I, P, r and t (representing interest, principal, rate and time, respectively). The third relates the three variable h, a, and b (representing the lengths of the hypotenuse and the two legs of a right triangle). The fourth relates the two variables x and y (representing the coordinates of a point on a particular parabola). Such relationships are common in the world, and formulas give a concise way of describing them.

A literal equation is *linear in a given variable* if the only operations involving that variable are addition, subtraction, and multiplication by a constant or another different variable. For example, the equation $I = Prt$ is linear in each of the four variables. The equation $y = x^2$ is linear in y, but not linear in x. The equation $h^2 = a^2 + b^2$ is not linear in any of the three variables.

You will notice that three of the four examples of literal equations are written so that one of the variables appears "by itself" on one side of the equation, with

[3]It is not coincidental that the English word *algorithm* derives from the Arabic *al-Khwarizmi*, a title referring to the 9th century mathematician Muhammad ibn Musa al-Khwarizmi. Al-Khwarizmi is best known for his textbook *Al-kitab al-mukhtasar fi hisab* ***al-gabr*** *wa'l-muqabala* ("The compendious book on calculation by completing and balancing"), from which the English word *algebra* derives.

no algebraic operations indicated. (In the equation $h^2 = a^2 + b^2$, the variable h is not "by itself" since it is being raised to the second power.) Many formulas are written in this format. It indicates that the value of the variable appearing "by itself" can be determined knowing the values for all the variables on the other side.

For example, given the "simple interest" formula $I = Prt$, suppose we are given that the principal amount is \$2,000 ($P = 2000$), the annual interest rate is 0.1% ($r = 0.001$) and the time invested is 3 years ($t = 3$). The formula implies that knowing this information, we should be able to determine the interest earned. Substituting the given values into the formula,

$$\begin{aligned} I &= (2000)(0.001)(3) \\ I &= 6. \end{aligned}$$

This is a linear equation whose only solution is 6. The interest earned on a principal amount of \$2,000 over three years at an annual interest rate of 0.1% is \$6.

You will notice in the example in the preceding paragraph that the mathematics involved in using a formula to determine the value of one variable when the others are given does not involve much algebra at all—at least when the variable to be determined is "by itself" on one side of the equation. All that remains, given values for the other variables, is evaluation.

However, for many reasons, it will often be convenient to rewrite a literal equation so that a given variable appears "by itself." When the equation is linear is the given variable, we can apply the method of Section 4.2 to "solve the equation in terms of the given variable." (Notice that this is an abuse of the meaning of the word "solve." In fact we are not solving the equation at all, which would involve finding values of all variables for which the equation is true. Nevertheless, the terminology is so common that it would be futile to avoid it.)

Example 4.3.1. *Solve for t: $I = Prt$.*

Answer. *The instructions, "Solve for t" mean, "Write an equivalent equation with the t by itself on one side of the equation." Notice that the equation is linear in t. The variable t is not "by itself," since it is multiplied by the variables P and r. Applying the multiplication principle, assuming that neither P nor r have the value 0:*

$$\begin{aligned} I &= Prt \\ \frac{I}{Pr} &= \frac{Prt}{Pr} \\ \frac{I}{Pr} &= t. \end{aligned}$$

Since the order of the equality does not matter, we can write it as

$$t = \frac{I}{Pr}.$$

The answer is $t = I/(Pr)$.

Example 4.3.2. *Solve for y: $3x + 4y = 12$.*

Answer. *The equation is linear in y. The variable y is not "by itself:" it is multiplied by 4 with $3x$ added to the result. We apply the algorithm of Section 4.2:*

$$\begin{array}{rcrcrl} 3x & + & 4y & = & & 12 \\ -3x & & & \vdots & -3x & \\ \hline & & 4y & = & -3x & +12 \\ & & \frac{4y}{4} & = & \frac{-3x+12}{4} & \\ & & y & = & \frac{-3x}{4} & +\frac{12}{4} \\ & & y & = & -\frac{3}{4}x & +3. \end{array}$$

The answer is $y = -\dfrac{3}{4}x + 3$.
A few things to notice about our use of the algorithm:

- *At the second-to-last step, dividing both sides of the equation by 4, we used the distributive property to divide each term on the right by 4.*
- *At the last step, we wrote the coefficient $-3/4$ of x more plainly. Notice that $-\dfrac{3}{4}x = -\dfrac{3}{4} \cdot \dfrac{x}{1} = -\dfrac{3x}{4}$. Also keep in mind that $\dfrac{-3}{4} = \dfrac{3}{-4} = -\dfrac{3}{4}$.*

Example 4.3.3. *Solve for y: $2x - 5y = 8$.*

Answer. *The equation is linear in y. Again we apply the algorithm of Section 4.2:*

$$\begin{array}{rcrcrl} 2x & - & 5y & = & & 8 \\ -2x & & & \vdots & -2x & \\ \hline & & -5y & = & -2x & +8 \\ & & \frac{-5y}{-5} & = & \frac{-2x+8}{-5} & \\ & & y & = & \frac{-2x}{-5} & +\frac{8}{-5} \\ & & y & = & \frac{2}{5}x & -\frac{8}{5}. \end{array}$$

The answer is $y = \dfrac{2}{5}x - \dfrac{8}{5}$.
Pay careful attention to the signs in studying this example!

4.3.1 Exercises

1. Solve for y: $3x - 2y = 6$.

2. Solve for y: $5x + 4y = 10$.

3. Solve for r: $I = Prt$.

4. Solve for C: $F = \frac{9}{5}C + 32$

5. (*) Solve for m: $y - y_0 = m(x - x_0)$.

4.4 Solving linear inequalities in one variable

A linear inequality, as its name implies, is an inequality in which the only operations involving the variables are addition, subtraction, and multiplying by a constant. In this section, we consider inequalities with just one variable.

Like equations, inequalities can be true or false. Solving an inequality involves finding all values for the variables which make the statement true.

A look at a very easy example of a linear inequality shows a significant difference compared to linear equations. Consider for example the linear inequality $x \leq 2$. (Compare this with Example 4.2.1.) Notice that the variable is "by itself" on one side of the inequality, with no other operations involved. We can see by inspection that 2 is a solution: $2 \leq 2$ is true. (Notice that $x \leq 2$ is a compound statement: it is true when EITHER $x < 2$ is true OR when $x = 2$ is true.) But 2 is not the ONLY solution! For example, 1 is also a solution: $1 \leq 2$ is true. -4 is another solution: $-4 \leq 2$ is true. 1.9999 is another solution: $1.9999 \leq 2$ is true. You can convince yourself pretty quickly that the very simple linear inequality $x \leq 2$ in fact has *infinitely many* solutions. This is typical for algebraic inequalities.

Most linear equations in one variable have exactly one solution. We have seen at least one situation (Example 4.2.6) where a linear equation might have infinitely many solutions, but in that case *every* real number is a solution. The "easy" inequality we were considering, $x \leq 2$, has infinitely many solutions, but not every real number is a solution. For example, 3 is not a solution: $3 \leq 2$ is false.

Even the simplest linear inequality poses the following question for us: How can we solve the inequality—find ALL solutions—when there are infinitely many of them? How can we indicate which numbers are solutions and which are not?

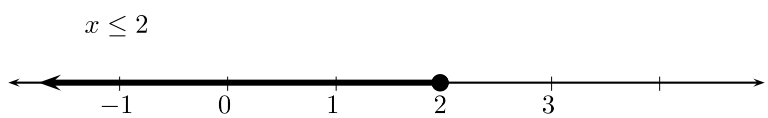
To answer these questions, we will introduce a technique that will be useful in a number of situations: graphing. To *graph* an algebraic statement means to draw a picture of all solutions. Typically our "picture" will involve our standard method of visualizing the real numbers: the number line. "Solving" and "graphing" are really the same type of problem (finding all solutions), but the answer is written differently.

Our basic method for graphing algebraic statements will be to indicate solutions on a number line with a solid circle (•). In the case that we have many solutions "infinitesimally close to each other," which will look like a solid (shaded) line. Here is the simple example we have been considering so far:

Example 4.4.1. *Graph all solutions of the inequality:* $x \leq 2$.

Answer. *Every number less than (on a number line, to the left of)* 2 *will be a solution, so we will shade the region of the number line to the left of two. In addition, we will use a solid circle for the "border" solution* 2*, to indicate that* 2 *is in fact a solution. We will use the term "border value" to denote the fact that for all values on one side of the value on the number line, the inequality is true, while for all values on the other side, the inequality is false.*

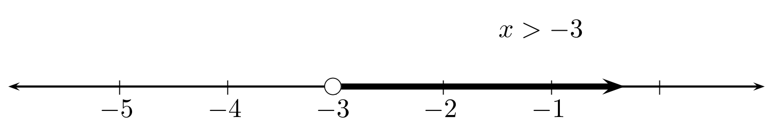
The graph of all solutions of $x \leq 2$ *is:*



Notice that the picture really describes ALL solutions of the inequality $x \leq 2$*; we have "solved" the inequality. We see whether a number is a solution or not by whether or not it is in the shaded region on the number line.*

In the graphical method for solving linear inequalities, special attention must be given to the "border" value. Consider for example the strict inequality $x > -3$. In this case, any number to the right of -3 on the number line will be a solution. The border value -3 is NOT a solution, however: $-3 > -3$ is false. In order to deal with the problem of shading every value to the right of -3 but not including -3 itself, we will indicate the border value with an empty circle (◦).

Hence the graph of all solutions of $x > -3$ is:



Summarizing:

Border values for linear inequalities in one variable

- For strict inequalities ($<$, $>$), the border value is indicated with an open circle ($\circ$);
- For non-strict inequalities ($\leq$, $\geq$), the border value is indicated with a solid circle ($\bullet$).

In order to solve more complicated linear inequalities in one variable, we will present two slightly different methods. Each has some advantages. Both will rely on the methods we have already developed to solve linear equations in one variable, namely the addition and multiplication principles. The main difference between the two methods will be in deciding which side of the border value to shade.

4.4.1 Solving linear inequalities in one variable: Test value method

As the above simple examples show, graphing an inequality has two key steps: finding the border value and deciding which side of the border value are solutions to the inequality. Our first method of solving inequalities separates these two steps. The main idea in this method is that the border value of a linear inequality divides the number line into a "greater than" side and a "less than" side (relative to this inequality); the border value corresponds to the solution of an equation. We will determine the appropriate side to shade by choosing a test value, which will determine which side is which.

Example 4.4.2. *Graph all solutions of the inequality:* $3(x+2)-4>2x+8$.

Answer. ***Step 1. Find the border value*** *In order to find the border value, we consider the corresponding equation:*

$$3(x+2)-4=2x+8.$$

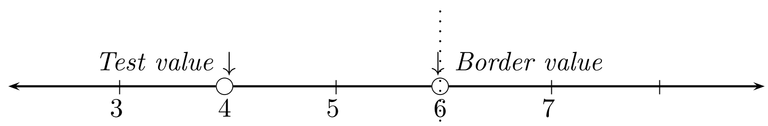
This is a linear equation in one variable. The border value is the solution of this equation.

$$\begin{array}{cccccccc}
3(x+2) & & & - & 4 & = & 2x & + & 8 \\
3x & + & 6 & - & 4 & = & 2x & + & 8 \\
3x & & & + & 2 & = & 2x & + & 8 \\
-2x & & & & & \vdots & -2x & & \\
\hline
x & + & 2 & & & = & & & 8 \\
 & - & 2 & & & \vdots & & - & 2 \\
\hline
x & & & & & = & & & 6.
\end{array}$$

The border value is 6. *Notice that we will represent the border value with an open circle (○) since the inequality we are solving is strict.*

Step 2. Use a test value to determine which side to shade. *For a test value, we choose any number EXCEPT the border value. We "test" this value by substituting it into the original inequality to determine whether it is a solution. If it is a solution, all values on the same side of the border value will be solutions; if it is not a solution, all values on the opposite side of the border value will be solutions.*

Suppose we choose 4 *as a test value.*



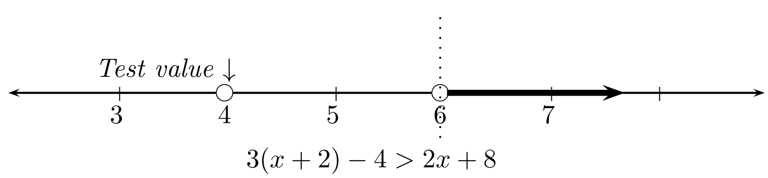
If 4 *is a solution of the inequality, we will shade all values* on the same side *of the border value* 6 *as the test value; if* 4 *is not a solution, we will shade all values* on the opposite side *of* 6 *as* 4.

Is 4 *as solution of* $3(x+2)-4>2x+8$*?*

$$\begin{aligned} 3((4)+2)-4 &> 2(4)+8 \\ 3(6)-4 &> 8+8 \\ 18-4 &> 16 \\ 14 &> 16. \end{aligned}$$

The inequality is false; the test value 4 *is not a solution of the inequality. Therefore we will shade all values on the* opposite *side of the border value as* 4.

The graph of all solutions of $3(x+2)-4>2x+8$ *is:*



For convenience, we summarize the test-value method:

Test-value method for graphing linear inequalities in one variable

- Find the border value by solving the corresponding linear *equation*.
- Determine which side of the border value to shade by choosing a test value and deciding whether it is a solution or not.

Represent the border value with an open or closed circle according to whether the inequality is strict or not.

Example 4.4.3. *Graph all solutions of the inequality:* $2(2x-3)+3x \leq 10x-5$.

Answer. ***Step 1: Find the border value.***

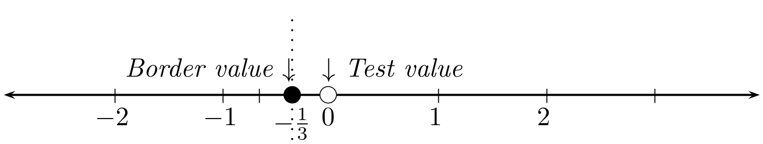
To find the border value, we solve the equation $2(2x-3)+3x = 10x-5$*:*

$$\begin{array}{ccccccccc}
2(2x-3) & & & + & 3x & = & 10x & - & 5 \\
4x & - & 6 & + & 3x & = & 10x & - & 5 \\
7x & - & 6 & & & = & 10x & - & 5 \\
-10x & & & & & \vdots & -10x & & \\ \hline
-3x & - & 6 & & & = & & -5 & \\
 & + & 6 & & & \vdots & & +6 & \\ \hline
-3x & & & & & = & & 1 & \\
\frac{-3x}{-3} & & & & & = & & \frac{1}{-3} & \\
x & & & & & = & & -\frac{1}{3}. &
\end{array}$$

The border value is $-1/3$*. We will indicate the border value with a solid circle (•) since the original inequality (*$\leq$*) is not strict.*

Step 2. Use a test value to determine which side to shade.

Let's choose 0 *as our test value.*

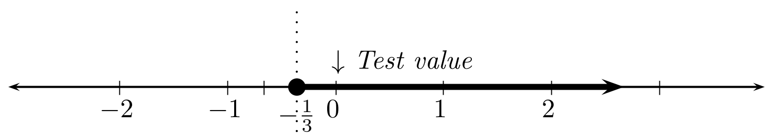


We test whether 0 *is a solution of* $2(2x-3)+3x \leq 10x-5$*:*

$$\begin{aligned} 2(2(0)-3)+3(0) &\leq 10(0)-5 \\ 2(0-3)+0 &\leq 0-5 \\ 2(-3) &\leq -5 \\ -6 &\leq -5. \end{aligned}$$

The inequality is true; the test value 0 *is a solution of the inequality. Therefore we will shade all values on the* same *side of the border value as* 0.

The graph of all solutions of $2(2x-3)+3x \leq 10x-5$ *is:*



$$2(2x-3)+3x \leq 10x-5$$

We close with one final example.

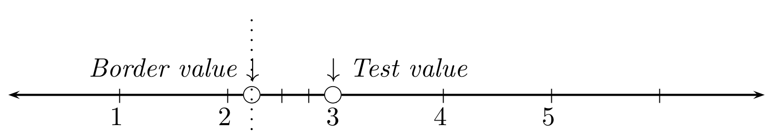
Example 4.4.4. *Graph all solutions of the inequality:* $4x-7<2$.

Answer. ***Step 1: Find the border value.*** *We solve the equation* $4x-7=2$*:*

$$\begin{array}{cccc} 4x & - & 7 & = & 2 \\ & + & 7 & \vdots & +7 \\ \hline 4x & & & = & 9 \\ \frac{4x}{4} & & & = & \frac{9}{4} \\ x & & & = & \frac{9}{4}. \end{array}$$

The border value is $9/4$, *which we will indicate with an open circle* ($\circ$) *since the original inequality is strict.*

Step 2. Use a test value to determine which side to shade. *Let's choose* 3 *as a test value:*

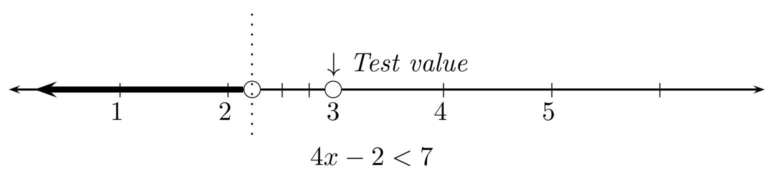


We test whether 3 *is a solution of the inequality* $4x - 7 < 2$*:*

$$\begin{aligned} 4(3) - 7 &< 2 \\ 12 - 7 &< 2 \\ 5 &< 2 \end{aligned}$$

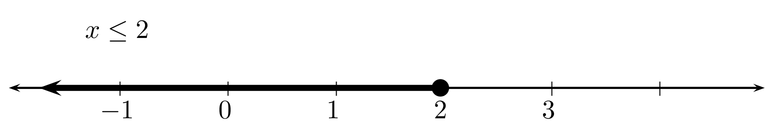
The inequality is false; 3 *is not a solution of* $4x - 2 < 7$*. Hence we shade on the opposite side of the border value.*

The graph of all solutions of $4x - 7 < 2$ *is:*



4.4.2 Solving linear inequalities in one variable: Standard form method

Another way of deciding which half of the number line to shade for a typical linear inequality in one variable is to take advantage of the special form $x <$ ___ or $x >$ ___. We will call this form "standard form," in which the variable appears on the LEFT side of the inequality by itself with no operations. We opened this section by considering an example of this form: $x \leq 2$. The graph consisted of shading all values less than (to the *left of*) 2, along with the border value 4 which we indicated with a solid circle (•):



On the other hand, we considered the inequality $x > -3$, also in standard form. The graph consisted of shading all values greater than (to the *right of*) -3, with the border value -3 indicated with an open circle (○) to indicate that it is not a solution of the inequality:

$x > -3$

Our second method of solving linear inequalities in one variable will be to convert the inequality into an equivalent one in the standard form with the variable by itself *on the left side of the inequality*, in almost exactly the same way as we did to solve a linear equation. In that case, we can follow the rules:

- For inequalities in the standard form $x \leq a$ or $x < a$, always shade *to the left of* the border value a;
- For inequalities in the standard form $x \geq a$ or $x > a$, always shade *to the right of* the border value a.

As usual, the border value a will be indicated with a solid circle ($\bullet$) or an open circle ($\circ$) depending on whether or not the inequality is strict.

The standard form method has the advantage of the word "always." In particular, because we take the trouble to write the inequality in the standard form, there is no need to choose a test value to determine which side of the border value to shade. There are two important points that need to be kept in mind, however.

The first is that the standard form presumes that the variable is by itself on the LEFT side of the inequality. When we solved equations, we were free to write the variable by itself on EITHER side of the equation. In fact, we saw examples, where it was more convenient to write an equation in an equivalent form with the variable by itself on the right side of the equation.

This problem is easy to solve as long as we keep in mind that writing $a < b$ is exactly the same as writing $b > a$. In other words, writing the inequality from right to left "changes the sense" of the inequality (in other words, the inequality symbol "points in the opposite direction.") So if we were to rewrite a linear inequality into the form $5 \geq x$, we would simply rewrite it in the standard form $x \leq 5$.

The second point to keep in mind is more serious. The careful reader might have noticed that this method was described as rewriting the inequality in standard form "in almost exactly the same way" as for linear equations. More specifically, that means using the addition and multiplication principals to "do the same thing to both sides" to obtain a simpler statement. In fact, the addition principle can be used in exactly the same way: Adding (or subtracting) the same quantity from both sides of an inequality produces an equivalent inequality (an inequality with the same solutions).

The multiplication principal requires some adjustment, however. To see this, let's start with a true inequality $2 < 3$. Multiplying both sides by 5, for example, we obtain the inequality $10 < 15$, which is still true. However, if we multiply both sides of the same inequality by -5, we obtain $-10 < -15$—which is false! Clearly, multiplying (and dividing, as you might guess) by a negative quantity has a different effect than multiplying (or dividing) by a positive quantity. Without going into a lengthy explanation for why this is so, just keep in mind that multiplying by a negative quantity involves some sense of an "opposite," and additive opposites on a number line are "mirror images" of each other, reflected across the point representing 0.

For this reason, we need to adjust the multiplication principal for inequalities:

The multiplication principal for inequalities

- Multiplying (or dividing) both sides of an inequality by a positive quantity will produce an equivalent inequality;
- Multiplying (or dividing) both sides of an inequality by a negative quantity will produce an equivalent inequality *after changing the sense of the original inequality.*

For example, dividing both sides of the inequality $-3x > 6$ by -3 gives the equivalent inequality $\frac{-3x}{-3} < \frac{6}{-3}$, or $x < -2$. The sense of the original inequality has changed from $>$ to $<$.

With these two small adjustments to our method of approaching linear equations, let's go back to the same examples as we saw in the test value method.

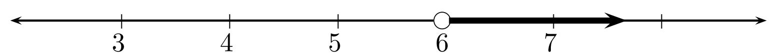
Example 4.4.5. *Graph the inequality:* $3(x + 2) - 4 > 2x + 8$.

Answer. *We apply the addition principal and the revised multiplication principle to obtain an equivalent inequality in standard form:*

$$
\begin{array}{cccccccc}
3(x+2) & & & - & 4 & > & 2x & + & 8 \\
3x & + & 6 & - & 4 & > & 2x & + & 8 \\
3x & & & + & 2 & > & 2x & + & 8 \\
-2x & & & & & \vdots & -2x & & \\
\hline
x & + & 2 & & & > & & & 8 \\
 & - & 2 & & & \vdots & & - & 2 \\
\hline
x & & & & & > & & & 6.
\end{array}
$$

Notice that at no point did we need to multiply or divide by a negative number.

The new, equivalent inequality is of the form $x > a$ *(where here* a *is* 6*). We will shade* to the right of 6, *indicating* 6 *with an open circle since the inequality is strict:*



$3(x + 2) - 4 > 2x + 8$

Summarizing the standard-form method:

Standard form method for graphing linear inequalities in one variable

- Use the addition principle and (modified) multiplication principle to re-write the inequality in the standard form, with the variable by itself on the *left* side of the inequality.
- For inequalities involving "less than" ($<$ or $\leq$), shade to the left of the border value. For inequalities involving "greater than" ($>$ or $\geq$), shade to the right of the border value.

Represent the border value with an open or closed circle according to whether the inequality is or is not strict.

Let's apply this method to the other two examples we saw previously.

Example 4.4.6. *Graph all solutions of the inequality:* $2(2x-3)+3x \leq 10x-5$.

Answer. *We again rewrite the inequality in standard form:*

$$
\begin{array}{rcrcrcrcr}
2(2x-3) & & & + & 3x & \leq & 10x & - & 5 \\
4x & - & 6 & + & 3x & \leq & 10x & - & 5 \\
7x & - & 6 & & & \leq & 10x & - & 5 \\
-10x & & & & & \vdots & -10x & & \\
\hline
-3x & - & 6 & & & \leq & & & -5 \\
 & + & 6 & & & \vdots & & & +6 \\
\hline
-3x & & & & & \leq & & & 1 \\
\frac{-3x}{-3} & & & & & \geq & & & \frac{1}{-3} \\
x & & & & & \geq & & & -\frac{1}{3}.
\end{array}
$$

This time, in our final step, we had to divide by a negative number, so the sense of the inequality changed from $\leq$ *to* $\geq$.

Since the final inequality has the standard form $x \geq a$ *(where* a *is* $-1/3$*), we will shade to the right of the border value; the border value* $-1/3$ *will be indicated with a solid circle since the inequality is not strict.*

The graph of all solutions of $2(2x-3)+3x \leq 10x-5$ *is:*

$2(2x-3)+3x \leq 10x-5$

Example 4.4.7. *Graph all solutions of the inequality:* $4x-7<2$.

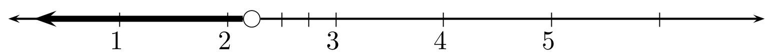
Answer. *Rewriting the inequality in standard form:*

$$
\begin{array}{rcrcr}
4x & - & 7 & < & 2 \\
 & + & 7 & \vdots & +7 \\
\hline
4x & & & < & 9 \\
\frac{4x}{4} & & & < & \frac{9}{4} \\
x & & & < & \frac{9}{4}.
\end{array}
$$

At no point did we multiply or divide by a negative number; the sense of the inequality remained the same.

Since the equivalent inequality has the standard form $x < a$ *(where here* a *is* $9/4$*), we will shade* to the left of *the border value* $9/4$*, indicating the border value with an open circle since the inequality is strict.*

The graph of all solutions of $4x - 7 < 2$ *is:*



$4x - 2 < 7$

To summarize:

- A typical linear inequality in one variable will have infinitely many solutions. For this reason, the solutions are typically indicated with a graph on a number line.
- The typical graph of a linear inequality will be half of a number line, all points to the left of or to the right of the border value, with the shaded region representing solutions to the inequality.
- The border value is indicated with either an open circle ($\circ$), in the case of a strict inequality ($<$ or $>$), or a solid circle ($\bullet$), in the case of a non-strict inequality ($\leq$ or $\geq$).
- Which side of the border value to shade can be determined either by the test value method or by the standard form method.

Keep in mind that an inequality, like an equation, can be an identity (and so all real numbers are solutions) or a contradiction (and so has no solution).

4.4.3 Exercises

In the problems below, "solve" means "graph all solutions." For each one, list five individual solutions.

1. Solve: $3x - 4 > 6$.
2. Solve: $2(x - 3) + 4 \leq x - 5$.
3. Solve: $3(2x - 1) + 4(3x + 5) > 2(x - 6)$.
4. Solve: $x - 5(2x + 1) \leq 6$.
5. (*) Solve: $-2(x - 3) \geq -4(x + 1) + 2x$.
6. (*) Solve: $2x + 3(x - 2) > 5(x - 1) - 1$.

4.5 Chapter summary

- A solution of an algebraic equation or inequality is a value for each variable which, when substituted, makes the statement true.
- To solve means to find all solutions.
- A linear equation is one in which the only operations involving variables are addition, subtraction, and multiplication by a constant.
- A typical linear equation in one variable has one solution. The exceptions are contradictions (statements that have no solution) and identities (statements which are true for all values of the variable).
- Solving a linear equation in one variable involves using the addition and multiplication principles to find an equivalent equation for the form

$$x = \_\_\_.$$

- "Solving" a literal equation for a given variable means writing an equivalent equation with the given variable "by itself" on one side of the equation.
- A typical linear inequality in one variable will have infinitely many solutions. To solve them, the solutions are graphed on a number line.

Chapter 5

Linear equations and inequalities in two variables

Vocabulary

- xy-plane
- Plotting ordered pairs
- Graph
- Intercepts (x– and y–intercept of a line in an xy-plane)
- Slope of a line
- Parallel lines
- Perpendicular lines
- Horizontal lines
- Vertical lines
- Slope-intercept form of a linear equation in two variables
- Point-slope form of a linear equation in two variables
- System of linear equations

5.1 Solving linear equations in two variables

We now turn our attention to linear equations with two variables, which we will assume to be called x and y. A linear equation in two variables can always be written in a standard form

$$Ax + By = C,$$

where A and B are constant coefficients and C is a constant. What is "standard" about this form is that the terms involving variables are on one side of the equation, while the constant term (not involving variables) is on the other side of the equation. However, a linear equation may not be written in this standard form. In fact, we will soon see several situations in which it is better to write a linear equation in another form.

As with any algebraic statement, a linear equation in two variables may be true or false, depending on the values for both variables x and y. As we saw earlier in Section 4.1, a solution to a linear equation in two variables consists of a value for each of the two variables, which we indicate by writing them together as an ordered pair.

Let's start by looking at a relatively easy example of a linear equation in two variables:

$$x + y = 5.$$

It's easy to see a few examples of solutions to this equation: $(1, 4)$, $(2, 3)$, and $(3, 2)$, for example. With a little more thought, more exotic solutions come to mind: $(-1, 6)$ and $\left(\frac{1}{2}, 4\frac{1}{2}\right)$, for example. On the other hand, not every ordered pair is a solution to this equation: $(2, 2)$ is not a solution, for example.

5.1.1 A method for producing solutions

In the case that the equation is more complicated, there is still a straightforward method to produce solutions. We illustrate this method in the following example.

Example 5.1.1. *Find three solutions to the equation* $2x - 5y = 10$.

Answer. *Our strategy will be to "eliminate" one of the variables and to solve the remaining linear equation in one variable. We eliminate a variable by choosing a value for that variable, then substituting the value into the original equation. The solution to the original equation will be an ordered pair consisting of the chosen value for the "eliminated" variable and the value obtained by solving the resulting (one-variable) equation.*

For example, let's choose the value 0 *for* x. *Substituting into the given equation for* x *gives* $2(0) - 5y = 10$*; the variable* x *has been "eliminated." We then solve:*

$$\begin{array}{rcrcl} 2(0) & - & 5y & = & 10 \\ 0 & - & 5y & = & 10 \\ & & -5y & = & 10 \\ & & \frac{-5y}{-5} & = & \frac{10}{-5} \\ & & y & = & -2. \end{array}$$

The solution corresponding to our choice of 0 *for* x *is* $(0, -2)$.

For another solution, let's choose the value 0 *for* y*. Substituting this value for* y *gives* $2x - 5(0) = 10$*. Solving:*

$$\begin{array}{rcrcl} 2x & - & 5(0) & = & 10 \\ 2x & - & 0 & = & 10 \\ 2x & & & = & 10 \\ \frac{2x}{2} & & & = & \frac{10}{2} \\ x & & & = & 5. \end{array}$$

The solution corresponding to our choice of 0 *for* y *is* $(5, 0)$*.*

Since we were asked for three solutions, we make one more choice. Let's choose the value 1 *for* y*. Substituting gives* $2x - 5(1) = 10$*. Solving:*

$$\begin{array}{rcrcl} 2x & - & 5(1) & = & 10 \\ 2x & - & 5 & = & 10 \\ & + & 5 & \vdots & +5 \\ \hline 2x & & & = & 15 \\ \frac{2x}{2} & & & = & \frac{15}{2} \\ x & & & = & \frac{15}{2}. \end{array}$$

The solution corresponding to our choice of 1 *for* y *is* $(15/2, 1)$*.*

The three solutions we obtained are $(0, -2)$*,* $(5, 0)$*, and* $(15/2, 1)$*.*

We will organize the data from finding solutions to a linear equation in two variables into a table. For example, we will summarize the three solutions above as:

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
5	$\boxed{0}$	$(5, 0)$
$15/2$	$\boxed{1}$	$(15/2, 1)$

Notice that we have indicated the value that was chosen with a boxed number, while the value obtained by solving the corresponding equation with an unboxed number.

We can summarize this method for finding solutions.

Finding solutions to an algebraic equation in two variables

To find solutions to an algebraic equation in two variables:

1. Choose a value for one of the variables;
2. Substitute the chosen value into the equation and solve the resulting equation in one variable.

The ordered pair corresponding to the chosen value with the value obtained by solving the resulting equation (in the appropriate order) will be a solution to the original equation in two variables.

One thing should be clear from the method described in the example above: A linear equation in two variables will typically have infinitely many solutions, one for each choice of value for x (or y). This will present some problems from the point of view of solving such equations—finding *all* solutions.

5.1.2 Graphing linear equations in two variables

In Section 4.4 on linear inequalities in one variable, we saw a powerful method for keeping track of solutions of algebraic statements with infinitely many solutions: graphing. However, in the case of algebraic statements in two variables, a number line is not sufficient. To keep track of the values of both variables, we will use the xy–plane (sometimes called the Cartesian plane, after one of the originators of the concept, the French philosopher and mathematician René Descartes).

For the sake of reference, we list here some of the most important properties of an xy-plane (see Figure 5.1):

- It is formed by two number lines placed at right angles and meeting where both are labeled 0. The number lines are called the x-axis (the horizontal number line) and the y-axis (the vertical number line). The point of intersection of the axes is called the origin.
- The positive x-direction is to the right. The positive y-direction is upwards.
- An ordered pair is represented by a point on the xy-plane by means of its *coordinates*. The first number (the x-coordinate) represents the number of units ("in the x-direction") from the y-axis to the point . The second number (the y-coordinate) represents the number of units ("in the y-direction") from the x-axis to the point.

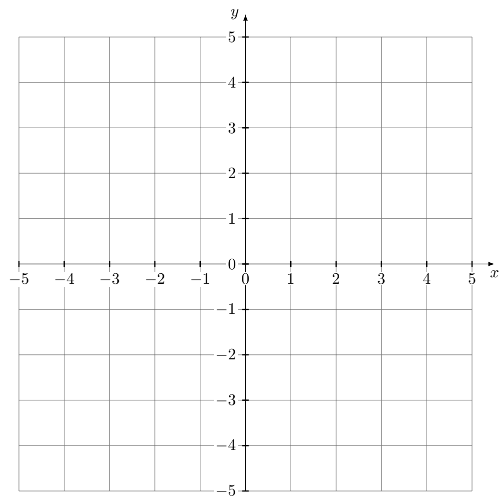
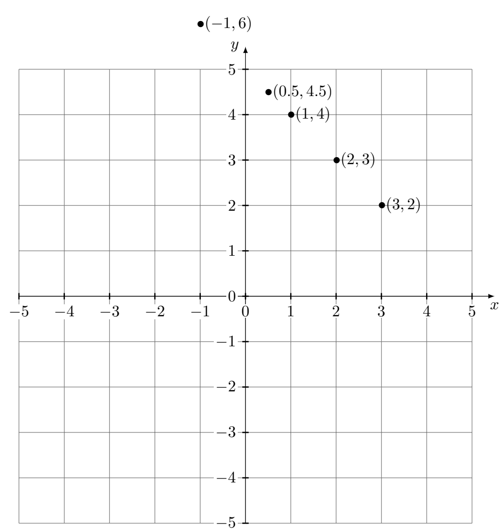


Figure 5.1: An xy-plane

- Points on the x-axis correspond to ordered pairs having 0 as a y-coordinate. Points on the y-axis correspond to ordered pairs having 0 as an x-coordinate.

Let's return to our example $x + y = 5$. Just by inspection, we found several solutions. We will now represent each ordered pair solution with a point in the xy-plane. (This is called *plotting* the ordered pairs.)



Five solutions of $x + y = 5$

This graph, obtained by plotting five solutions of the same linear equation in two variables, points to a crucial fact that will be central to our treatment of linear equations in two variables:

BIG FACT: The geometry of solutions to linear equations in two variables

The points corresponding to plotting all solutions to a linear equation in two variables all lie on a single line. Every point on this line corresponds to a solution to the equation.

This fact, combined with some basic geometry, gives a powerful technique to solve a linear equation in two variables in the form of a graph.

General method to graph linear equations in two variables

To graph all solutions of a linear equation in two variables:

1. Find at least two solutions.
2. Plot the solutions.
3. Draw the line passing through the chosen solutions.

Notice that geometry comes into the picture due to the fact, written down as far back as Euclid, that two (different) points determine a unique line passing through them. This fact is what allows us to “buy two solutions, get infinitely many solutions free.”

Combined with our method for producing solutions to linear equations in two variables above, we are hence able to graph any linear equation in two variables.

Example 5.1.2. *Graph the equation* $2x - 5y = 10$.

Answer. *Recall in Example 5.1.1 above, we found three solutions to* $2x - 5y = 10$*, given in the table*

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
5	$\boxed{0}$	$(5, 0)$
$15/2$	$\boxed{1}$	$(15/2, 1)$

We plot these solutions in Figure 5.2.

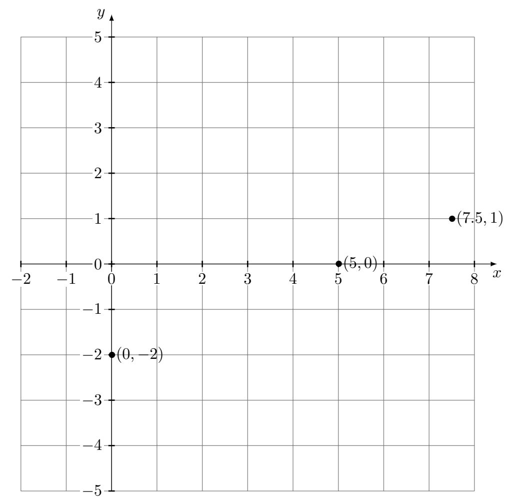


Figure 5.2: Three solutions of $2x - 5y = 10$

Notice that the three solutions appear to lie on the same line, as we expected from our Big Fact. All that remains is to "connect the dots" in Figure 5.3.

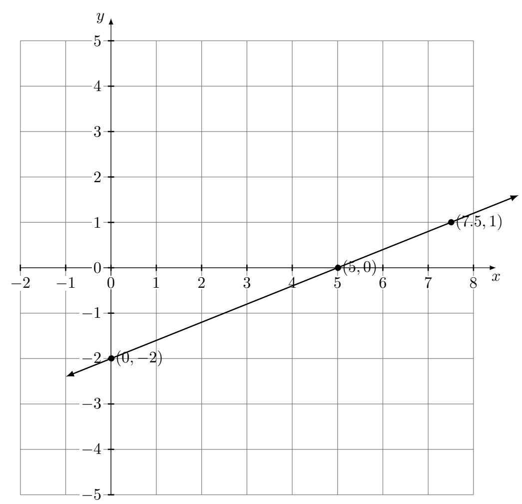


Figure 5.3: All solutions of $2x - 5y = 10$.

It is important to emphasize that the last "connect the dots" step, simplest from the procedural point of view, is also the most significant. We have gone from three solutions to infinitely many solutions—one for each point on the line.

Let's look at two more examples.

Example 5.1.3. *Graph the solutions of* $3x + 4y = 12$.

Answer. *We first find three solutions.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcl} 3(0) & + & 4y & = & 12 \\ 0 & + & 4y & = & 12 \\ & & 4y & = & 12 \\ \frac{4y}{4} & & & = & \frac{12}{4} \\ y & & & = & 3. \end{array}$$

So $(0,3)$ *is a solution.*

Choosing 0 *for* y*, we substitute and solve:*

$$\begin{array}{rcrcl} 3x & + & 4(0) & = & 12 \\ 3x & + & 0 & = & 12 \\ 3x & & & = & 12 \\ \frac{3x}{3} & & & = & \frac{12}{3} \\ x & & & = & 4. \end{array}$$

So $(4,0)$ *is a solution.*

Choosing -3 *for* y*, we substitute and solve:*

$$\begin{array}{rcrcl} 3x & + & 4(-3) & = & 12 \\ 3x & - & 12 & = & 12 \\ & + & 12 & \vdots & +12 \\ \hline 3x & & & = & 24 \\ \frac{3x}{3} & & & = & \frac{24}{3} \\ x & & & = & 8. \end{array}$$

So $(8,-3)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
$\boxed{0}$	3	$(0,3)$
4	$\boxed{0}$	$(4,0)$
8	$\boxed{-3}$	$(8,-3)$

We now plot the three solutions and connect them with a line. See Figure 5.4.

Notice that choosing 0 first for x and then for y is useful for more than just the ease of working with the number 0. The point whose x-coordinate is 0 (the point $(0,3)$ in the previous example) is the *y-intercept* of the line: the point

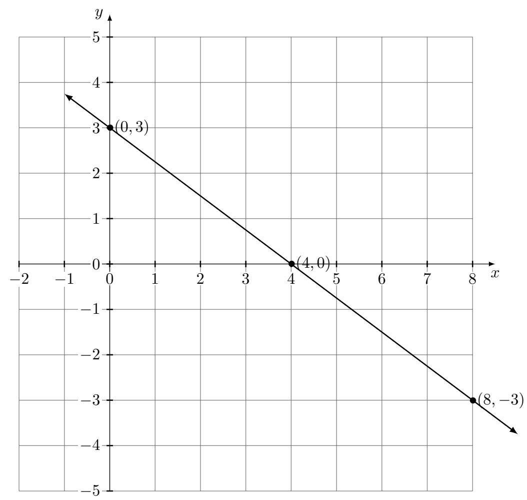


Figure 5.4: All solutions of $3x + 4y = 12$.

where the line intersects the y-axis. Likewise, the point whose y-coordinate is 0 (the point $(4, 0)$ in the previous example) is the x-*intercept* of the line, or the point where the line intersects the x-axis. We will often refer to these two special points on a line in the xy-plane, as they stand out on the graph.

Example 5.1.4. *Graph the solutions of* $y = \frac{1}{4}x - 2$.

Answer. *As usual, we will make three choices to find three solutions. This time, however, we will take advantage of the form in which the equation is written, with the* y *by itself on one side of the equation, and only choose values of* x.

Choosing 0 *for* x, *we substitute and solve:*

$$\begin{aligned} y &= \tfrac{1}{4}(0) - 2 \\ y &= 0 - 2 \\ y &= -2. \end{aligned}$$

So $(0, -2)$ *is a solution.*

Choosing 4 *for* x, *we substitute and solve:*

$$\begin{aligned} y &= \tfrac{1}{4}(4) - 2 \\ y &= 1 - 2 \\ y &= -1. \end{aligned}$$

So $(4, -1)$ *is a solution.*

Choosing 8 *for* x, *we substitute and solve:*

$$\begin{aligned} y &= \tfrac{1}{4}(8) - 2 \\ y &= 2 - 2 \\ y &= 0. \end{aligned}$$

So $(8,0)$ is a solution.
Hence we have the table:

x	y	*Solution*
0	-2	$(0,-2)$
4	-1	$(4,-1)$
8	0	$(8,0)$

(Can you see why we chose the values of x that we did?)
Plotting the solutions and connecting them with a line gives Figure 5.5.

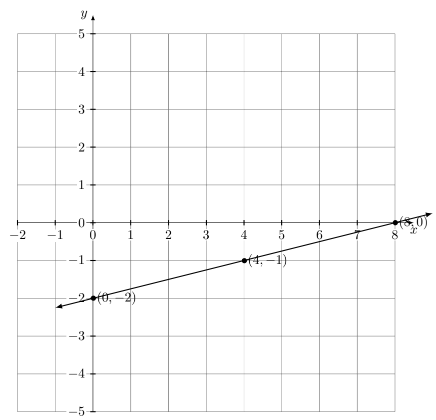


Figure 5.5: All solutions of $y = \frac{1}{4}x - 2$.

5.1.3 Exercises

For each of the linear equations in two variables below, graph the solutions.

1. $x - y = 4$
2. $2x + 3y = -6$
3. $5x - y = 2$
4. $-4x + 3y = 12$
5. $-x + 3y = 9$
6. $y = 2x - 1$
7. $y = \frac{1}{3}x - 2$
8. $y = -\frac{3}{4}x + 1$

5.2 A detour: Slope and the geometry of lines

We saw in the last section how geometry can be helpful in solving a linear equation in two variables. In particular, using the fact that two points determine a line, we were able to find all solutions of a linear equation in two variables (as a graph) just by knowing any two different solutions.

In this section, we continue the theme of how geometry can help us study linear equations in two variables. After defining the slope of a line, we will show how we can use this concept to develop another method for graphing the solutions to such equations. We will also show how this concept allows us to write an equation for a line in the xy-plane.

The *slope* will give a way to measure a line. It will be a single number that is designed to measure the "steepness" of a line.

Consider for example the lines shown in Figure 5.6. Line A is steeper than line B. (Imagine yourself riding a bicycle up two hills represented by the lines. It will be harder to pedal up line A than line B!) So we will want to assign a larger number as the slope of line A than for the slope of line B. Line C is not steep at all; it is "flat." We will want to assign a slope of 0 to this line. Line D appears to be about as steep as line A, but in different "directions." Line A is slanted upwards (from left to right), while line D is slanted downwards. We will assign a positive number as the slopes for lines A and B, but a negative number for the slope of line D. Vertical lines are special in that they do not have a slope. (Don't try to ride your bike down a vertical cliff!)

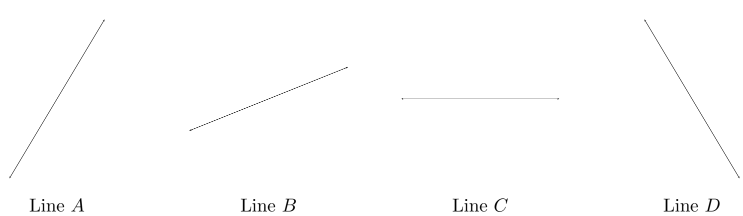
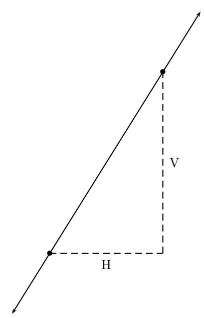


Figure 5.6: Four lines with different slopes.

How do we make this measurement called slope? It turns out than an effective way to assign a number that matches exactly with our expectations from the previous paragraph is to define the slope as the ratio of the vertical change in distance between two points on the line to the horizontal change in distance between the same two points, with the understanding that a change from upper to lower (going from left to right) will be negative[1]. See Figure 5.7.



$$m = \frac{V}{H}$$

Figure 5.7: The definition of slope m.

Notice that we have defined the slope without reference to a coordinate system, i.e. without an xy-plane. In the case that the line is drawn with reference to a coordinate system, the vertical and horizontal distances in the definition of the slope can be written in terms of the coordinates of two points on the line with coordinates (x_1, y_1) and (x_2, y_2):

The slope of a line in an xy-plane

The slope of a line in an xy-plane passing through the points with coordinates (x_1, y_1) and (x_2, y_2) is given by the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

(See Figure 5.8.)

It should be pointed out that in this context, the notation Δy and Δx are

[1]This definition in itself is based on an important fact from geometry. Recall that two triangles are *similar* if their corresponding angles have equal measurements. The ratio of corresponding sides of similar triangles are equal. For that reason, the slope does not depend on the two points chosen. Can you see why?

sometimes used to represent the change in x and y respectively, so the slope can be remembered as

$$m = \frac{\Delta y}{\Delta x}.$$

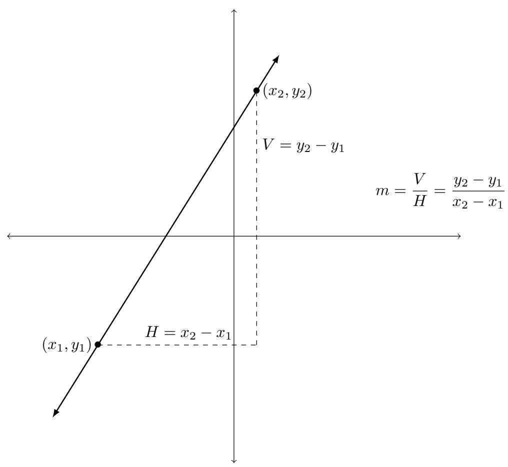


Figure 5.8: The slope defined relative to an xy-plane.

In order to use the formula defining the slope, the coordinates of (any!) two points on the line are needed.

Example 5.2.1. *Find the slope of the line passing through the points with coordinates* $(6, -2)$ *and* $(3, 7)$.

Answer. *Since we are given the coordinates of two points on the line, all that remains to do is to label the coordinates, substitute into the formula defining the slope, and evaluate.*

Labelling,

$$\begin{matrix} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& 6 & , & -2 &), & (& 3 & , & 7 &) \end{matrix}$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(7) - (-2)}{(3) - (6)} \\ &= \frac{7 + 2}{3 + (-6)} \\ &= \frac{9}{-3} \\ &= -3. \end{aligned}$$

The slope is -3.

For the sake of the reader who is seeing the slope formula in action for the first time, let's re-do the previous example, but labeling the coordinates in the opposite way:

$$(\overset{x_1}{3}, \overset{y_1}{7}), (\overset{x_2}{6}, \overset{y_2}{-2})$$

Then substituting,

$$\begin{aligned} m &= \frac{(-2)-(7)}{(6)-(3)} \\ &= \frac{(-2)+(-7)}{3} \\ &= \frac{-9}{3} \\ &= -3. \end{aligned}$$

We obtain the same answer, the slope being -3. This is a special case of the point that we made in the definition: *the slope does not depend on which two points on the line are chosen*, and in particular, does not depend on the order that the points are used.

Although a graph is not necessary for the purpose of computing the slope of a line, the reader might want to plot the two given ordered pairs $(6,-2)$ and $(3,7)$ to visualize the line passing through the corresponding points to verify that the line slants downwards going from left to right, as we would expect from a line with a negative slope.

We next illustrate an example where the required information to compute the slope from the definition is not given directly. We will see shortly that there is another, more effective way to approach this example.

Example 5.2.2. *Use the definition to find the slope of the line given by the equation $2x + y = 2$.*

Answer. *Although we are not given the coordinates of two points on the line, in some ways we have better: we have an equation for the line. We have already seen a method for obtaining as many solutions to this equation as we want—two will be enough.*

Choosing 0 *for* y*, we substitute and solve:*

$$\begin{array}{ccccc} 2x & + & (0) & = & 2 \\ 2x & & & = & 2 \\ \frac{2x}{2} & & & = & \frac{2}{2} \\ x & & & = & 1. \end{array}$$

So $(1,0)$ *is a solution.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{ccccc} 2(0) & + & y & = & 2 \\ 0 & + & y & = & 2 \\ & & y & = & 2. \end{array}$$

So $(0, 2)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
1	$\boxed{0}$	$(1, 0)$
$\boxed{0}$	2	$(0, 2)$

Now, labeling the coordinates of our two solutions,

$$(\overset{x_1}{1}, \overset{y_1}{0}), (\overset{x_2}{0}, \overset{y_2}{2})$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(2) - (0)}{(0) - (1)} \\ &= \frac{2}{0 + (-1)} \\ &= \frac{2}{-1} \\ &= -2. \end{aligned}$$

The slope is -2.

This example also gives us a way to illustrate even more surely that the slope does not depend on the points chosen. Suppose your classmate's choices are different from yours, and they obtain two different solutions $(-1, 4)$ and $(2, -2)$. (Check that these are really solutions to $2x + y = 2$!) In that case, they would label:

$$(\overset{x_1}{-1}, \overset{y_1}{4}), (\overset{x_2}{2}, \overset{y_2}{-2}).$$

Substituting and evaluating would give:

$$\begin{aligned} m &= \frac{(-2) - (4)}{(2) - (-1)} \\ &= \frac{(-2) + (-4)}{2 + (1)} \\ &= \frac{-6}{3} \\ &= -2. \end{aligned}$$

The two points were chosen differently, but the slope of the line is the same!

We conclude this subsection with an example that will lead in to the next main use of the slope concept.

Example 5.2.3. *Find the slope of the line given by the equation* $y = \frac{2}{3}x - 4$.

Answer. *As in the last example, we first find any two solutions.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{2}{3}(0) & - & 4 \\ y & = & 0 & - & 4 \\ y & = & -4. & & \end{array}$$

So $(0, -4)$ *is a solution.*

Choosing 3 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{2}{3}(3) & - & 4 \\ y & = & 2 & - & 4 \\ y & = & -2. & & \end{array}$$

So $(3, -2)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
$\boxed{0}$	-4	$(0, -4)$
$\boxed{3}$	-2	$(3, -2)$

Labeling our two solutions,

$$(\overset{x_1}{0}, \overset{y_1}{-4}),\ (\overset{x_2}{3}, \overset{y_2}{-2})$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(-2) - (-4)}{(3) - (0)} \\ &= \frac{(-2) + (4)}{3)} \\ &= \frac{2}{3}. \end{aligned}$$

The slope is $2/3$*. We will see very shortly that this answer is no surprise.*

The previous example 5.2.3 is a special case of an important fact relating the slope to linear equations in two variables:

Slope-intercept form of a linear equation in two variables

Suppose that a linear equation is written in the special form

$$y = mx + b,$$

with the variable y by itself on one side of the equation. Then m (the coefficient of x) is the slope of the line, and the y-intercept has coordinates $(0, b)$.

This special form of writing a linear equation in two variables, where the variable y is written by itself on one side of the equation, is known as the *slope-intercept form* of the equation of a line, since both the slope and the y-coordinate of the y-intercept can be read directly from the equation.

Notice that in Example 5.2.3, the equation $y = \frac{2}{3}x - 4$ was written in slope-intercept form. The slope $2/3$ was indeed the coefficient of x. Notice also, although we didn't mention it at the time, that the y-intercept has coordinates $(0, -4)$, a fact that we could also read from the form of the equation. (Keep in mind that the b term in the special slope-intercept form is *added*, so we should think of the equation as being written $y = \frac{2}{3}x + (-4)$.)

If a linear equation in two variables is not written in slope-intercept form, then there is no way to read off the information so easily. However, by changing the form of the equation, we can take advantage of the special slope-intercept form for any equation.

Example 5.2.4. *Find the slope and y-intercept of the line given by the equation $3x - 4y = 12$.*

Answer. *The equation is not written in slope-intercept form, since the variable y is not by itself. However, we can solve for y in terms of x:*

$$\begin{array}{cccccc} 3x & - & 4y & = & & 12 \\ -3x & & & \vdots & -3x & \\ \hline & & -4y & = & -3x & + \ 12 \\ & & \frac{-4y}{-4} & = & \frac{-3x+12}{-4} & \\ & & y & = & \frac{-3x}{-4} & + \ \frac{12}{-4} \\ & & y & = & \frac{3}{4}x & - \ 3. \end{array}$$

The slope is $3/4$ and the y-intercept has coordinates $(0, -3)$.

We will see several more examples of this procedure in a different context in the following subsection.

5.2.1 Using the slope as an aid in graphing

In this subsection, we show how the slope gives an alternative method to the problem of graphing the solutions to a linear equation with two variables, apart from making a table of values to find solutions. It is based on the following principal:

> The slope, considered as a ratio of the change in the y-coordinates to the change in the x-coordinates of points on the line, gives a way to obtain a new point on the line from a given one.

Specifically, we will think of the slope as a fraction which gives instructions to move "up and to the right" or "down and to the right," depending on whether the slope is positive or negative.[2]

Example 5.2.5. *Find three other points on the line passing through the point with coordinates* $(-3,-2)$ *and having slope* 2*.*

Answer. *The slope is* $2 = \frac{2}{1}$*. So, beginning from the given point's coordinates* $(-3,-2)$*, we will move our pencil on the graph one unit to the right and two units upwards to obtain our first new point. See Figure 5.9. This new point has coordinates* $(-2,0)$*, as should be clear from the graph.*

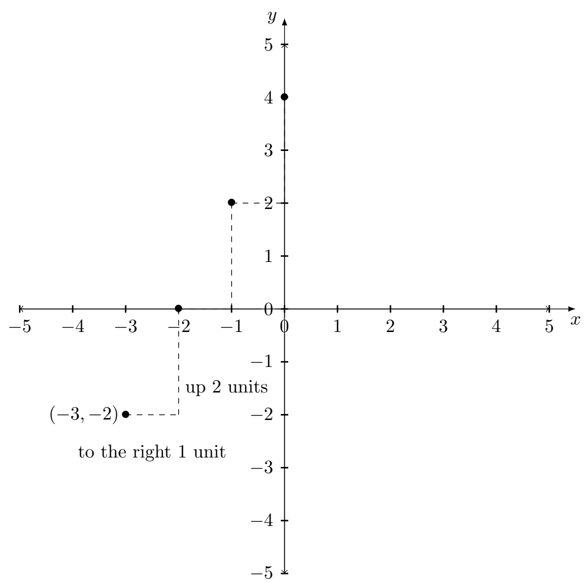


Figure 5.9: Using the slope to find a second point on a line.

Repeating the procedure two more times gives two other new points with coordinates $(-1,2)$ *and* $(0,4)$*. (Even though we could write down a "formula" to obtain the numerical coordinates of one point from the next, it is by far simpler in the cases we will encounter to just read the coordinates from the xy-plane.)*

Using the method of the previous example gives us an effective way to graph the solutions of a linear equation in two variables—especially if the equation is written in slope-intercept form.

[2]More properly, we should think of moving "in the same direction" or "in the opposite direction," so that, for example, we can also obtain a second point from a given one on a line with positive slope by moving down and to the left.

Example 5.2.6. *Graph the solutions of $y = -x + 3$.*

Answer. *Notice that the equation is written in slope-intercept form; y is by itself on one side of the equation. The slope is -1 (the coefficient of x), while the y-intercept has coordinates $(0, 3)$.*

Using the slope $m = -1 = \frac{-1}{1}$, we start at the given point with coordinates $(0, 3)$ and "move" one unit downward and one unit to the right in order to obtain a second point having coordinates $(1, 2)$. This gives us two solutions; the graph will consist of all points on the line passing through these two points.

The graph is given in Figure 5.10.

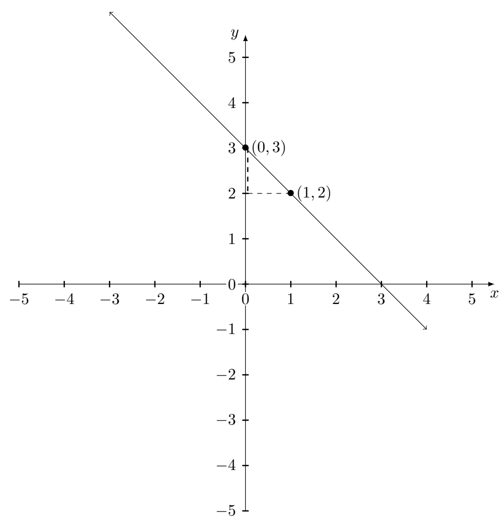


Figure 5.10: All solutions of $y = -x + 3$.

While the previous example was straightforward due to the fact that the equation was written in slope-intercept form to begin with, we have already seen that it doesn't take much effort to rewrite an equation in slope-intercept form if it isn't written that way to begin with, by solving for y.

Example 5.2.7. *Graph the solutions of $2x - y = 6$.*

Answer. *The equation is not written in slope-intercept form, since y is not by itself on one side of the equation. Solving for y in terms of x:*

$$\begin{array}{rcrcccc}
2x & - & y & = & & & 6 \\
-2x & & & \vdots & -2x & & \\
\hline
 & & -y & = & -2x & + & 6 \\
 & & \frac{-y}{-1} & = & \frac{-2x+6}{-1} & & \\
 & & y & = & \frac{-2x}{-1} & + & \frac{6}{-1} \\
 & & y & = & 2x & - & 6.
\end{array}$$

We now see that the slope is 2 *and the* y*-intercept has coordinates* $(0, -6)$.

Using the slope $m = 2 = \dfrac{2}{1}$, *we start at the point representing* $(0, -6)$ *and "move" upwards two units and to the right one unit in order to obtain a second solution* $(1, -4)$.

The graph is given in Figure 5.11.

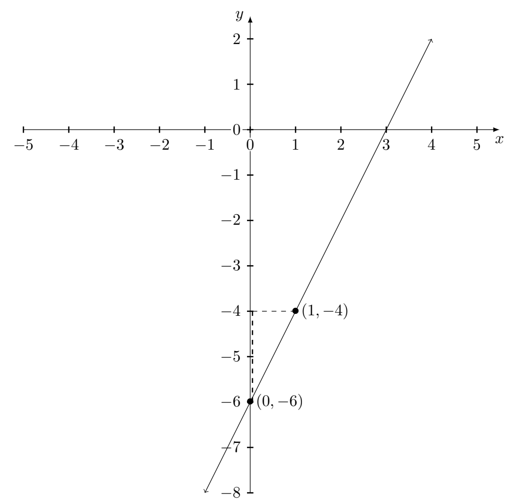


Figure 5.11: All solutions of $2x - y = 6$.

The only possible difficulty in this method of graphing is that when following the method too literally, we will occasionally be forced to plot points with fractional coordinates, as the next example illustrates.

Example 5.2.8. *Graph the solutions of* $3x + 2y = 5$.

Answer. *The equation is not written in slope-intercept form, since* y *is not by itself on one side of the equation. Solving for* y*:*

$$
\begin{array}{rcrcrcrcr}
3x & + & 2y & = & & & 5 \\
-3x & & & \vdots & -3x & & \\
\hline
 & & 2y & = & -3x & + & 5 \\
 & & \frac{2y}{2} & = & \frac{-3x+5}{2} & & \\
 & & y & = & \frac{-3x}{2} & + & \frac{5}{2} \\
 & & y & = & -\frac{3}{2}x & + & \frac{5}{2}.
\end{array}
$$

We see that the slope is $-3/2$ *and the* y*-intercept has coordinates* $(0, 5/2)$.

Since $5/2 = 2\ \frac{1}{2}$, *the point representing* $(0, 5/2)$ *is plotted halfway between those representing* $(0, 2)$ *and* $(0, 3)$. *Using the slope* $m = \dfrac{-3}{2}$, *we start at the* *point representing* $(0, 5/2)$ *and "move" downwards three units and to the right two units to obtain a second solution* $(2, -1/2)$.

The graph is given in Figure 5.12.

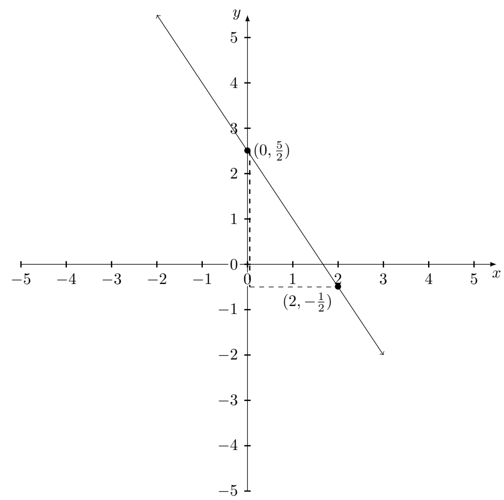


Figure 5.12: All solutions of $3x + 2y = 5$.

(Notice that we encountered fractional coordinates in this example because the y*-intercept had a fractional* y*-coordinate. If we had used a solution with integer coordinates like* $(1, 1)$*, we could have avoided this inconvenience—but then we would have been on our way to constructing a table.)*

5.2.2 Finding an equation of a given line

So far, we have concentrated on the relationship between the slope and the graph of a linear equation in two variables. The sign of the slope indicates which "direction" the line is slanted. The magnitude of the slope measures the ratio of the vertical change to the horizontal change, and so given one point on the line, the slope indicates how to determine other points on the same line.

However, the slope concept also opens the door to a answering a new kind of question. Suppose we are given a line (in an xy-plane) described by some geometric data. How can we find an equation whose solutions correspond to the given line[3]?

What is meant by describing a line with geometric data? We will consider the following situations:

- A line described by one point on the line and the slope;
- A line described by two points on the line;

[3]Notice that we do not ask for "the" equation of a line. The reader can check, for example, that the equations $x + y = 1$ and $2x + 2y = 2$ have the same solutions, and so describe the same line in an xy-plane.

- A line described by one point on the line, given that it is parallel to another line;
- A line described by one point on the line, given that it is perpendicular to another line.

The simplest example will show that we already have tools to answer this question.

Example 5.2.9. *Find an equation for the line passing through the point with coordinates* $(0, -2)$ *and having slope* 3.

Answer. *Notice that in this case, the point given happens to be the* y*-intercept! (That can be seen even without plotting the point by noticing that the* x*-coordinate is* 0.*) Hence we can treat the slope-intercept form of a line, which we have written as* $y = mx + b$, *as a formula, and substitute the values of* m *and* b.

In this case, $m = 3$ *and* $b = -2$, *so an equation of the line, in slope-intercept form, would be*

$$y = 3x - 2.$$

"That was too good to be true!" Of course, we had been given exactly the data needed to substitute into the slope-intercept "formula" for a line. In the next example, we show that the previous method still applies in a more general context. We also illustrate a second method which is better adapted to the more general setting.

Example 5.2.10. *Find an equation for the line passing through the point with coordinates* $(1, -2)$ *and having slope* -4.

Answer. *This time, the given point is not the* y*-intercept (the* x*-coordinate is not* 0*!), so we cannot proceed as directly as in the previous example.*

Method 1

Even though we do not have all the information needed to substitute into the slope-intercept "formula," we can proceed in two steps.

The first, easy step is to substitute the information we do have, which is the slope ($m = -4$)*, into the formula:*

$$y = -4x + b.$$

This time, b *is still unknown.*

In the second step, we will use the coordinates $(1, -2)$ *of the given point to solve for* b, *by substituting the coordinates for* x *and* y *in the equation we have obtained so far:*

$$\begin{array}{ccccc} y & = & -4x & + & b \\ (-2) & = & -4(1) & + & b \\ -2 & = & -4 & + & b \\ \\ +4 & \vdots & +4 & & \\ \hline 2 & = & & & b. \end{array}$$

The solution for b is 2.

Now, since we have values for m AND b, we can substitute into the slope-intercept "formula" as above. The answer is

$$y = -4x + 2.$$

Method 2

Instead of trying to use the slope-intercept "formula," the second method will use the definition of the slope directly. Namely, we will substitute the coordinates for the given point $(1, -2)$ along with the coordinates of a second unknown point (x, y), *along with the value of the slope, into the formula defining the slope* $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. *Namely, we label:*

$$\left(\overset{x_1}{1} \;,\; \overset{y_1}{-2} \right), \quad \left(\overset{x_2}{x} \;,\; \overset{y_2}{y} \right).$$

After substituting these values, we solve for y in terms of x:

$$\begin{array}{rrcrrr} & -4 & = & \frac{(y)-(-2)}{(x)-(1)} & & \\ & -4 & = & \frac{y+2}{x-1} & & \\ & -4 \cdot (x-1) & = & \frac{y+2}{x-1} \cdot (x-1) & & \\ -4x & +4 & = & y & + & 2 \\ & -2 & \vdots & & & -2 \\ \hline -4x & +2 & = & y. & & \end{array}$$

The answer is $y = -4x + 2$.

Notice that in the key step to this method, multiplying both sides by $(x - 1)$ *to "cancel" the denominator in the definition of the slope, we assumed that* $x - 1 \neq 0$. *This is permitted since we were supposing (x, y) to be the coordinates of a second point on the line different from* $(1, -2)$.

While Method 1 functions well, it is somewhat artificial in that we are using a "formula" that doesn't match the data we are given. That is why Method 1 is a two-step method.

Method 2, on the other hand, used exactly the information we were given: the slope and the coordinates of *any* one point on the line. Because it applies in the more general setting, we summarize from Method 2 a "formula" for an equation of the line passing through a given point with a given slope.

The point-slope form of a linear equation in two variables

An equation for the line with slope m and passing through the point with coordinates (x_0, y_0) is given by

$$y - y_0 = m(x - x_0).$$

This is known as the *point-slope form of a line.* As indicated in Method 2 of the last example, it derives from the definition of the slope, where we have incorporated the step of "canceling the denominator" into the formula.

Unlike the slope-intercept form of a line, which is useful because we can "read off" geometric data from the equation, the point-slope form of a line is almost exclusively used as a "formula" to find an equation for a line, where values of m, x_0 and y_0 are substituted to obtain an equation involving x and y.

In the remaining examples, we will use the point-slope form of the line to find an equation for the given line.

Example 5.2.11. *Find an equation for the line passing through the points with coordinates* $(4, 1)$ *and* $(-2, 5)$*.*

Answer. *Unlike the previous examples in this section, this time we are not given the slope. Fortunately, since we have the coordinates of two points on the line, we can use the definition to find the slope.*

Step 1: Find the slope *Labelling*

$$(\overset{x_1}{4}, \overset{y_1}{1}), \quad (\overset{x_2}{-2}, \overset{y_2}{5}),$$

we substitute into the definition:

$$\begin{aligned} m &= \frac{(5) - (1)}{(-2) - (4)} \\ &= \frac{4}{-6} \\ &= -\frac{2}{3}. \end{aligned}$$

Step 2: Use the point-slope formula *We now have* $m = -2/3$*. We can choose the coordinates of either of the given points to use in the point-slope formula; let's use the first. Labeling,*

$$(\overset{x_0}{4}, \overset{y_0}{1}),$$

we can substitute into the point-slope formula and solve for y in terms of x:

$$\begin{array}{rcrcrcl} y & - & (1) & = & (-\frac{2}{3})(x & - & (4)) \\ y & - & 1 & = & (-\frac{2}{3})(x & - & 4) \\ y & - & 1 & = & (-\frac{2}{3})(x) & - & (-\frac{2}{3})(4) \\ y & - & 1 & = & -\frac{2}{3}x & - & (-\frac{8}{3}) \\ y & - & 1 & = & -\frac{2}{3}x & + & \frac{8}{3} \\ & + & 1 & \vdots & & + & 1 \\ \hline y & & & = & -\frac{2}{3}x & + & \frac{11}{3}. \end{array}$$

(Notice that solving for y in terms of x amounts to writing the answer in slope-intercept form.)

Since this is the first example of its type, let's verify that the result does not depend on which of the two points we choose. If we had instead chosen the second point, we would have obtained

$$\begin{array}{ccccc} & x_0 & & y_0 & \\ (& -2 & , & 5 &). \end{array}$$

We can now substitute into the point-slope formula and solve for y in terms of x:

$$\begin{array}{rcrcrcl} y & - & (5) & = & (-\frac{2}{3})(x & - & (-2)) \\ y & - & 5 & = & (-\frac{2}{3})(x & + & 2) \\ y & - & 5 & = & \left(-\frac{2}{3}\right)(x) & + & \left(-\frac{2}{3}\right)(2) \\ y & - & 5 & = & -\frac{2}{3}x & + & \left(-\frac{4}{3}\right) \\ y & - & 5 & = & -\frac{2}{3}x & - & \frac{4}{3} \\ & + & 5 & \vdots & & + & 5 \\ \hline y & & & = & -\frac{2}{3}x & + & \frac{11}{3}. \end{array}$$

While the equation looked different immediately after substituting into the point-slope formula, the slope-intercept form of the equation is the same.

The answer, in slope-intercept form, is $y = -\dfrac{2}{3}x + \dfrac{11}{3}$.

The last two examples of this subsection will rely on the the following translation of geometric facts into the language of slopes. Recall that two lines in a plane are **parallel** if they have no point of intersection; two lines in a plane are **perpendicular** if they intersect at right angles. These geometric definitions can be translated (with some work) into algebraic facts by means of the slope.

Parallel and perpendicular lines described by slope

- Two lines are **parallel** if they have *the same slopes.*
- Two lines are **perpendicular** if *the product of their slopes is* -1.

In algebraic terms, suppose two lines have slopes m_1 and m_2. If the lines are parallel, then $m_1 = m_2$. If the lines are perpendicular, then $m_2 = -1/m_1$ (where $m_1 \neq 0$). (It might be helpful to think of $m_2 = -1/m_1$ in words: "m_2 is the opposite of the reciprocal of m_1.")

Example 5.2.12. *Find an equation for the line passing through the point with coordinates* $(-3, 2)$ *and which is parallel to the line* $x + 6y = 1$.

Answer. *We are not given the slope of the line in question. However, we are given the equation of a parallel line. Let's find the slope of the parallel line, then use the same slope for the line in question.*

Step 1: Find the slope of the parallel line.

Since we are given an equation for the parallel line, let's rewrite it in slope-intercept form:

$$\begin{array}{rcrcrcr} x & + & 6y & = & & & 1 \\ -x & & & \vdots & -x & & \\ \hline & & 6y & = & -x & + & 1 \\ & & \frac{6y}{6} & = & \frac{-x+1}{6} & & \\ & & y & = & \frac{-x}{6} & + & \frac{1}{6} \\ & & y & = & -\frac{1}{6}x & + & \frac{1}{6}. \end{array}$$

The slope of the parallel line is $-1/6$.

Step 2: Use the point-slope formula.

We will substitute $m = -1/6$ *(using the* ***same slope*** *as the parallel line) and the coordinates of the given point*

$$\begin{array}{ccccc} & x_0 & & y_0 & \\ (& -3 & , & 2 &) \end{array}$$

into the point-slope formula, and solve for y in terms of x.

$$\begin{array}{ccccccc}
y & - & (2) & = & (-\frac{1}{6})(x & - & (-3)) \\
y & - & 2 & = & (-\frac{1}{6})(x & + & 3) \\
y & - & 2 & = & \left(-\frac{1}{6}\right)(x) & + & \left(-\frac{1}{6}\right)(3) \\
y & - & 2 & = & -\frac{1}{6}x & + & \left(-\frac{3}{6}\right) \\
y & - & 2 & = & -\frac{1}{6}x & - & \frac{1}{2} \\
 & + & 2 & \vdots & & + & 2 \\
\hline
y & & & = & -\frac{1}{6}x & + & \frac{3}{2}.
\end{array}$$

The answer, in slope-intercept form, is $y = -\dfrac{1}{6}x + \dfrac{3}{2}$.

Example 5.2.13. *Find an equation for the line passing through the point with coordinates $(3, 5)$ which is perpendicular to the line $3x - 2y = 12$.*

Answer. *Again, we are not given the slope of the line in question.*

Step 1: Find the slope of the perpendicular line.

We rewrite the equation of the perpendicular line in slope-intercept form:

$$\begin{array}{cccccc}
3x & - & 2y & = & & 12 \\
-3x & & & \vdots & -3x & \\
\hline
 & & -2y & = & -3x & + \; 12 \\
 & & \frac{-2y}{-2} & = & \frac{-3x+12}{-2} & \\
 & & y & = & \frac{-3x}{-2} & + \; \frac{12}{-2} \\
 & & y & = & \frac{3}{2}x & - \; 6.
\end{array}$$

The slope of the perpendicular line is $3/2$.

Step 2: Use the point-slope formula.

For the slope of the line in question, we will use the ***opposite of the reciprocal*** *of the slope of the perpendicular line: we will substitute $m = -2/3$ along with and the coordinates of the given point*

$$\begin{array}{ccccc}
 & x_0 & & y_0 & \\
(& 3 & , & 5 &)
\end{array}$$

into the point-slope formula:

$$\begin{array}{rrrcrrr} y & - & (5) & = & (-\frac{2}{3})(x & - & (3)) \\ y & - & 5 & = & (-\frac{2}{3})(x & - & 3) \\ y & - & 5 & = & \left(-\frac{2}{3}\right)(x) & - & \left(-\frac{2}{3}\right)(3) \\ y & - & 5 & = & -\frac{2}{3}x & - & \left(-\frac{6}{3}\right) \\ y & - & 5 & = & -\frac{2}{3}x & + & 2 \\ & + & 5 & \vdots & & + & 5 \\ \hline y & & & = & -\frac{2}{3}x & + & 7. \end{array}$$

The answer, in slope-intercept form, is $y = -\dfrac{2}{3}x + 7$.

Notice that in none of the examples in this section were we asked to graph the lines in question. Having done the work of writing their equations in slope-intercept form, however, doing so would have not been much extra effort.

5.2.3 Special cases: Horizontal and vertical lines

In the beginning of our discussion of linear equations in two variables, we mentioned that such an equation (in variables x and y) could always be written in the form $Ax + By = C$, where A, B, and C are constants. We did not specify that these constants were not 0 (although if they are *both* 0, the equation is no longer a linear equation!). In the case that either A or B is zero, the corresponding term is "missing," and it appears that the equation has only one variable. However, the context determines whether we should consider the equation in a one-variable setting or a two-variable setting.

Let's start with the case of horizontal lines. When we first introduced the slope concept, we specified that horizontal lines should have slope 0.

Example 5.2.14. *Find an equation of the horizontal line with slope* 0 *and passing through the point with coordinates* $(-3, -7)$.

Answer. *We have been given exactly the information needed to use the point-slope formula. So we will substitute and solve for* y *in terms of* x.

$$\begin{array}{rrrcrrr} y & - & y_0 & = & m(x & - & x_0) \\ y & - & (-7) & = & (0)(x & - & (-3)) \\ y & + & 7 & = & (0)(x & + & 3) \\ y & + & 7 & = & 0 & & \\ & - & 7 & \vdots & -7 & & \\ \hline y & & & = & -7. & & \end{array}$$

The answer is $y = -7$. Notice that although the example was clearly stated in the setting of two variables (an ordered pair was given!), only one variable appears in the equation describing the line.

Let's consider the equation $y = -7$ from the previous example more carefully. A solution to this equation, which in this context will be an ordered pair (x, y), must make the equation $y = -7$ after substituting its coordinates into the equation. However, there is no place to substitute x-values. In other words, the equation $y = -7$ imposes no restrictions at all on x! A table might look like:

x	y	Solution
$\boxed{0}$	-7	$(1, -7)$
$\boxed{-4}$	-7	$(-4, -7)$
$\boxed{29}$	-7	$(29, -7)$
$\boxed{-0.717}$	-7	$(-0.717, -7)$
$\boxed{3}$	-7	$(3, -7)$

Whatever x value we choose, the equation requires that the y-coordinate be -7.

Turning our attention to vertical lines, we immediately run into the problem that a vertical line does not have a slope (roughly speaking, the slope of a vertical line is "infinite"). Because of this, our strategy of relying on the point-slope formula would lead nowhere.

However, our discussion of a table of values for solutions to an equation in two variables with one variable missing still applies.

Example 5.2.15. *Graph the equation $x = -1$ in an xy-plane.*

Answer. *We will make a table to find solutions. Since the equation $x = -1$ does not involve the variable y, we will be free to choose any value of for y. However, the only x-value that will make the equation true will be -1. One possible table might be:*

x	y	*Solution*
-1	$\boxed{0}$	$(-1, 0)$
-1	$\boxed{-2}$	$(-1, -2)$
-1	$\boxed{1}$	$(-1, 1)$

Plotting these three solutions and drawing the line through them, we obtain Figure 5.13.

Notice that the line given by the equation $x = -1$ is a vertical line.

There are some obvious patterns in the previous two examples, which we can summarize as follows:

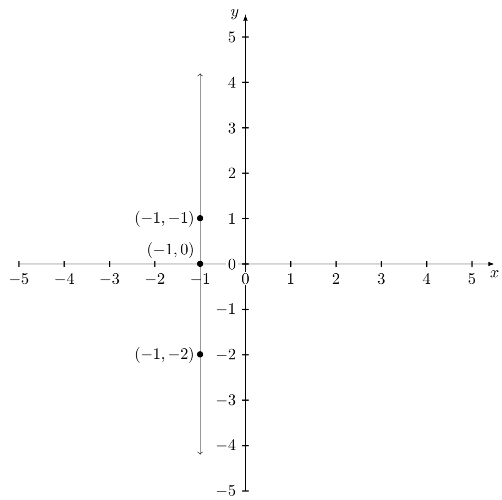


Figure 5.13: All solutions to $x = -1$.

Horizontal and vertical lines

- An equation for a *horizontal* line passing through the point with coordinates (a, b) is $y = b$.
- An equation for a *vertical* line passing through the point with coordinates (a, b) is $x = a$.

For future reference, it is worth remembering two special cases of this pattern in an xy-plane:

- An equation for the x-axis is $y = 0$.
- An equation for the y-axis is $x = 0$.

5.2.4 Exercises

1. Find the slope of the lines in an xy-plane described by the following information:

   (a) Passing through the points $(-2, 4)$ and $(1, -2)$.
   (b) Passing through the points $(0, -3)$ and $(4, 5)$.
   (c) Passing through the points $(-1, 4)$ and $(-1, -2)$.
   (d) Having equation $x - 3y = 4$.

(e) Having equation $2x + 3y = -6$.

(f) Having equation $5x - y = 2$.

(g) Having equation $y = 2x - 1$.

(h) Having equation $y = \frac{1}{3}x - 2$.

(i) Having equation $y = -\frac{3}{4}x + 1$.

(j) Having equation $y = 4$.

(k) Having equation $y = -x$

2. Find the slope and y-intercept of the line given by the equation $y = -\frac{3}{4}x + 1$.

3. Find the slope and y-intercept of the line given by the equation $5x - y = 2$.

4. Find an equation of the line having slope $3/4$ and passing through the point $(3, -2)$.

5. Find an equation of the line passing through the points $(2, -1)$ and $(5, 1)$.

6. Find an equation of the line passing through the point $(4, -2)$ and parallel to the line given by $3x - 4y = 6$.

7. Find an equation of the line passing through the point $(1, 0)$ and perpendicular to the line given by $x + 4y = 2$.

The following exercises give an alternate method to approach problems of the type in Examples 5.2.12 and 5.2.13.

8. (*) Show that for any values of A, B, C_1, C_2, $(A \neq 0)$ the line described by the equations $Ax + By = C_1$ is parallel to the line described by $Ax + By = C_2$.

9. Use the result of the previous exercise to find an equation of the line parallel to $3x + 5y = 8$ and passing through the point with coordinates $(2, -3)$.

10. (*) Show that for any values of A, B, C_1, C_2, $(A, B \neq 0)$ the line described by the equations $Ax + By = C_1$ is perpendicular to the line described by $-Bx + Ay = C_2$.

11. Use the result of the previous exercise to find an equation of the line perpendicular to $-x + 5y = 7$ and passing through the point with coordinates $(-1, 5)$.

5.3 Solving linear inequalities in two variables

We will approach linear inequalities in two variables in the same way as we approached linear inequalities in one variable. The reader should review Section 4.4.2 on one-variable inequalities briefly before proceeding; just like in that section, we will outline two approaches to solving two-variable inequalities.

As we have seen, a solution to a linear inequality in two variables is a value for each of the two variables which, when substituted into the inequality, make the inequality true. As in the case of linear equations in two variables, we will represent a solution with an ordered pair.

Let's look at an example: $x + y < 3$. Given any ordered pair, we can test to see whether or not it is a solution by substituting and evaluating. For example, $(3, 4)$ is not a solution since $(3) + (4) < 3$ is false. On the other hand, $(0, 1)$ is a solution, since $(0) + (1) < 3$ is true. You should check that $(-1, 1)$, $(2, -3)$ and $(0, 0)$ are also solutions to $x + y < 3$, while $(3, 3)$ and $(1, 2)$ are not solutions. After checking these ordered pairs, it is not hard to believe that the inequality has infinitely many solutions—as well as infinitely many ordered pairs which are not solutions.

As usual in the case when we have infinitely many solutions, we will attempt to draw a graph to represent all the solutions. However, plotting the solutions (and non-solutions) to the inequality $x + y < 3$ shows that coming up with a "pattern" will take a little more thought, see Figure 5.14.

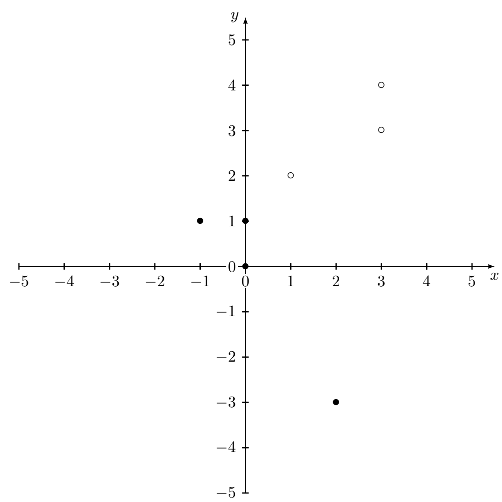


Figure 5.14: Four solutions (•) and three non-solutions (∘) to $x + y < 3$.

The key to seeing a pattern here is to take a step back and remember that solutions to a linear *equation* all lie on a line. Points not on the line do not represent solutions to the linear equation—or, equivalently, represent solutions to a linear *inequality*. In other words, if an ordered pair (a, b) is not a solution to the equation $Ax + By = C$ (and so the corresponding point is not on the line given by $Ax+By=C$), then the ordered pair (a,b) is a solution to $Ax+By \neq C$.

Now there are two ways that the inequality $Ax+By \neq C$ can be true: either $Ax+By<C$ is true, or $Ax+By>C$ is true. It is an important fact about an xy-plane that *all points representing solutions to $Ax+By<C$ lie on the same side of the line $Ax+By=C$ in an xy-plane, and all points representing solutions to $Ax+By>C$ lie on the other side of the same line.* Figure 5.15 is the same as Figure 5.14, except with the "border" line $x+y=3$ indicated. (Notice that the ordered pair $(1,2)$, which is not a solution to $x+y<3$, is represented by a point on the border line described by $x+y=3$.)

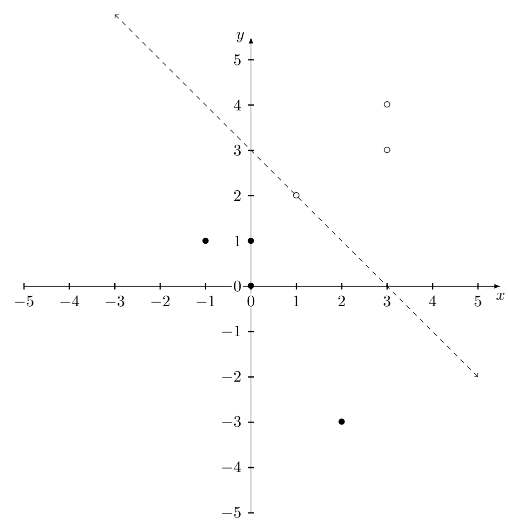


Figure 5.15: Solutions (•) and non-solutions (∘) to $x+y<3$, with border line $x+y=3$.

We can summarize the above discussion as follows: *The graph of all solutions to a typical linear inequality in two variables will consist of all points on one side of a line in an xy-plane. The border line will not (or will) be included depending on whether the inequality is strict (or not).*

Our strategy to solve a linear inequality in two variables will then be the following:

General strategy to solve linear inequalities in two variables

To solve a linear inequality in two variables:

1. Draw the border line. Use a dotted line for strict inequalities (so that points on the border line do not represent solutions) or a solid line for non-strict inequalities (so that the border points do represent solutions).

2. Shade the side of the border line that consists of solutions.

As in Section 4.4.2, we will discuss two methods to decide which side of the border line to shade.

Method 1: Test point method

The idea of this method is to choose any point in the xy-plane *not on the border line*. Test whether the chosen point represents a solution to the inequality. If it does represent a solution, shade all points on the *same side* of the border line as the test point. If it does not, shade all points on the *opposite side* of the border line.

We will give three examples using this method.

Example 5.3.1. *Graph the solutions of* $x - 3y < 6$.

Answer. *The first step is to graph the border line represented by* $x - 3y = 6$*; notice that we will draw the border as a* dotted *line since the inequality is strict (and the points on the border do not represent solutions to the inequality). To do that, we can use either of our methods for graphing linear equations. We list here a possible table of values to find two solutions:*

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
6	$\boxed{0}$	$(6, 0)$

Now we choose a test point to determine which side of the border line to shade. Let's choose one with coordinates $(1, 1)$*. To test it, we substitute these coordinates into the original inequality* $x - 3y < 6$*:*

$$\begin{array}{rcrcl} (1) & - & 3(1) & < & 6 \\ 1 & - & 3 & < & 6 \\ & & -2 & < & 6. \end{array}$$

The inequality is true, and so $(1, 1)$ *is a solution. We shade all points on the* same side *of the border line as the one representing* $(1, 1)$ *to represent all solutions of* $x - 3y < 6$*. See Figure 5.16.*

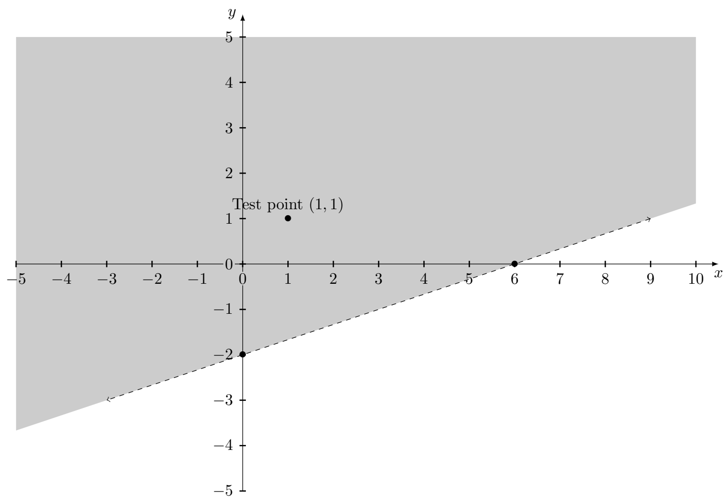


Figure 5.16: All solutions of $x - 3y < 6$.

Example 5.3.2. *Graph the solutions of* $2x + 5y \geq 10$.

Answer. *We first graph the border line represented by* $2x + 5y = 10$*; we will draw the line as a* solid *line since the inequality is non-strict, and so points on the border do represent solutions to the inequality. In order to graph the border line, we might use the following table of values:*

x	y	*Solution*
$\boxed{0}$	2	$(0, 2)$
5	$\boxed{0}$	$(5, 0)$

Now we choose a test point; this time let's choose the origin, with coordinates $(0, 0)$. *Substituting these coordinates into the original inequality* $2x + 5y \geq 10$,

$$\begin{array}{ccccc} 2(0) & + & 5(0) & \geq & 10 \\ 0 & + & 0 & \geq & 10 \\ & & 0 & \geq & 10. \end{array}$$

The inequality is false; $(0, 0)$ *is not a solution to* $2x + 5y \geq 10$. *We shade all points on the* opposite side *of the border line as the origin to represent all solutions of* $2x + 5y \geq 10$. *See Figure 5.17.*

Example 5.3.3. *Graph all solutions of* $y < -\dfrac{1}{3}x + 1$.

Answer. *First, as always, we graph the border line represented by* $y = -\dfrac{1}{3}x + 1$. *We will draw it using a dashed line since the inequality is strict. This time, since the equation is written in slope-intercept form, we see that the y-intercept has*

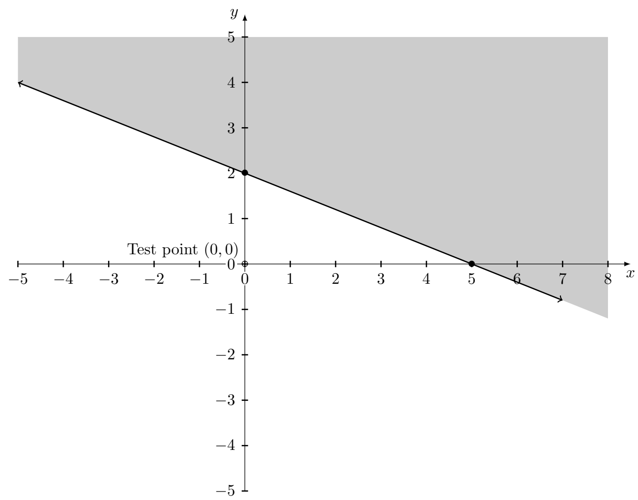


Figure 5.17: All solutions of $2x + 5y \geq 10$.

coordinates $(0, 1)$. *A second solution can be obtained by "moving" down one unit and to the right three units to give* $(3, 0)$.

Now we choose a test point; since zero is a nice number to work with let's choose the origin with coordinates $(0, 0)$ *again. To decide whether it is a solution, we substitute into* $y < -\dfrac{1}{3}x + 1$*:*

$$\begin{array}{ccccc} (0) & < -\frac{1}{3}(0) & + & 1 & \\ 0 & < & 0 & + & 1 \\ 0 & < & 1. & & \end{array}$$

The inequality is true; $(0, 0)$ *is a solution of* $y < -\dfrac{1}{3}x + 1$. *We will shade all points on the* same side *of the border line as the origin* $(0, 0)$. *See Figure 5.18.*

Method 2: Standard form method

Many students look at a few examples of linear inequalities and try to find patterns, or "shortcuts," to the test point method. "Wouldn't it be great," someone might say, "if every 'less than' inequality had a graph shaded below the border line! Then I don't have to waste my time with test points." However, look back at Examples 5.3.1 and 5.3.3; if there is a pattern, it is not so simple. In fact, since the an inequality can be written in so many equivalent forms, there is really no hope for an easy "shortcut."

However, if we make the effort of writing the inequality in a standard form, it is possible to make "rules" for which side of the border line to shade. Here is an example of such rules:

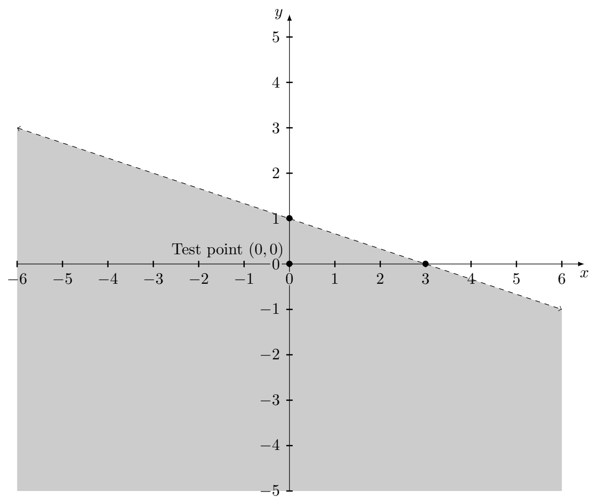


Figure 5.18: All solutions of $y < -\dfrac{1}{3}x + 1$.

1. The points representing solutions to a linear inequality of the form $y < mx + b$ (or $y \leq mx + b$) lie *below* the border line given by $y = mx + b$.

2. The points representing solutions to a linear inequality of the form $y > mx + b$ (or $y \geq mx + b$) lie *above* the border line given by $y = mx + b$.

Notice what is "standard" about this standard form: The y variable is by itself on the *left* side of the inequality. As with linear equalities in one variable, where the standard form consisted of having the x variable by itself on the left side of the inequality, if the inequality is not in standard form, we can use our basic addition and multiplication principals to rewrite the inequality in standard form. Keep in mind that as always, multiplying or dividing both sides of an inequality by a negative quantity requires changing the sense of the inequality.

Example 5.3.3 already gave an example of these standard form rules, since in that example the y variable was already by itself on the left side. Notice in Figure 5.18 that the shaded region is *below* the border line, as the rules for standard form dictate for the inequality $<$.

Here are two more examples illustrating the standard form method for graphing linear inequalities in two variables.

Example 5.3.4. *Graph the solutions of* $3x + 4y \geq 12$.

Answer. *In this case, the inequality is not in our standard form. We solve for* y*:*

$$\begin{array}{rcrcrcr} 3x & + & 4y & \geq & & & 12 \\ -3x & & & \vdots & -3x & & \\ \hline & & 4y & \geq & -3x & + & 12 \\ & & \frac{4y}{4} & \geq & \frac{-3x+12}{4} & & \\ & & y & \geq & \frac{-3x}{4} & + & \frac{12}{4} \\ & & y & \geq & -\frac{3}{4}x & + & 3. \end{array}$$

Notice that at no point did we divide by a negative number; the sense of the inequality $\geq$ *does not change.*

One advantage of the standard form we have chosen is that the equation of the border line $y = -\dfrac{3}{4}x + 3$ *is in slope-intercept form. The* y*-intercept has coordinates* $(0, 3)$ *and the slope is* $-\dfrac{3}{4}$*, so the coordinates of a second point on the line is obtained by "moving" down three units and to the right four units from* $(0, 3)$*, giving* $(4, 0)$*. We draw the border line with a solid line since the inequality* $\geq$ *is not strict.*

Since the inequality in standard form is $\geq$*, we will shade* above *the border line. See Figure 5.19.*

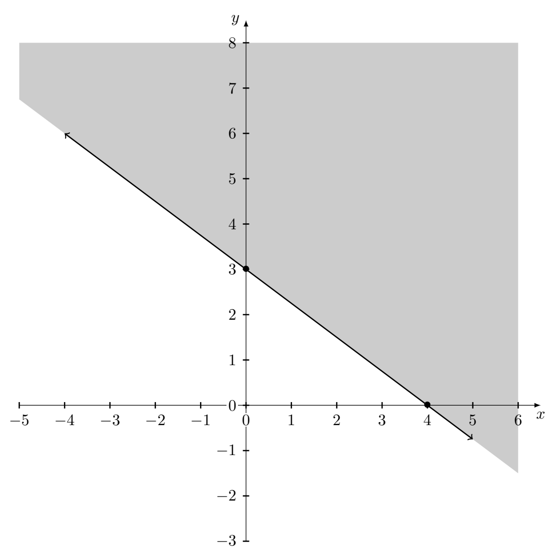


Figure 5.19: All solutions of $3x + 4y \geq 12$.

Example 5.3.5. *Graph the solutions of* $4x - 2y > 5$.

Answer. *We will first write the inequality in standard form:*

$$\begin{array}{ccccccc} 4x & - & 2y & > & & & 5 \\ -4x & & & \vdots & -4x & & \\ \hline & & -2y & > & -4x & + & 5 \\ & & \frac{-2y}{-2} & < & \frac{-4x+5}{-2} & & \\ & & y & < & \frac{-4x}{-2} & + & \frac{5}{-2} \\ & & y & < & 2x & - & \frac{5}{2}. \end{array}$$

This time, when we divided by -2 on the fourth line, the sense of the inequality changes from $>$ to $<$.

The border line, represented by $y = 2x - \dfrac{5}{2}$, has y-intercept $\left(0, -\dfrac{5}{2}\right)$ and slope $m = 2 = \dfrac{2}{1}$. We can obtain a second point by starting from $\left(0, -\dfrac{5}{2}\right)$ and "moving" up 2 units and to the right 1 unit to give $\left(1, -\dfrac{1}{2}\right)$. (Notice that $-5/2 = -2.5$) We will draw the border line with a dashed line since the original inequality $>$ is strict.

Even though the original inequality was $>$, in standard form the inequality changed to $<$ (when we divided by a negative number). For that reason, we will shade below *the border line. See Figure 5.20.*

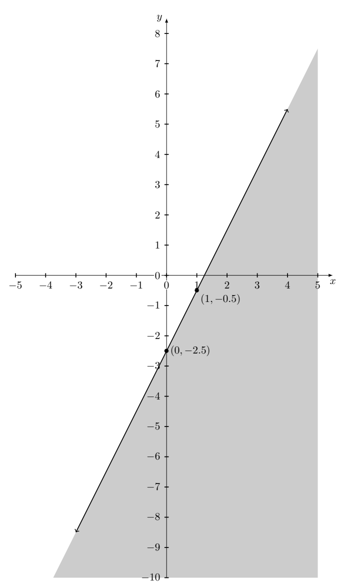


Figure 5.20: All solutions of $4x - 2y > 5$.

5.3.1 Exercises

Solve the following linear inequalities in two variables. In each case, graph all solutions and list five individual solutions.

1. $-x - y > 6$
2. $2x + 5y \leq 10$
3. $3x - 2y \geq 12$
4. $-4x + y > 4$
5. $y \geq -\frac{1}{2}x + 4$
6. $y < 1$

5.4 Solving systems of linear equations

A *system* of equations represents a situation where a solution must make all of several equations true, as opposed to just one equation. In this section we will consider only systems of two linear equations in two unknowns. A solution to such a system will be an ordered pair which, when substituted, makes both equations true.

For example, the following is a typical system of linear equations:

$$\begin{cases} 2x & + & 5y & = & 13 & \circledcirc \\ x & - & 2y & = & 2 & \circledast \end{cases} \tag{5.1}$$

There are a few things to notice about our notation in writing systems of linear equations:

- The system is indicated by the symbol $\{$. This indicates that a solution must make both equations true. (Caution: Not every text uses this symbol.)

- We write both equations is the general form $Ax + By = C$, where all variable terms are on the left side of the equations and all constant terms are on the right side. (If an equation is not written in this form, it can be rewritten as an equivalent one in the general form by using the addition principle.)

- We have written the equations so that like terms are in the same "column," with x-terms written above x-terms and y-terms written above y-terms.

- We use the symbols $\circledcirc$ and $\circledast$ to represent the two equations. For example, in this case, "Equation $\circledcirc$," or just $\circledcirc$, will refer to the equation $2x + 5y = 13$.

Let's look at some potential solutions for System (5.1). The reader should check the validity of the statements below:

- $(9, -1)$ is a solution to Equation $\circledcirc$, but $(9, -1)$ is not a solution to Equation $\circledast$. So $(9, -1)$ IS NOT a solution to System (5.1).

- $(6, 2)$ is a NOT solution to Equation $\circledcirc$, but $(6, 2)$ is a solution to Equation $\circledast$. So $(6, 2)$ IS NOT a solution to System (5.1).

- $(4, 1)$ is a solution to Equation $\circledcirc$, and $(4, 1)$ is not a solution to Equation $\circledast$. So $(4, 1)$ IS a solution to System (5.1).

- $(0, 0)$ is a not solution to Equation $\circledcirc$, and $(0, 0)$ is not a solution to Equation $\circledast$. So $(0, 0)$ IS NOT a solution to System (5.1).

The preceding paragraph should convince the reader that to find solutions to a system of equations, it is not enough to solve the two equations separately. At this point, we found one solution to System (5.1), but we can't be sure that it is the *only* solution. To do that, we need a method to solve systems of linear equations.

Before discussing a general method, we can again let geometry give us a guide as to what to expect. We know, for example, that Equation $\circledcirc$ has infinitely many solutions, which form a line when plotted in an xy-plane. We also know that Equation $\circledast$ also has infinitely many solutions, which form a different line when plotted in an xy-plane. So, if we graph both equations in the same xy-plane, a point will represent a solution to *both* equations if it lies on both lines—in other words, if it is a point of intersection of the two lines. But we know from elementary geometry that two non-parallel lines have exactly one point of intersection. So we have the following conclusion:

> A typical system of two linear equations in two variables will have exactly one solution. The solution, when plotted on an xy-plane, represents the point of intersection of the lines represented by the two equations.

In fact, this discussion already gives one method to solve a system of linear equations: Graph both equations on the same xy-plane, and the solution will be the coordinates of the point of intersection. However, this method requires a high degree of accuracy in plotting, and we will not generally rely on this method to solve systems of linear equations.

There is another, more algebraic way to solve systems of linear equations. Beginning with one of the equations, we could solve for one of the variables, say y, in terms of the other variable x. Then we could substitute this expression for y in terms of x into the second equation to obtain a new equation in just one variable x. This new linear equation (in one variable) will typically have one solution. Substituting this solution into the first equation (for y in terms of x) will give a corresponding value of y. The solution will be an ordered pair consisting of the solutions for x and y.

The method in the preceding paragraph is sometimes known (for obvious reasons) as the *substitution method*. Despite the fact this method applies to a wide variety of systems beyond those that we are considering here, we will not pursue this method any further. In many situations it requires detailed calculations with fractions, which as it turns out can be avoided in most cases we will encounter.

Solving systems of linear equations: Elimination method

We are going to outline a method for solving systems of two linear equations in two variables x and y, both of which have *integer* coefficients for both variables (this can always be arranged using the method of Section 4.2.4). We will arrive at the method by considering several examples, from simpler to more general.

Example 5.4.1. *Solve:*

$$\begin{cases} x + y = 4 & \circledcirc \\ x - y = 1 & \circledast \end{cases} \tag{5.2}$$

Answer. *Looking at System (5.2), we can notice that the y-terms have a special form in Equations ⊚ and ⊛: they are "opposites," in the sense that their coefficients (1 and −1) have the same magnitude but opposite sign. Let's apply the addition principle, which as a reminder states that we can add the same quantity to both sides of an equation without changing the solutions. For a solution to Equation ⊛, both sides are equal, so we will "add Equation ⊛ to Equation ⊚," meaning add the left sides and right sides of the equations.*

Eliminate y:

$$\begin{array}{ccccccc} x & + & y & = & 4 & & \circledcirc \\ x & - & y & = & 1 & & \circledast \\ \hline 2x & & & = & 5 & & \circledcirc + \circledast \end{array}$$

Notice that the new equation, which we denote ⊚ + ⊛, is an equation in one variable, with solution $5/2$. We have learned so far that if (x, y) is a solution to System (5.2), then x must have the value $5/2$.

What is the corresponding y-value for the solution? One way to find this would be to substitute the x-value $5/2$ into either Equation ⊚ or ⊛ to obtain a new equation in one variable y and then solve. However, let's stay in the spirit of "elimination."

The preceding step of eliminating y worked so well because the original coefficients of y were so nice. If we wanted to eliminate x, adding the equations directly does not work, as we just saw. While the coefficients of x, which are both 1, do have the same magnitude, they have the same sign, and so are not "opposites."

But why not, instead of adding the two equations, subtract them—or, what is the same, add the opposite of Equation ⊛ to Equation ⊚?

Eliminate x:

$$\begin{array}{cccccccc} x & + & y & = & 4 & \circledcirc & & \\ -x & + & y & = & -1 & \circledast & \times & (-1) \\ \hline & & 2y & = & 3 & \circledcirc & - & \circledast \end{array}$$

Notice that we multiplied every term on both sides *of Equation ⊛ by -1. We represent this with the notation ⊛ × (-1).*

So after adding to obtain ⊚ − ⊛ (which is the same as ⊚ + (-1) × ⊛), we obtain the equation in one variable $2y = 3$, which has solution $3/2$. This tells us that if (x, y) is a solution to System (5.2), then y must have the value $3/2$.

From our preceding discussion, we expect that System (5.2) has one solution. We conclude:

The solution to System (5.2) is $\left(\frac{5}{2}, \frac{3}{2}\right)$.

It is worth pointing out from this first example that we did encounter fractions in our solution, even though the system involved equations without fractional coefficients. This is completely normal. However, we did not encounter fractions until the very last step of each elimination, and in fact, we never had to perform operations with these fractions. As we will see, this is typical for systems with integer coefficients and a major advantage of the elimination method.

From our first example, we can already see the outlines of the elimination method: Combine the two equations in such a way that one of the variables is "eliminated" in order to find a value for the other variable. Then repeat the process, eliminating the other variable to find the value for the remaining unknown. The solution is the ordered pair formed by the two values obtained in this way.

What remains to investigate is exactly how to combine the equations in such a way that one variable is always eliminated. The next example is a step in that direction.

Example 5.4.2. *Solve:*

$$\begin{cases} 3x & + & y & = & 9 & \circledcirc \\ x & - & 2y & = & -6 & \circledast \end{cases} \tag{5.3}$$

Answer. *In this example, unfortunately, the coefficients of neither variable are "opposites." In fact, neither adding nor subtracting the equations will eliminate either of the variables this time.*

However, we don't give up hope. Notice that even though the coefficients of y are not opposites, at least they have opposite signs! If there was only a way to change the equations in such a way that the magnitudes were equal...

Actually, the notation we used in the first example already had the clue to a way around this problem. If we multiply both sides of Equation $\circledcirc$ by 2, then the new y term will be opposite that of the y-term in Equation $\circledast$; adding the resulting equations will eliminate y!

Eliminate y:

$$\begin{array}{cccccccc} 6x & + & 2y & = & 18 & \circledcirc & \times & 2 \\ x & - & 2y & = & -6 & \circledast & & \\ \hline 7x & & & = & 12 & 2\times\circledcirc & + & \circledast \end{array}$$

After eliminating y, we obtain an equation in just one variable (x) whose solution is $12/7$. The conclusion is that if (x, y) is a solution to System (5.3), then x must be $12/7$.

Turning now to the y-coordinate of the solution, we want to eliminate x. This time, the coefficients of x not only have different magnitudes (1 and 3), but they have the same sign. They are far from being opposites. A little thought, though, can convince us that again, we already have the idea of how to cope with this: why not multiply Equation $\circledast$ by the negative *number -3. That way, the resulting coefficients of x will have the same magnitudes but opposite signs:*

Eliminate x:

$$\begin{array}{rcrcrcr} 3x & + & y & = & 9 & & \circledcirc \\ -3x & + & 6y & = & 18 & & \circledast \times (-3) \\ \hline & & 7y & = & 27 & & \circledcirc \quad + \quad (-3) \times \circledast \end{array}$$

In the resulting equation $\circledcirc + (-3) \times \circledast$*, we have eliminated* x *to obtain an equation in one variable* y *with solution* $27/7$*.*

The solution to System (5.3) is

$$\left(\frac{12}{7}, \frac{27}{7}\right).$$

One thing should be clear from these examples so far: Be careful of signs when multiplying both sides by a negative number!

Example 5.4.3. *Solve:*

$$\begin{cases} 2x + 5y = 13 & \circledcirc \\ x - 2y = 2 & \circledast \end{cases} \tag{5.4}$$

Answer. *This is the example that we used in the opening of the section (System (5.1). We already saw the solution at that time, when we were checking if various ordered pairs were solutions. Now we will apply the elimination method to actually find the solution "from scratch."*

In looking at the system, we see that we can eliminate x *exactly as in the previous example. We will multiply Equation* $\circledast$ *by* -2 *(notice that the coefficients of* x *initially have the same sign, so we need to multiply by a negative number in order to make the resulting coefficients "opposite."*

Eliminate x:

$$\begin{array}{rcrcrcr} 2x & + & 5y & = & 13 & & \circledcirc \\ -2x & + & 4y & = & -4 & & \circledast \times (-2) \\ \hline & & 9y & = & 9 & & \circledcirc \quad + \quad (-2) \times \circledast \end{array}$$

The equation $9y = 9$ *has* 1 *as a solution, so the* y*-coordinate of the solution to System (5.4) is* 1*.*

When we turn to eliminating y*, we encounter a new problem. The good news is that the coefficients of* y *(*5 *and* -2*) have opposite signs. But there is no way to multiply just one of the equations by an integer to make the coefficients of* y *"opposites," as we need to eliminate* y*.*

It turns out that the way around this difficulty is not hard: we will use the multiplication principle on both *equations. First, we find a common multiple of the magnitudes* 2 *and* 5*. That is, we find an integer that both* 2 *and* 5 *divide evenly. The least common multiple of* 2 *and* 5 *is* 10*. (The reader might notice that finding a common multiple of* 2 *and* 5 *is exactly the same mental process as finding a common denominator*[4] *for two fractions with denominators* 2 *and* 5*.)*

[4] In fact, a common denominator is just a common multiple of the denominators.

Once the common multiple 10 *is found, we will multiply both equations by a number so that the magnitude of the coefficient of* y *is* 10*. That is, in this case, we will multiply Equation* ⊚ *by* 2 *and Equation* ⊛ *by* 5*.*

Eliminate y*:*

$$\begin{array}{rcrcrccc} 4x & + & 10y & = & 26 & \circledcirc & \times & 2 \\ 5x & - & 10y & = & 10 & \circledast & \times & 5 \\ \hline 9x & & & = & 36 & 2\times\circledcirc & + & 5\times\circledast \end{array}$$

The resulting equation $9x = 36$ *has* 4 *as a solution, so the* x*-coordinate of the solution to System (5.4) is* 4*.*

Putting this together with the result of the previous elimination step, we find that the solution to System (5.4) is $(4, 1)$*.*

With the three preceding examples as guides, we can write down a general method that describes the "elimination" that is at the heart of the elimination method.

> **In order to eliminate a variable from a system of linear equations**:
>
> - Find a common multiple of the magnitudes of the coefficients of the variable to be eliminated;
> - If the coefficients of the desired variable originally had different signs, multiply each equation by a positive number so that the magnitude of the coefficients of the desired variable in the resulting equations is the common multiple;
> - If the coefficients of the desired variable originally had the same sign, multiply one equation by a positive number and one equation by a negative number so that the magnitude of the coefficients of the desired variable is the common multiple.
>
> After these preparations, adding the resulting equations will result in a new equation that does not involve the variable to be eliminated.

The next example illustrates the general method.

Example 5.4.4. *Solve:*

$$\begin{cases} 6x + 4y = 16 & \circledcirc \\ 9x - 5y = 7 & \circledast \end{cases} \tag{5.5}$$

Answer. *To eliminate* x*, we see that the least common multiple of the coefficients of* x *(*6 *and* 9*) is* 18*. Since the signs of the coefficients are the same, we will multiply one equation (say Equation* ⊛*) by a negative number. Specifically, we will multiply Equation* ⊚ *by* 3 *and Equation* ⊛ *by* -2*:*

Eliminate x:

$$\begin{array}{rcrcrll} 18x & + & 12y & = & 48 & \circledcirc \times 3 & \\ -18x & + & 10y & = & -14 & \circledast \times (-2) & \\ \hline & & 22y & = & 34 & 3 \times \circledcirc & + \ (-2) \times \circledast \end{array}$$

The solution to the resulting equation $22y = 34$ *is* $17/11$ *(after reducing), so the* y*-coordinate of the solution to System (5.5) is* $17/11$

Now to eliminate y*, we see that the least common multiple of the magnitudes of the coefficients of* y *in System (5.5) (*4 *and* -5*) is* 20*. Since the signs of the coefficients are already different, we will multiply both equations by positive numbers to achieve the common multiple. Specifically, we will multiply Equation* $\circledcirc$ *by* 5 *and Equation* $\circledast$ *by* 4*:*

Eliminate y:

$$\begin{array}{rcrcrll} 30x & + & 20y & = & 80 & \circledcirc \times 5 & \\ 36x & - & 20y & = & 28 & \circledast \times 4 & \\ \hline 66x & & & = & 108 & 5 \times \circledcirc & + \ 4 \times \circledast \end{array}$$

The solution to the equation $66x = 108$ *is* $18/11$ *(after reducing), so the* x*-coordinate of the solution to System (5.5) is* $18/11$.

Together with the first elimination step, we see that the solution to System (5.5) is

$$\left(\frac{18}{11}, \frac{17}{11}\right).$$

5.4.1 Systems that do not have exactly one solution

By thinking of a system of two linear equations in two unknowns graphically, we came to the conclusion that a "typical" such system will have exactly one solution, just like a "typical" linear equation in one variable will have exactly one solution. However, keeping in mind Section 4.2.3, we might expect that not every system is "typical."

To see what might go wrong, consider following example.

Example 5.4.5. *Solve:*

$$\begin{cases} x & + & 2y & = & 9 & \circledcirc \\ 3x & + & 6y & = & 10 & \circledast \end{cases} \tag{5.6}$$

Answer. *We will apply our elimination method, as usual.*

To eliminate x*, we see that the least common multiple of the coefficients of* x *(*1 *and* 3*) is* 3*. The signs of the coefficients are the same, so we will multiply one equation (say Equation* $\circledcirc$*) by a negative number. Specifically, we will multiply Equation* $\circledcirc$ *by* -3 *and Equation* $\circledast$ *by* 1*:*

Eliminate x:

$$\begin{array}{rcrcrl} -3x & - & 6y & = & -27 & \circledcirc \times (-3) \\ 3x & + & 6y & = & 10 & \circledast \times 1 \\ \hline & & 0 & = & -17 & (-3) \times \circledcirc \; + \; \circledast \end{array}$$

Although we were aiming to eliminate x, both variables were eliminated in the resulting equation!

As in Section 4.2.3, the question in such cases is whether the new equation is true or false.

Since the equation $0 = -17$ is false, System (5.6) has no solution.

What went "wrong" in the previous example? Why did our elimination procedure end up eliminating both variables, instead of the "typical" one variable at a time?

To investigate Example 5.2.12 more closely, let's rewrite both equations in slope-intercept form. Solving Equation $\circledcirc$ for y gives:

$$\begin{array}{rcrcrcl} x & + & 2y & = & & & 9 \\ -x & & & \vdots & -x & & \\ \hline & & 2y & = & -x & + & 9 \\ & & \frac{2y}{2} & = & \frac{-x+9}{2} & & \\ & & y & = & \frac{-x}{2} & + & \frac{9}{2} \\ & & y & = & -\frac{1}{2}x & + & \frac{9}{2}. \end{array}$$

We see that the slope of the line given by Equation $\circledcirc$ is $-1/2$ and the y-intercept is $(0, 9/2)$.

Turning to Equation $\circledast$, we solve for y:

$$\begin{array}{rcrcrcl} 3x & + & 6y & = & & & 10 \\ -3x & & & \vdots & -3x & & \\ \hline & & 6y & = & -3x & + & 10 \\ & & \frac{6y}{6} & = & \frac{-3x+10}{6} & & \\ & & y & = & \frac{-3x}{6} & + & \frac{10}{6} \\ & & y & = & -\frac{1}{2}x & + & \frac{5}{3}. \end{array}$$

The slope of the line given by Equation $\circledast$ is $-1/2$ and the y-intercept is $(0, 5/3)$.

Comparing, we see that the two lines represented by Equations $\circledcirc$ and $\circledast$ have the same slope, but different y-intercepts. In other words, the Equations represent parallel lines!

In fact, this shouldn't be a big surprise. Our geometric thinking that led us to the conclusion that the typical system of two linear equations in two variables had a single solution was that two different lines in a plane typically intersect in one point—except when the two lines are parallel, in which case they have no point of intersection.

Keeping in mind that we saw two "unusual" situations in Section 4.2.3, let's look at one last example.

Example 5.4.6. *Solve:*

$$\begin{cases} 2x & - & y & = & 5 & \circledcirc \\ 4x & - & 2y & = & 10 & \circledast \end{cases} \tag{5.7}$$

Answer. *Let's eliminate x first. The least common multiple of the coefficients of x (2 and 4) is 4. The signs of the coefficients are the same, so we will multiply one equation (say Equation $\circledcirc$) by a negative number. Specifically, we will multiply Equation $\circledcirc$ by -2 and Equation $\circledast$ by 1:*

Eliminate x:

$$\begin{array}{rcrcrl} -4x & + & 2y & = & -10 & \circledcirc \times (-2) \\ 4x & + & 2y & = & 10 & \circledast \times 1 \\ \hline & & 0 & = & 0 & (-2) \times \circledcirc \; + \; \circledast \end{array}$$

Again, we have eliminated both variables. This time, though, the resulting equation is true.

In the one-variable situation in Section 4.2.3, this would have led us to conclude that all real numbers were solutions. However, in this case, not every ordered pair is a solution. For example, the reader can check that $(0,0)$ is not a solution to System (5.7).

To understand what the resulting true equation is telling us, let's again rewrite the equations in slope-intercept form to see what some geometry can tell us.

Solving Equation $\circledcirc$ for y gives:

$$\begin{array}{rcrcrcrcr} 2x & - & y & = & & & 5 \\ -2x & & & \vdots & -2x & & \\ \hline & & -y & = & -2x & + & 5 \\ & & \frac{-y}{-1} & = & \frac{-2x+5}{-1} & & \\ & & y & = & \frac{-2x}{-1} & + & \frac{5}{-1} \\ & & y & = & 2x & - & 5. \end{array}$$

The slope of the line given by Equation $\circledcirc$ is 2 and the y-intercept is $(0,-5)$.

Solving Equation $\circledast$ *for* y *gives:*

$$\begin{array}{rcrcrcrcr}
4x & - & 2y & = & & & 10 \\
-4x & & & \vdots & -4x & & \\
\hline
& & -2y & = & -4x & + & 10 \\
& & \frac{-2y}{-2} & = & \frac{-4x+10}{-2} & & \\
& & y & = & \frac{-4x}{-2} & + & \frac{10}{-2} \\
& & y & = & 2x & - & 5.
\end{array}$$

The slope of the line given by Equation $\circledast$ *is* 2 *and the* y*-intercept is* $(0, -5)$*.*

Notice that the two equations represent lines with the same slope *and the* same y-intercept*—they actually represent the same line.*

In other words, System ([5.7](#)) *has* infinitely many solutions*, all of which are represented by the points on the line given by either Equation* $\odot$ *or Equation* $\circledast$*. We could graph the solutions: See Figure* [5.21](#)*.*

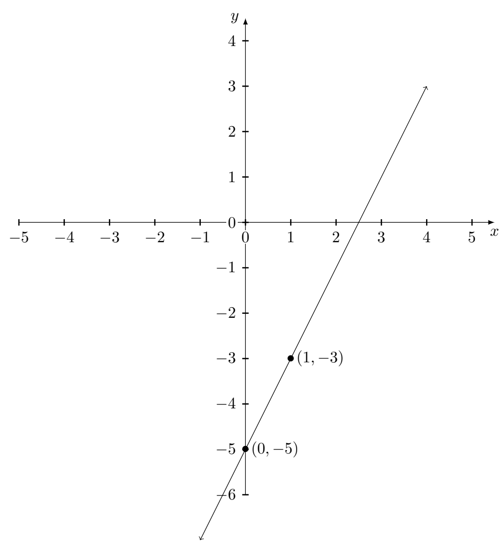


Figure 5.21: All solutions of

$$\begin{cases} 2x - y = 5 \\ 4x - 2y = 10 \end{cases}$$

The unusual cases in this section are detected using the elimination method when both variables are eliminated. We can summarize our results as follows:

Suppose that, in using the elimination method to solve a system of two linear equations in two unknowns, an equation results which involves neither of the two variables.

- If the resulting equation is false, the system has no solution. The lines represented by the two equations are parallel lines.
- If the resulting equation is true, the system has infinitely many solutions, represented by the points on the graph of either equation. The lines represented by the two equations are the same.

5.4.2 Exercises

Solve the following systems of linear equations.

1. $\begin{cases} x - y = 8 \\ 2x + y = 1 \end{cases}$

2. $\begin{cases} 3x - 2y = -1 \\ 2x + y = -3 \end{cases}$

3. $\begin{cases} x - 2y = 4 \\ 5x + 6y = 3 \end{cases}$

4. $\begin{cases} 2x - 3y = 6 \\ x + 4y = 8 \end{cases}$

5. $\begin{cases} 3x + 2y = 4 \\ -x + 5y = 10 \end{cases}$

6. $\begin{cases} 4x - y = 6 \\ 2x + 3y = 8 \end{cases}$

7. $\begin{cases} -2x + y = 5 \\ 4x - 2y = 8 \end{cases}$

8. $\begin{cases} y = 2x - 5 \\ y = \dfrac{x+4}{2} \end{cases}$

5.5 Chapter summary

- A typical linear equation in two variables has infinitely many solutions. When graphed on an xy-plane, the points corresponding to solutions of a linear equation in two variables form a line.
- The most basic strategy to graph all solutions of a linear equation in two variables is to plot two solutions, then draw the line passing through these two.
- An alternate method to graph all solutions to a linear equation is to use the slope of the line and the coordinates of one point on the line. This is most useful when the equation is written in slope-intercept form.
- Given the slope of a line and the coordinates of one point on the line, the point-slope form of a line gives a "formula" to write an equation of the line.
- Linear inequalities in two variables typically have infinitely many solutions. The points corresponding to these solutions in an xy-plane all lie on the same half the xy-plane with border line given by the corresponding linear equation.
- A typical system of two linear equations in two variables has one solution.

Part III

Polynomials

Chapter 6

Polynomials

Vocabulary

- Term
- Polynomial
- Coefficient
- Degree of a term
- Degree of a polynomial
- Leading term
- Descending order
- Like terms
- Scientific notation
- Distributive law

6.1 Introduction to polynomials

Up to this point, we have been looking at algebra from the point of view of solving equations and inequalities. Indeed, this is a major point of distinction between arithmetic and algebra. In arithmetic, there is no such thing as a conditional statement: every equation or inequality is either true or false. In algebra, a typical equation or inequality may be true or false, depending on the values of the variables involved. For that reason, the word "solve," in the sense of finding all solutions, only has meaning in the context of algebra.

There is another way of looking at algebra, however, that has nothing to do with solving equations or inequalities. In this view, algebra is a kind of "arithmetic of symbols." Variables, which up to this point have been treated as unknown numbers, will be viewed as symbols which can themselves by added, subtracted, multiplied or divided according to fixed rules. These rules should, of course, correspond to the rules of arithmetic of numbers when values are substituted for the variables. For instance, we would like the commutative and associative laws of addition and multiplication to apply, and we would like multiplication to distribute over addition.

We have already seen this arithmetic of symbols in the course of solving linear equations when we "combined like terms." In this chapter, we carry out this symbolic arithmetic in a more general setting.

We will start by setting up the terminology that we will use throughout this chapter.

Terms

A *term* is an algebraic expression which is not itself written as the sum (or difference) of two or more expressions. It may involve products or quotients of constants or variables.

An algebraic expression can be written as a sum of terms. In this case, a *term* is a quantity which appears in an algebraic expression as part of a sum (a "summand").

Phrased differently, terms are expressions which are *added*.

For example, the algebraic expression

$$x^2 + 4x + 5$$

has three terms: x^2, $4x$, and 5. One might say that "terms are separated by the addition symbol (+)."

From now on, it will be especially important to distinguish between subtraction and "adding the opposite." For example, we will consider the expression

$$x^3 - 3x^2 - 5x - 4$$

as being

$$x^3 + (-3x^2) + (-5x) + (-4).$$

In particular, this expression has four terms (which are added): x^3, $-3x^2$, $-5x$ and -4.

A word of caution in our terminology. We will encounter more complicated expressions like

$$(x - 3)(x^2 + 5x + 6).$$

This expression has only one term! It is formed by two grouped expressions, $(x-3)$ and (x^2+5x+6), which are *multiplied*, not added. In a similar way, the expression

$$(2x+3)(4x-1)+(3x+2)(x-5)$$

has two terms: the first term is $(2x+3)(4x-1)$ and the second term is $(3x+2)(x-5)$. The point is that terms can be quite complicated, but they must appear as part of a sum.

A polynomial will be an algebraic expression having a particular form.

Polynomials in one variable

A *polynomial* in one variable (say x) is an algebraic expression, which can be written in such a way that each of its terms has the form

$$ax^n,$$

where a represents any number and n represents a whole number.

(It is easy to make a definition for polynomials with more than one variable. For example, a polynomial in two variables x and y should have terms of the form ax^my^n, where both m and n are whole numbers.)

The most important feature of a polynomial is the exponent of the variable part of each term. To say that the exponent must be a whole number means, for instance, that the exponent cannot be negative, nor can it be fractional.

Here are some examples of polynomials:

- x^2-6x-7;
- $\dfrac{3x-5}{2}$;
- t^5+1;
- $x^7+x^6+x^5+x^4+x^3+x^2+x+1$.

Here are some examples of algebraic expressions that are not polynomials:

- $\dfrac{1}{x}$;
- x^{-2} (we will see what this means shortly);
- $\dfrac{x+1}{x^2-3x+5}$;
- $\sqrt{x}$;

- $2^x + x^2$.

Exercise 6.1.1. *For each of the examples listed above of algebraic expressions which are not polynomials, identify what it is about them that prevents them from being considered polynomials.*

Notice that each term of a polynomial is formed by multiplying a "number part," called the *coefficient*, with a "variable part." The variable part is completely described by the exponents of the variables involved.

Degree

For a polynomial in one variable, the *degree of a term* is the exponent of the variable part of the term. The *degree of a polynomial* is the highest degree of any of its terms.

Pay attention to the fact that this definition is really *two* definitions: the degree of a term is (usually) different than the degree of the polynomial in which it appears.

In the case of polynomials in more than one variable, the definition requires a little more care. In that case, the degree of a term is defined to be the sum of all the exponents having a variable base. So the degree of the term $-5x^2y$ would be 3.

Notice that a polynomial of degree one corresponds to what we referred to in the preceding chapter as *linear*.

Exercise 6.1.2. *For each of the polynomials below, identify the terms. For each term, identify the coefficient and the degree. Then determine the degree of the polynomial.*

(a) $x^2 - 6x - 7$;

(b) $\dfrac{3x-5}{2}$;

(c) $t^5 + 1$;

(d) $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

A polynomial can be classified by the number of terms it involves. For example, a *monomial* is a polynomial with one term, a *binomial* is a polynomial with two terms, and a *trinomial* is a polynomial with three terms.

Normally, we will write a polynomial in *descending order*: terms with higher degree will be written to the left of terms with lower degree. In case a polynomial is not written in descending order, we will take the trouble to rewrite it by using the commutative law of addition, as the following example illustrates. The terms with the highest degree is called the *leading term*.

Example 6.1.3. *Write the polynomial* $4x^3 - 5x^6 + 6 - 3x + x^2$ *in descending order.*

Answer. *The polynomial not written in descending order. For example, a degree* 3 *terms* $(4x^3)$ *is written to the left of a degree* 6 *term* $(-5x^6)$.

There are five terms: $4x^3$, $-5x^6$, 6, $-3x$ *and* x^2. *We arrange them in descending order, from highest degree term (the degree* 6 *term) to the lowest degree term (the degree* 0 *term).*

Written additively, the polynomial is written

$$(-5x^6) + (4x^3) + (x^2) + (-3x) + (6),$$

where we have used parentheses only to highlight the separate terms. More simply, the polynomial would be written in descending order as

$$-5x^6 + 4x^3 + x^2 - 3x + 6.$$

The above example emphasizes again the importance of the ability to interchange between subtracting and "adding the opposite."

6.1.1 Exercises

Decide whether or not the following algebraic expressions are polynomials.

1. $x^{100} - \pi x^2$

2. $6x - 18x^3$

3. $\dfrac{9x+2}{4}$

4. $x^2 + x + 3 - \frac{2}{x^2}$

5. $5^x - 2^x$

For each of the following polynomials, (a) list the terms; (b) for each term, identify the degree of the term and the coefficient of the term; (c) rewrite the polynomial in descending order; and (d) identify the degree of the polynomial.

6. $3 - x$

7. $5 + x^2 - 3x$

8. $1 - x + x^2 - x^3 + \dfrac{x^4}{2}$

6.2 Adding and subtracting polynomials

With this introduction to the terminology of polynomials, we now proceed to the "arithmetic of polynomials" that we discussed in the introduction. In other words, we will combine two polynomials using the basic operations of addition, subtraction, multiplication, and division, to obtain (most of the time!) a new polynomial.

The key idea to understand how to add and subtract polynomials is the idea of "like terms." We have already seen this concept several times. We saw it in the course of solving linear equations: we combined (meaning added or subtracted) variable terms with variable terms and constant terms with constant terms. We even saw the idea earlier: it is the reason why, when adding fractions, that we need to have a common denominator.

The basic idea of combining like terms can be seen by looking at some simple examples using whole number coefficients, where we can represent multiplication as repeated addition. For example, $3x^4$ simply means $x^4 + x^4 + x^4$. Thinking in this way, we could represent the sum $3x^4 + 2x^4$ as

$$(x^4 + x^4 + x^4) + (x^4 + x^4),$$

which (by virtue of the associative law) is the same as $x^4 + x^4 + x^4 + x^4 + x^4$, or just $5x^4$.

The equation[1] $3x^4 + 2x^4 = 5x^4$ has another justification, which is better suited to explaining a general rule. Applying the distributive law[2],

$$3x^4 + 2x^4 = (3 + 2)x^4 = 5x^4.$$

The advantage of this way of looking at the sum is that it holds for *any* coefficients, not just for whole number coefficients.

Of course, both of the above approaches depended on the fact that both terms being added were "x^4-terms." If we tried to apply either of the above strategies to the sum $3x^4+2x^3$, both would fail. In fact, these two terms cannot be "combined" at all.

The essence of the above discussion can be summarized in the following points:

- Two terms are called *like terms* if they have the same variable part. In other words, the terms should involve the same variables, and each variable should have the same exponent.

- Like terms can be *combined* by adding (or subtracting) their coefficients. Symbolically, $ax^n + bx^n = (a + b)x^n$. Notice the variable part does not change as a result of adding like terms.

[1]Notice that this equation, unlike most of the equations we saw in Chapter 2, is an *identity*: it is true for all values of x.

[2]Thinking of the variable as an (unknown) number, the distributive law needs no extra justification. Considering the variable as a symbol, however, the distributive law technically must be "extended" to apply in the setting of variables as well as of numbers.

- Terms that are not like terms cannot be combined. Their sum must be represented by two (unlike) terms.

6.2.1 Adding polynomials

With this said, addition of polynomials is nothing more than "combining like terms."

Example 6.2.1. *Add:* $(3x^2 - 2x + 1) + (4x^2 + 6x - 7)$.

Answer. *The parentheses in this problem are grouping symbols, written to emphasize that we are adding two polynomials,* $3x^2 - 2x + 1$ *and* $4x^2 + 6x - 7$. *However, thanks to the associative and commutative properties of addition, the parentheses have no impact whatsoever on the problem: we can rearrange and group the terms any way we like. In particular,*

$$\begin{aligned} &(3x^2 - 2x + 1) + (4x^2 + 6x - 7) && \\ &3x^2 - 2x + 1 + 4x^2 + 6x - 7 && \textit{Removing parentheses} \\ &(3x^2 + 4x^2) + (-2x + 6x) + (1 - 7) && \textit{Grouping like terms} \\ &7x^2 + 4x - 6 && \textit{Combining like terms} \end{aligned}$$

The answer is $7x^2 + 4x - 6$.

It is common practice when adding polynomials to take advantage of "column notation." Just like when adding numbers with many digits, different columns represent different "place values" (powers of ten), columns can be used in adding polynomials so that different columns represent different like terms. We will always write polynomials in descending order when using column notation.

For example, in the previous example, we can write

$$\begin{array}{rrrrrrr} & 3x^2 & - & 2x & + & 1 \\ + & 4x^2 & + & 6x & - & 7 \\ \hline & 7x^2 & + & 4x & - & 6 \end{array}$$

(We will many times not write the + symbol in front of the second polynomial when using column notation, with addition being assumed.)

Notice also using column notation that all like terms are *added*. So in the second column, for example, we are adding the terms $-2x$ and $6x$. Speaking loosely, we could say "the minus sign applies to the coefficient of the term."

Using column notation, it is especially important to pay attention to "missing terms," as the following example illustrates.

Example 6.2.2. *Add:* $(x^4 - 5x^2 + 2x - 6) + (x^3 - 8x - 3)$.

Answer. *Writing the sum in column notation,*

$$\begin{array}{ccccccccc} x^4 & & & - & 5x^2 & + & 2x & - & 6 \\ & & x^3 & & & - & 8x & - & 3 \\ \hline x^4 & + & x^3 & - & 5x^2 & - & 6x & - & 9 \end{array}$$

Notice again that

- *The addition of the two polynomials is not explicitly written; it is assumed that the polynomials in the two rows are added;*
- *The columns are always added, with a minus sign considered as the sign of the coefficient of the term involved.*

The answer is $x^4 + x^3 - 5x^2 - 6x - 9$.

Example 6.2.3. *Add:* $(x^3 - x^2 + 4x - 7) + (3x^2 - 4x + 1)$.

Answer. *Rewriting the sum in column notation:*

$$\begin{array}{ccccccc} x^3 & - & x^2 & + & 4x & - & 7 \\ & & 3x^2 & - & 4x & + & 1 \\ \hline x^3 & + & 2x^2 & & & - & 6 \end{array}$$

Notice that in the third column, the sum of $4x$ *and* $-4x$ *is* $0x$*, which is simply* 0 *(this property of* 0*, familiar in the setting of numbers, extends to variables as well). Since adding* 0 *does not change the quantity being added, we do not need to write the* 0 *term, unless it is the only term in the polynomial remaining.*

The answer is $x^3 + 2x^2 - 6$.

6.2.2 Subtracting polynomials

To subtract polynomials, we will follow the same strategy that we used to subtract signed numbers: we will think of subtraction as "adding the opposite." We only need to think carefully of what we mean by the opposite of a polynomial.

We have used the word "opposite" in the sense that two numbers are opposites if their sum is zero. We will use the word in exactly the same way for polynomials: *two polynomials are opposites if their sum is zero.*

Example 6.2.4. *The following are examples of polynomials which are opposites:*

- *The opposite of* $2x - 4$ *is* $-2x + 4$.
- *The opposite of* $x^3 - 6x^2 - 7x + 2$ *is* $-x^3 + 6x^2 + 7x - 2$.
- *The opposite of* $1 - t^3$ *is* $t^3 - 1$.

Exercise 6.2.5. *For each pair of polynomials in the previous example, add the polynomials to show that their sum is zero in order to confirm that the polynomials are opposites.*

In the case of numbers, we used the symbol $-$ to represent "the opposite of." So -2 means the same as "the opposite of 2," and $-(-6)$ means the same as "the opposite of -6."

We will use the same understanding of the $-$ symbol in the case of polynomials. Rewriting the example above, we have

- $-(2x-4)$ means "the opposite of $2x-4$," or $-2x+4$.
- $-(x^3-6x^2-7x+2)$ means "the opposite of x^3-6x^2-7x+2," or $-x^3+6x^2+7x-2$.
- $-(1-t^3)$ means "the opposite of $1-t^3$," or t^3-1.

From even these few examples, you should be able to see an important pattern that we will always use in practice: *To find the opposite of a polynomial, change the sign of the coefficient of every term in the polynomial.*

With this background, we can interpret subtraction of polynomials.

Subtraction of polynomials

To subtract two polynomials, add the first polynomial to the opposite of the second polynomial.

Example 6.2.6. *Subtract:* $(3x-5)-(2x+1)$.

Answer. *We rewrite the subtraction problem as "adding the opposite:"*

$$(3x-5)+(-2x-1).$$

Notice that the first polynomial remains the same; we add the opposite of the second polynomial, which was originally $2x+1$.

Now, combining like terms,

$$\begin{aligned}(3x-5)+(-2x-1)&\\ 3x-5+(-2x)+(-1)&\\ [3x+(-2x)]+[-5+(-1)]&\qquad \textit{Grouping like terms}\\ x-6&\qquad \textit{Combining like terms}\end{aligned}$$

The answer is $x-6$.

Example 6.2.7. *Subtract:* $(t^3-2t^2+t-5)-(2t^3+t-4)$.

Answer. *Rewriting,*

$$(t^3-2t^2+t-5)+(-2t^3-t+4).$$

In column notation, we add

$$\begin{array}{rcrcrcr} t^3 & - & 2t^2 & + & t & - & 5 \\ -2t^3 & & & - & t & + & 4 \\ \hline -t^3 & - & 2t^2 & & & - & 1 \end{array}$$

The answer is $-t^3 - 2t^2 - 1$.

As a final example, we remind the reader that when a subtraction is indicated by a sentence of the form, "Subtract X from Y," the first quantity appears second in the difference, as $Y - X$. Unlike addition, subtraction is not commutative—the order that we write the terms does affect the outcome.

Example 6.2.8. *Subtract* $x^2 - 2x - 5$ *from* $8x + 2$.

Answer. *Translated into algebra, the problem asks to perform the following subtraction problem:*

$$(8x + 2) - (x^2 - 2x - 5).$$

Now, rewriting as an addition problem,

$$\begin{array}{rl} (8x + 2) + (-x^2 + 2x + 5) & \\ 8x + 2 + (-x^2) + 2x + 5 & \\ -x^2 + (8x + 2x) + (2 + 5) & \quad \textit{Grouping like terms} \\ -x^2 + 10x + 7 & \quad \textit{Combining like terms} \end{array}$$

The answer is $-x^2 + 10x + 7$.

6.2.3 Exercises

Perform the indicated operations.

1. $(x^2 - 5x - 6) + (2x^2 + 2x + 4)$
2. $(5x^3 + 2x^2 - 3x + 3) + (-x^3 - 3x^2 + 2x + 4)$
3. $(y^2 + 5y - 1) - (-3y^2 + 2y - 4)$
4. $(x^3 - x^2 - x + 1) - (4x^3 - 3x^2 + 2x + 4)$
5. $(3x^3 - 2x + 1) - (-x^3 - 3x^2 + 2x + 4)$
6. Subtract $4x + 2$ from $-x + 15$
7. Subtract $x^2 - 4x - 1$ from $2x^2 - 2x + 5$

6.3 Properties of exponents

Up to this point, the exponents with a variable base appearing in polynomials have mainly served to distinguish between like and unlike terms. Going further, however, we will need to pay more attention to how the terms involving exponents interact when multiplied or divided.

The following table summarizes key properties of exponents. Here, in general, x and y represent bases, which will be either a number or a variable. The exponents a and b will for now represent numbers, but we will be a little vague here about exactly what kind of numbers they are (in the case of polynomials, the exponents will be whole numbers).

Properties of exponents

(E1) $x^a \cdot x^b = x^{a+b}$.

(E2) $\dfrac{x^a}{x^b} = x^{a-b}$ (as long as $x \neq 0$).

(E3) $x^0 = 1$ (as long as $x \neq 0$).

(E4) $(x \cdot y)^a = x^a \cdot y^a$.

(E5) $(x^a)^b = x^{a \cdot b}$.

(E6) $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ (as long as $y \neq 0$).

Before we provide examples to show these properties "in action," it is worth making some comments about these properties.

- Notice that the properties all involve the operations of multiplication or division. One might say, "Exponents behave nicely with multiplication and division." The interaction between exponents and addition (or subtraction) is more complicated, as we will see below.
- The key feature of properties (E1) and (E2) is that the factors being multiplied or divided have *the same base* (denoted by x).
- The key feature of properties (E4), (E5), and (E6) is that the base of the exponential on the left hand side involves only the operations of multiplication, division, or exponentiation.
- Most of the properties (with the exception of (E3)) are easy to justify when the exponents are positive whole numbers. In that case, writing

exponents as "repeated multiplication," the properties follow directly from the commutative and associative properties of multiplication along with the definition of division as an inverse operation to multiplication.

- Property (E3) is of a different nature than the others; for this reason, it is sometimes hardest to justify. The reason is that it has no interpretation as a "repeated multiplication," since the phrase, "Multiply the base x by itself 0 times" is meaningless. Instead, (E3) follows from formally extending property (E2), in the following sense. Consider the expression $\frac{x^a}{x^a}$, where for the moment we will consider a to be a positive whole number and x any nonzero number (so that the denominator is not zero!). On the one hand, any nonzero number divided by itself is 1: $\frac{x^a}{x^a} = 1$. On the other hand, if we insist that (E2) must hold, we have $\frac{x^a}{x^a} = x^{a-a} = x^0$. For this reason, if (E2) is to hold, the only way to define x^0 in a consistent way is $x^0 = 1$, as in (E3).

Properties (E1)–(E6) are most useful when the bases involved are variables. In the following examples, we will use the common word "simplify" to mean "use the relevant properties of exponents to write in an equivalent, simpler form."

Example 6.3.1. *Simplify:* $(w^5x^8)(w^2x^3)$.

Answer. *The only operation involved is multiplication, so we can change the order and grouping of the factors at will, relying on the commutative and associative properties of multiplication.*

$$\begin{aligned} &(w^5x^8)(w^2x^3) \\ &(w^5w^2)(x^8x^3) && \textit{Grouping factors with the same base} \\ &(w^{5+2})(x^{8+3}) && \textit{Property (E1)} \\ &w^7x^{11}. \end{aligned}$$

The answer is w^7x^{11}. Notice the the bases of the remaining exponentials (w and x) are different, and so no further simplification is possible.

Example 6.3.2. *Simplify:* $\frac{x^3y^5}{x^3y}$.

Answer. *This time we have an expression involving multiplication and division. The strategy will be the same: grouping factors with the same base.*

$$\frac{x^3y^5}{x^3y}$$
$$\left(\frac{x^3}{x^3}\right)\cdot\left(\frac{y^5}{y}\right) \quad \textit{Grouping factors with the same base}$$
$$(x^{3-3})(y^{5-1}) \quad \textit{Property (E2)}$$
$$x^0y^4$$
$$1\cdot y^4 \quad \textit{Property (E3)}$$
$$y^4.$$

The answer is y^4. *Notice that the in the factor* $\frac{y^5}{y}$, *the exponent of the denominator* y *is* 1.

Example 6.3.3. *Simplify:* $\left(\frac{x^8y^4}{x^2}\right)^3$.

Answer. *In this case, we will simplify the expression inside the grouping symbols first.*

$$\left(\frac{x^8y^4}{x^2}\right)^3$$
$$\left(\frac{x^8}{x^2}\cdot\frac{y^4}{1}\right)^3 \quad \textit{Grouping factors with the same base}$$
$$(x^{8-2}\cdot y^4)^3 \quad \textit{Property (E2)}$$
$$(x^6y^4)^3$$
$$(x^6)^3\cdot(y^4)^3 \quad \textit{Property (E4)}$$
$$x^{6\cdot3}y^{4\cdot3} \quad \textit{Property (E5)}$$
$$x^{18}y^{12}.$$

The answer is $x^{18}y^{12}$.

Example 6.3.4. *Simplify:* $(3x^2y^7)^4$.

Answer. *In this example, the base is a product of three factors:* 3, x^2, *and* y^7. *Property (E4), applied to this situation, implies that each factor separately must be raised to the fourth power*[3].

$$(3x^2y^7)^4$$
$$(3)^4\cdot(x^2)^4\cdot(y^7)^4 \quad \textit{Property (E4)}$$
$$81\cdot x^{2\cdot4}\cdot y^{7\cdot4} \quad \textit{Property (E5)}$$
$$81x^8y^{28}.$$

[3]Property (E4) is stated for a base which is the product of two factors. However, in the case of three factors, we can apply the property twice: $(x\cdot(y\cdot z))^a = x^a\cdot(y\cdot z)^a = x^a\cdot y^a\cdot z^a$.

The answer is $81x^8y^{28}$.

Example 6.3.5. *Simplify:* $\dfrac{(2xy^4)^3}{x^2y^5}$.

Answer. *We will first simplify the numerator, then separate the factors according to common bases.*

$$\begin{aligned}
&\frac{(2xy^4)^3}{x^2y^5} \\
&\frac{(2)^3(x)^3(y^4)^3}{x^2y^5} && \textit{Property (E4)} \\
&\frac{8x^3y^{12}}{x^2y^5} && \textit{Property (E5)} \\
&\left(\frac{8}{1}\right)\cdot\left(\frac{x^3}{x^2}\right)\cdot\left(\frac{y^{12}}{y^5}\right) && \textit{Grouping factors with the same base} \\
&(8)(x^{3-2})(y^{12-5}) && \textit{Property (E2)} \\
&8xy^7.
\end{aligned}$$

The answer is $8xy^7$. *Notice that in the last step, as usual, we do not write the exponent* 1*:* $x^1 = x$. *In the grouping step, we grouped the whole number* 8 *in the numerator, writing it as a fraction* 8/1 *for the sake of seeing the multiplication more clearly.*

6.3.1 Integer exponents

The properties of exponents listed above are enough for most of the work we will do with polynomials. However, it is worth pointing out that they also give a way to define negative exponents in a way consistent with our understanding of exponents as repeated multiplication, similar to the way that the definition $x^0 = 1$ is required if the other properties are to be satisfied.

Namely, we make the following definition:

For any nonzero base $x \neq 0$ and any exponent a, we define

$$x^{-a} = \frac{1}{x^a}.$$

In particular,

$$x^{-1} = \frac{1}{x}.$$

To see that this definition is consistent with the properties above, notice that on the one hand, $x^{0-a} = x^{-a}$. On the other hand, if properties (E2) and (E3) are to hold, we have $x^{0-a} = \dfrac{x^0}{x^a} = \dfrac{1}{x^a}$. In other words, to be consistent with the standard properties of whole number exponents, the only possible definition for negative exponents is the one we have stated, $x^{-a} = 1/x^a$.

Since this definition is less intuitive than our usual understanding of whole number exponents, we list some numerical examples of this definition.

Example 6.3.6. *Find the values of each of the following exponentials.*

(a) 3^{-1}

(b) 2^{-5}

(c) 10^{-4}

(d) $(-2)^{-3}$

(e) $\left(\dfrac{2}{5}\right)^{-1}$

(f) $\left(\dfrac{3}{4}\right)^{-2}$

Answer. *(a)* $3^{-1} = \dfrac{1}{3^1} = \dfrac{1}{3}$.

(b) $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$.

(c) $10^{-4} = \dfrac{1}{10^4} = \dfrac{1}{10000} = 0.0001$.

(d) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$.

(e) $\left(\dfrac{2}{5}\right)^{-1} = \dfrac{1}{\left(\frac{2}{5}\right)^1} = \dfrac{1}{\left(\frac{2}{5}\right)} = \dfrac{1}{1} \cdot \dfrac{5}{2} = \dfrac{5}{2}$.

(f) $\left(\dfrac{3}{4}\right)^{-2} = \dfrac{1}{\left(\frac{3}{4}\right)^2} = \dfrac{1}{\frac{9}{16}} = \dfrac{1}{1} \cdot \dfrac{16}{9} = \dfrac{16}{9}$.

The previous examples provide evidence that negative exponents do not affect the sign of the result, but instead indicate a reciprocal. This is not surprising if we think of exponents as repeated multiplication; the "opposite" sign indicates the "opposite" in the sense of multiplication, which is the notion of reciprocal. Note especially in the last example that $\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$.

Since we have defined negative exponents in such a way to be consistent with the familiar properties of exponents, we can manipulate and simplify expressions involving negative exponents exactly as with positive exponents, as the following examples illustrate.

Example 6.3.7. *Simplify:* $\dfrac{x^7 \cdot x^{-9}}{x^{-4}}$. *Write the answer using only positive exponents.*

Answer.

$$\begin{aligned}
&\frac{x^7 \cdot x^{-9}}{x^{-4}} && \\
&\frac{x^{7+(-9)}}{x^{-4}} && \textit{Property (E1)} \\
&\frac{x^{-2}}{x^{-4}} && \\
&x^{(-2)-(-4)} && \textit{Property (E2)} \\
&x^{(-2)+4} && \\
&x^2. &&
\end{aligned}$$

The answer is x^2.

As usual, when we subtract negative numbers, we took the trouble to rewrite the subtraction as "adding the opposite."

Example 6.3.8. *Simplify:* $\left(\dfrac{x^{-3}y}{x^{-5}y^8}\right)^{-2}$. *Write the answer using only positive exponents.*

Answer. *We will simplify inside the grouping symbols first:*

$$\begin{aligned}
&\left(\frac{x^{-3}y}{x^{-5}y^8}\right)^{-2} && \\
&\left(\frac{x^{-3}}{x^{-5}} \cdot \frac{y}{y^8}\right)^{-2} && \textit{Grouping common bases} \\
&\left(x^{(-3)-(-5)} \cdot y^{1-8}\right)^{-2} && \textit{Property (E2)} \\
&\left(x^{(-3)+(5)}y^{1+(-8)}\right)^{-2} && \textit{Subtraction as "adding the opposite"} \\
&\left(x^2y^{-7}\right)^{-2} && \\
&(x^2)^{-2}(y^{-7})^{-2} && \textit{Property (E4)} \\
&x^{(2)(-2)}y^{(-7)(-2)} && \textit{Property (E5)} \\
&x^{-4}y^{14} && \\
&\frac{1}{x^4} \cdot \frac{y^{14}}{1} && \textit{Rewriting negative exponent as a reciprocal} \\
&\frac{y^{14}}{x^4}. &&
\end{aligned}$$

The answer is y^{14}/x^4. *Notice that while the equivalent expression* $x^{-4}y^{14}$ *is just as "simple" as the final answer, we took the extra effort to write the* *answer using only the more familiar whole number exponents, as requested in the problem.*

The reader should be advised that the above approach to simplifying is not the only possible route to the final answer. For example, Property (E6) of exponents could be applied to the expression as a first step. (Check to see that the final answer is the same!)

6.3.2 Exercises

Use properties of exponents to simplify the following algebraic expressions.

1. $(x^2)(x^5)$

2. $(z^4)^3$

3. $(4x^3)^{-2}$

4. $\dfrac{(a^2b^3)^2}{a^3b^5}$

5. $\dfrac{x^5 \cdot y^2}{(y^2)^3}$

6. $\dfrac{y^5 \cdot y^2}{(y^2)^3}$

The following exercises illustrate the error of the common mistake of applying a "rule" to equate $(a+b)^2$ with a^2+b^2. For the given values of a and b below, evaluate (a) $(a+b)^2$ and (b) a^2+b^2.

7. $a=2$, $b=3$

8. $a=-1$, $b=2$

6.4 A detour: Scientific notation

The distance from the sun to Earth is, on average, approximately 93,000,000 miles. The speed of light (in a vacuum) is approximately 300,000,000 meters per second. The radius of a hydrogen atom is approximately 0.000000000053 meters.

In many fields of science, we are faced with either very large or very small quantities, like the ones in the previous paragraph. Scientific notation is a convenient way of treating such numbers. In this section, we briefly review scientific notation. Even though it is not related to polynomials (or even algebra) as such, it will give us an opportunity to practice using the properties of exponents.

Scientific notation

A number is written in *scientific notation* if it has the form

$$a \times 10^n,$$

where

- a is a number whose magnitude is between 1 and 10, possibly being 1 but strictly less than 10. Symbolically, $1 \leq |a| < 10$.
- n is any *integer* (positive, negative or zero).

Notice the unfortunate but completely standard use of the "times" symbol $\times$ representing the multiplication involved in scientific notation. We will refer to the number a as the "number part" (or coefficient) of the number written in scientific notation to distinguish it from the "exponential part." (Technically, of course, all "parts" of a number written in scientific notation are numbers.)

For example, the following numbers are written in scientific notation:

- 2×10^8;
- -4.5×10^{15};
- 3.14×10^{-4}.

The following numbers are not written in scientific notation (can you see why?):

- 22×10^{-14};
- 755.88;
- 10×10^4;
- $-8 \times 10^{3/2}$.

Converting between scientific notation and standard (decimal) notation is accomplished with the help of the nice properties of powers of ten.

Example 6.4.1. *Convert each of the following numbers in scientific notation into standard (decimal) notation.*

(a) 2×10^8*;*

(b) -4.5×10^{15}*;*

(c) 3.14×10^{-4}*.*

Answer. *(a)* $2 \times 10^8 = 2(100000000) = 200,000,000$.

(b) $-4.5 \times 10^{15} = -4.5(1000000000000000) = -4{,}500{,}000{,}000{,}000{,}000.$

(c) $3.14 \times 10^{-4} = 3.14\left(\dfrac{1}{10^4}\right) = 3.14\left(\dfrac{1}{10000}\right) = 3.14(0.0001) = 0.000314.$

The clever reader can certainly see a way to describe a shortcut based on the three examples above in terms of "moving the decimal place."

Example 6.4.2. *Write each of the following numbers in scientific notation.*

(a) $93{,}000{,}000$

(b) $300{,}000{,}000$

(c) 0.000000000053.

Answer. *In order to determine the correct power of ten in writing a number in scientific notation, first identify what we will call the* leading digit*, meaning the first non-zero digit appearing in the number reading from left to right. So the leading digit in (a) would be* 9*; the leading digit in (b) would be* 3*; and the leading digit in (c) would be* 5*.*

The exponent of 10 *of the number written in scientific notation will be determined by the place value of the leading digit. (This can be determined by counting the digits between the leading digit and the units digit, not including the units digit but including the leading digit if it is not the same as the units digit.) So in (a), the* 9 *is in the position with place value* 10^7*; in (b), the* 3 *is in the position with place value* 10^8*; and in (c), the* 5 *is in the position with place value* 10^{-11}.

Putting this together, we obtain:

(a) $93{,}000{,}000 = 9.3 \times 10^7$;

(b) $300{,}000{,}000 = 3 \times 10^8$;

(c) $0.000000000053 = 5.3 \times 10^{-11}$.

Keep in mind that "big" numbers have positive powers of ten in scientific notation, while "small" numbers have negative powers of ten.

6.4.1 Multiplication and division of numbers in scientific notation

Because scientific notation has multiplication "built in" to the notation, performing the operations of multiplication and division with numbers in scientific notation are particularly simple using the rules of exponents. We will illustrate this sentence with the following examples.

Example 6.4.3. *Multiply:* $(3 \times 10^7)(1.5 \times 10^{-2})$.

Answer. *The first thing to notice is that since the only operations appearing are multiplication, we can use the commutative and associative properties to re-order and re-group the factors:*

$$(3 \times 10^7)(1.5 \times 10^{-2}) = (3)(1.5)(10^7)(10^{-2}).$$

Since scientific notation always involves multiples of the same base (ten), we can apply property (E1), adding the exponents:

$$\begin{aligned}(3 \times 10^7)(1.5 \times 10^{-2}) &= (3)(1.5)(10^7)(10^{-2}) \\ &= (4.5)(10^{7+(-2)}) \\ &= 4.5 \times 10^5.\end{aligned}$$

The answer is 4.5×10^5.

Summarizing the previous example, multiplying numbers in scientific notation involves multiplying their "number part" and adding the exponents of the powers of ten.

The following example shows that the product of two numbers written in scientific notation will not automatically result in a number written in scientific notation.

Example 6.4.4. *Multiply:* $(5 \times 10^8)(3 \times 10^4)$.

Answer. *Following the previous strategy:*

$$\begin{aligned}(5 \times 10^8)(3 \times 10^4) &= (5)(3)(10^8)(10^4) \\ &= (15)(10^{8+4}) \\ &= 15 \times 10^{12}.\end{aligned}$$

Unfortunately, the result is not in scientific notation, since the "number part" 15 *has magnitude greater than* 10. *We will approach this problem by writing the number part in scientific notation, then again using the fact of having a common base of* 10 *to apply Property (E1) of exponents again:*

$$\begin{aligned}(5 \times 10^8)(3 \times 10^4) &= 15 \times 10^{12} \\ &= (1.5 \times 10^1)(10^{12}) \\ &= 1.5 \times 10^{1+12} \\ &= 1.5 \times 10^{13}.\end{aligned}$$

The answer, written in scientific notation, is 1.5×10^{13}.

Our approach to dividing numbers written in scientific notation is similar to multiplication, but we will use Property (E2) of dividing exponentials with a common base.

Example 6.4.5. *Divide:* $\dfrac{6 \times 10^{-2}}{4 \times 10^{-5}}$.

Answer. *The only difference between division and multiplication is that we need to group the numerators and denominators carefully:*

$$\begin{aligned}\frac{6\times 10^{-2}}{4\times 10^{-5}} &= \frac{6}{4}\cdot\frac{10^{-2}}{10^{-5}}\\ &= 1.5\times 10^{(-2)-(-5)}\\ &= 1.5\times 10^{(-2)+(5)}\\ &= 1.5\times 10^{3}.\end{aligned}$$

The answer is 1.5×10^{3}. *Notice that we took the trouble to rewrite the subtraction of exponents as addition of the opposite.*

Example 6.4.6. *Divide:* $\dfrac{4\times 10^{-5}}{8\times 10^{12}}$.

Answer. *Following the procedure of the previous examples,*

$$\begin{aligned}\frac{4\times 10^{-5}}{8\times 10^{12}} &= \frac{4}{8}\cdot\frac{10^{-5}}{10^{12}}\\ &= 0.5\times 10^{(-5)-(12)}\\ &= 0.5\times 10^{(-5)+(-12)}\\ &= 0.5\times 10^{-17}.\end{aligned}$$

Unfortunately, we again are in the situation where the result is not written in scientific notation; the "number part" has magnitude less than 1. *Applying a similar strategy as the one in Example 6.4.4,*

$$\begin{aligned}\frac{4\times 10^{-5}}{8\times 10^{12}} &= 0.5\times 10^{-17}\\ &= (5\times 10^{-1})\times 10^{-17}\\ &= 5\times 10^{(-1)+(-17)}\\ &= 5\times 10^{-18}.\end{aligned}$$

The answer is 5×10^{-18}.

6.4.2 Exercises

Write the following numbers in scientific notation.

1. $7,500,000,000,000,000,000$ (the approximate number of grains of sand on the planet)
2. 0.000000000275 (the approximate diameter of a water molecule, measured in meters)

Write the following numbers in standard (decimal) notation.

3. 6.022×10^{23} (the approximate number of carbon atoms in 12 grams of pure carbon)
4. 1×10^{-3} (the number of liters in one milliliter)

Perform the indicated operation. Write your answer in scientific notation.

5. Multiply: $(6 \times 10^4)(3 \times 10^3)$
6. Divide: $\dfrac{4 \times 10^6}{2 \times 10^8}$
7. Multiply: $(7 \times 10^{-4})(1.3 \times 10^{-10})$
8. Divide: $\dfrac{-1.53 \times 10^1}{-3 \times 10^{-3}}$
9. Multiply: $(3.4 \times 10^8)(2.1 \times 10^{-5})$
10. Divide: $\dfrac{3 \times 10^{-12}}{6 \times 10^{-4}}$
11. Multiply: $(1.55 \times 10^{10})(8.1 \times 10^{-10})$
12. Divide: $\dfrac{-2 \times 10^5}{8 \times 10^{-5}}$

6.5 Multiplying polynomials

Returning to the subject of the arithmetic of polynomials, we now turn to multiplication. Since polynomials (in one variable x) are made up of terms having the form ax^n, the results of the previous section will apply, especially Property (E1) of exponents.

Notice that multiplying a monomial by a monomial involves nothing more that applying Property (E1) directly, as the following example shows.

Example 6.5.1. *Multiply:* $(15x^3)(4x^2)$.

Answer. *The two polynomials being multiplied,* $15x^3$ *and* $4x^2$*, each have one term. To multiply them, we will apply the associative and commutative properties to regroup the factors, the apply Property (E1):*

$$\begin{aligned} (15x^3)(4x^2) & \\ (15 \cdot 4)(x^3 \cdot x^2) & \quad \textit{(Re-grouping the factors)} \\ 60x^{3+2} & \quad \textit{(Property (E1))} \\ 60x^5. & \end{aligned}$$

The answer is $60x^5$.

Normally, we will not illustrate the regrouping as a separate step.

In order to multiply polynomials with more than one term, we will need to remember the *distributive law*. The distributive law describes an important relationship between multiplication and addition (which, in the case of operations with whole numbers, is based on multiplication as repeated addition).

Symbolically, the distributive law is often summarized by the identity

$$a(b+c) = ab + ac. \tag{6.1}$$

It's worth paying a little more attention to what Equation 6.1 is really saying. On the left hand side, there are two operations, addition and multiplication; the order of operations dictates that the sum, which is grouped, is performed before the multiplication. On the right hand side, there are three operations: two multiplications and one addition. The order of operations on the right dictates that the two multiplications are performed first, followed by the addition. The distributive law gives a precise way that the order of operations between addition and multiplication can be changed. In words, the product of two factors, one of which is a sum of two terms, is the same as the sum of the product of the first factor with each of the two terms involved in the sum.

Before going further, let's apply the straightforward expression of the distributive law to the product of a monomial with a binomial.

Example 6.5.2. *Multiply:* $(3x^2)(2x+7)$.

Answer.

$$\begin{aligned}(3x^2)(2x+7) &= (3x^2)(2x) + (3x^2)(7) && \textit{(The distributive law)}\\ &= 6x^{2+1} + 21x^2 && \textit{(Property (E1))}\\ &= 6x^3 + 21x^2.\end{aligned}$$

The answer is $6x^3 + 21x^2$.

It's also worth noting that even though the distributive law is written with only one of the factors involving a sum, it in fact applies more generally, keeping again the commutative and associative laws in mind. For example, try to justify each step in the following sequence of identities (each step involves applying one of either the commutative law, the associative law, or the distributive law as stated above):

$$\begin{aligned}(a+b+c)(x+y) &= (a+b+c)(x) + (a+b+c)(y)\\ &= (x)(a+b+c) + (y)(a+b+c)\\ &= (x)(a+(b+c)) + (y)(a+(b+c))\\ &= (x)(a) + (x)(b+c) + (y)(a) + (y)(b+c)\\ &= xa + (xb+xc) + ya + (yb+yc)\\ &= ax + bx + cx + ay + by + cy.\end{aligned}$$

What is most important about the above sequence identities is not really the in-between steps (although pointing out the different laws at work would make your fourth-grade math teacher smile!). We started with the product of a factor with three terms $a + b + c$ with a factor with two terms $x + y$. The final expression involves the sum of six ($= 3 \times 2$) multiplications. Each of the six multiplications involves one term from the first expression and one term from the second expression. Moreover, each term from the first expression is "matched" with each term in the second expression, which is why we ended up with six multiplications.

Let's summarize the distributive law in the following way:

The distributive law

The product of two factors, each of which is a sum of several terms, is the same as the sum of terms obtained by multiplying each term of the first factor by each term of the second factor.

Let's see a few examples of the distributive law in action in multiplying polynomials. You will notice that in these cases, after applying the distributive law and properties of exponents, like terms often appear—which can (and should) then be combined!

Example 6.5.3. *Multiply:* $(x + 4)(x + 2)$.

Answer. *Notice that in multiplying a polynomial with two terms by a polynomial with two terms will result, applying the distributive law, to* $2 \times 2 = 4$ *multiplications:*

$$\begin{aligned} &(x+4)(x+2) \\ &(x)(x) + (x)(2) + (4)(x) + (4)(2) && \textit{(Distributing)} \\ &x^2 + 2x + 4x + 8 && \textit{(Multiplying in each term)} \\ &x^2 + 6x + 8. && \textit{(Combining like terms)} \end{aligned}$$

The answer is $x^2 + 6x + 8$.

Example 6.5.4. *Multiply:* $(2x + 5)(3x + 2)$.

Answer.

$$\begin{aligned} &(2x+5)(3x+2) \\ &(2x)(3x) + (2x)(2) + (5)(3x) + (5)(2) && \textit{(Distributing)} \\ &6x^2 + 4x + 15x + 10 && \textit{(Multiplying in each term)} \\ &6x^2 + 19x + 10. && \textit{(Combining like terms)} \end{aligned}$$

The answer is $6x^2 + 19x + 10$.

The following example illustrates the way that we will approach the distributive law involving subtraction.

Example 6.5.5. *Multiply:* $(2x - 1)(x - 6)$.

Answer. *Both of the two binomials involve subtraction. However, as usual, we can consider this as a sum by writing the subtraction as "adding the opposite." In particular, the two terms in the first binomial are* $2x$ *and* -1*, while the two terms from the second binomial are* x *and* -6*. We will write this explicitly when distributing.*

$$\begin{aligned} &(2x-1)(x-6) \\ &(2x)(x) + (2x)(-6) + (-1)(x) + (-1)(-6) && \textit{(Distributing)} \\ &2x^2 - 12x - x + 6 && \textit{(Multiplying in each term)} \\ &2x^2 - 13x + 6. && \textit{(Combining like terms)} \end{aligned}$$

The answer is $2x^2 - 13x + 6$.

(Notice that in the third line, we switched back to writing the polynomial using subtraction, instead of writing $2x^2 + (-12x) + (-x) + 6$ *. Either way of writing would be acceptable, since they both have the same terms, but it is typical to write it as we have done in the answer above.)*

Example 6.5.6. *Multiply:* $(x + 4)(x - 4)$.

Answer.

$$\begin{aligned} &(x+4)(x-4) \\ &(x)(x) + (x)(-4) + (4)(x) + (4)(-4) && \textit{(Distributing)} \\ &x^2 - 4x + 4x - 16 && \textit{(Multiplying in each term)} \\ &x^2 - 16. && \textit{(Combining like terms)} \end{aligned}$$

The answer is $x^2 - 16$*. The "middle" like terms "cancelled" (their sum is zero).*

When we have whole number powers of a polynomial, it is best to take the 10 seconds required to rewrite the problem as a repeated multiplication.

Example 6.5.7. *Multiply:* $(3x + 4)^2$.

Answer. *We start by making the multiplication explicit.*

$$\begin{aligned} &(3x+4)^2 \\ &(3x+4)(3x+4) && \textit{(Exponent as repeated multiplication)} \\ &(3x)(3x) + (3x)(4) + (4)(3x) + (4)(4) && \textit{(Distributing)} \\ &9x^2 + 12x + 12x + 16 && \textit{(Multiplying in each term)} \\ &9x^2 + 24x + 16. && \textit{(Combining like terms)} \end{aligned}$$

The answer is $9x^2 + 24x + 16$. *Notice that the answer is NOT the same as* $(3x)^2 + (4)^2$ *(refer to the last two exercises in Section 6.3.2)!*

Example 6.5.8. *Multiply:* $(2x - 1)(x^2 - 5x + 2)$.

Answer. *The distributive law, applied to this multiplication of a binomial with a trinomial, will involve* $2 \times 3 = 6$ *multiplications.*

$$\begin{aligned}
&(2x-1)(x^2-5x+2) \\
&(2x)(x^2)+(2x)(-5x)+(2x)(2)+(-1)(x^2)+(-1)(-5x)+(-1)(2) && \textit{(Distributing)} \\
&2x^3-10x^2+4x-x^2+5x-2 && \textit{(Multiplying in each term)} \\
&2x^3-11x^2+9x-2. && \textit{(Combining like terms)}
\end{aligned}$$

The answer is $2x^3 - 11x^2 + 9x - 2$.

The last example illustrates the fact that when multiplying a product of three (or more) factors, we still have to apply the distributive law "one multiplication at a time."

Example 6.5.9. *Multiply:* $(x - 3)(x + 2)(2x + 5)$.

Answer. *We will introduce square brackets to group the first multiplication:*

$$\begin{aligned}
&[(x-3)(x+2)]\,(2x+5) \\
&[(x)(x)+(x)(2)+(-3)(x)+(-3)(2)]\,(2x+5) \\
&\left[x^2+2x-3x-6\right](2x+5) \\
&(x^2-x-6)(2x+5) \\
&(x^2)(2x)+(x^2)(5)+(-x)(2x)+(-x)(5)+(-6)(2x)+(-6)(5) \\
&2x^3+5x^2-2x^2-5x-12x-30 \\
&2x^3+3x^2-17x-30.
\end{aligned}$$

The answer is $2x^3 + 3x^2 - 17x - 30$.

To summarize, multiplying polynomials involves the distributive law, property (E1) of exponents, and combining like terms.

6.5.1 Exercises

Multiply the following polynomials.

1. $(x - 3)(x + 2)$
2. $(3x - 4)(2x - 1)$
3. $(2x + 3)^2$

4. $(x-1)(x^2+x+1)$
5. $(x+3)(x^3+2x^2-3x-2)$
6. $(x^2-3x-1)(x^2+2x+3)$
7. $(3x-2)^2$.
8. $(4x-3)(2x-1)(x+2)$.
9. $(2x-5)(x^2+4x-6)$
10. (*) $(2x-1)^3$.
11. (*) Verify the identity $(a+b)^2 = a^2+2ab+b^2$ by applying the distributive law on the left hand side.
12. (*) For what values of a and b does the equality $(a+b)^2 = a^2+b^2$ hold?

6.6 Dividing a polynomial by a monomial

Just as in division of whole numbers, division of polynomials raises certain very fundamental issues. The most basic problem is the fact that *the quotient of two polynomials may not be a polynomial.* This can be seen from a very simple example. Consider the quotient of the polynomial x^2 divided by the polynomial x^3. By the Property (E2) of exponents, above, we have $x^2 \div x^3 = x^{2-3} = x^{-1}$. Even though we can now make perfect sense of the meaning of x^{-1} (as $1/x$), it is not a polynomial, since the exponent is not a whole number.

We will not attempt a full treatment of division of polynomials here. In order to do so, we would need to consider "fractions of polynomials," what are known as *rational expressions* (rational since they are formed by ratios of polynomials). This is usual treated in an "intermediate algebra" course. A more detailed treatment would involve the *division algorithm* and a corresponding "long division" of polynomials. This is usually treated in a precalculus course.

We are interested in those division problems that can be handled using just the properties of exponents and the distributive law. It turns out that this can be done, provided that we divide a polynomial by a *monomial.*

We will be using the distributive law in the following form:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

In words, *the quotient of several terms (in the numerator) by a single term (in the denominator) is the same as the sum of the quotients of each term (in the numerator) by the term in the denominator.* As an exercise, you may derive this version of the distributive law from the usual one by expressing "division by c" as "multiplication by $1/c$."

Here are a few examples of division of polynomials, when the divisor (in the denominator) is a monomial.

Example 6.6.1. *Divide:* $\dfrac{27x^6+18x^4}{3x^2}$.

Answer.

$$\begin{aligned}\frac{27x^6+18x^4}{3x^2} &= \frac{27x^6}{3x^2}+\frac{18x^4}{3x^2} && \textit{(Distributing)}\\ &= 9x^{6-2}+6x^{4-2} && \textit{Dividing, using property (E2)}\\ &= 9x^4+6x^2.\end{aligned}$$

The answer is $9x^4+6x^2$.

Example 6.6.2. *Divide:* $\dfrac{2y^3-12y^2-18y}{2y}$.

Answer.

$$\begin{aligned}\frac{2y^3-12y^2-18y}{2y} &= \frac{2y^3}{2y}+\frac{-12y^2}{2y}+\frac{-18y}{2y} && \textit{(Distributing)}\\ &= y^2-6y-9.\end{aligned}$$

The answer is y^2-6y-9. *Notice that we maintained our custom of writing the distributive law using addition, in this case involving adding terms with negative coefficients. Also notice that* $\dfrac{y^2}{y}=y^{2-1}=y^1=y$, *while* $\dfrac{y}{y}=y^{1-1}=y^0=1$.

Example 6.6.3. *Divide:* $\dfrac{5x^5-10x^4+5x^2}{5x^2}$.

Answer.

$$\begin{aligned}\frac{5x^5-10x^4+5x^2}{5x^2} &= \frac{5x^5}{5x^2}+\frac{-10x^4}{5x^2}+\frac{5x^2}{5x^2} && \textit{(Distributing)}\\ &= x^3-2x^2+1.\end{aligned}$$

The answer is x^3-2x^2+1. *Pay special attention to the last term:*

$$\begin{aligned}\frac{5x^2}{5x^2} &= \frac{5}{5}\cdot\frac{x^2}{x^2}\\ &= 1\cdot x^{2-2}\\ &= 1\cdot x^0\\ &= 1\cdot 1 = 1.\end{aligned}$$

Many times this phenomenon is referred to as "canceling." The main thing to remember is that "canceling" does not mean "disappearing!"

Example 6.6.4. *Divide:* $\dfrac{11x^4-8x^3+10x^2}{4x^3}$.

Answer.

$$\begin{aligned}\frac{11x^4 - 8x^3 + 10x^2}{4x^3} &= \frac{11x^4}{4x^3} + -8x^3 4x^3 + 10x^2 4x^3 && \textit{(Distributing)}\\ &= \frac{11}{4}x^{4-3} + \frac{-8}{4}x^{3-3} + \frac{10}{4}x^{2-3}\\ &= \frac{11}{4}x - 2 + \frac{5}{2}x^{-1}.\end{aligned}$$

The answer is $(11/4)x - 2 + (5/2)x^{-1}$.
Two things to notice about this example:

1. *Because the division of whole number coefficients did not end up so "clean" this time, in the sense that the result did not end up being an integer, we wrote the division step more explicitly.*

2. *The fact that the result includes an* x^{-1}*-term (with a negative exponent) is another reminder: the quotient of two polynomials may not be a polynomial!*

6.6.1 Exercises

Perform the indicated division problems.

1. $\dfrac{10x^2 - 20x}{-2x}$

2. $\dfrac{x^2 - 3x + 9}{3x}$

3. $\dfrac{3x^5 - 12x^4 - 9x^2}{3x}$

4. $\dfrac{2x^8 - 2x^5 - 8x^4}{2x^4}$

5. $\dfrac{25x^3 - 35x^2 + 5x}{-5x}$

6.7 Chapter summary

- A polynomial is a special algebraic expression, all of whose terms involve only whole-number powers of the variable or variables.

- Exponents (originally defined in terms of repeated multiplication) have a number of properties which follow the principle, "Exponents work well with multiplication and division."

- Adding polynomials involves simply combining like terms.

- Subtracting polynomials (similar to subtracting signed numbers) is best accomplished by rewriting the difference as "adding the opposite," with the opposite of a polynomial understood as "changing the sign of every term."
- Multiplying polynomials involves both the distributive property and the simplest property of exponents, $x^a \cdot x^b = x^{a+b}$. After distributing and multiplying, be sure to check to see if there are any like terms which can be combined.
- Dividing a polynomial by a monomial can also be accomplished using a version of the distributive law, dividing each term of the dividend by the divisor and applying the rule of exponents $x^a/x^b = x^{a-b}$.
- The rules of exponents are also helpful in working with numbers written in scientific notation. In particular, two numbers written in scientific notation can be easily multiplied or divided, with the understanding that it sometimes takes an extra step to ensure that the final answer is written in scientific notation.

Chapter 7

Factoring

Vocabulary

- A factor (of an integer)
- A factor (of a polynomial)
- To factor
- Greatest common divisor
- To factor completely
- Difference of squares
- Quadratic trinomial
- Monic polynomial

7.1 Introduction to factoring

There are a number of circumstances when it is convenient to see a polynomial not as a sum of terms but as a product of factors. The process of writing a polynomial as a product of factors is called "factoring." The main use we will see for factoring will be in Chapter 9, in order to solve some quadratic equations. However, factoring is also a basic technique for working with rational expressions (in intermediate algebra) and in solving higher-degree polynomial equations (in precalculus).

In this chapter, we will outline some basic techniques for factoring. By the end, we will have developed a kind of "checklist" that we can apply to try to factor any polynomial expression.

The word "factor," in the context of mathematics, always implies the operation of multiplication. Just like we use the word "term" to mean a quantity being added, the word "factor" will be used to represent a quantity being multiplied.

Example 7.1.1. • *In the expression* $4 \cdot 7$*, there are two factors,* 4 *and* 7*.*

- *In the expression* $(3.79)(-1.2)(5.9)$*, there are three factors,* 3.79*,* -1.2 *and* 5.9*.*
- *In the expression* $3x$*, there are two factors,* 3 *and* x*.*

One thing to keep in mind is that factors might be more complicated that those in the previous example.

Example 7.1.2. • *In the expression* $4x(x^2+3x+1)$*, there are three factors:* 4*,* x *and* x^2+3x+1*. Notice that the entire expression in parentheses (which is one "group" having three terms) is* one factor*.*

- *In the expression* $(x+1)(x+4)$*, there are two factors,* $x+1$ *and* $x+4$*. Notice that each of the two factors has two terms.*

Unfortunately, the word "factor" is used in other ways as well. While the basic meaning of a factor as "an expression appearing in a product" is not lost, in other uses of the word this basic meaning is hidden in the background.

For example, the integer 12, considered as an arithmetic expression, does not appear as a factor at all, in the sense that there is no multiplication indicated. However, we can write $12 = 3 \cdot 4$. So we can say that 3 and 4 are *factors of* 12. Likewise, -2 is also a factor of 12, since 12 can also be written as $(-2)(-6)$. Notice that, from this point of view, factors of an integer come in pairs: -2 is a factor of 12, and so is -6.

In summary, a **factor of an integer** is an integer which can be multiplied by another integer to give the original number. (Sometimes it is said that a factor of an integer "divides the original number evenly," but we want to emphasize that the word factor implies the operation of multiplication.) While the word "factor" in this sense is sometimes meant to only refer to positive numbers, we will need to consider both positive and negative factors.

Here are a few things to remember about factors of integers:

- 1 is a factor of every integer. (And so is -1.)
- Every integer is a factor of itself. (And so is its opposite a factor of itself.)
- A *prime* number is a positive integer with exactly two positive factors: 1 and itself.[1]

Example 7.1.3. *List all the factors of each of the following integers:*

(a) 12*;*

(b) -140*;*

[1] For this reason, the number 1 is not a prime number, since it has only one positive factor.

(c) 25;

(d) 17.

Answer. *(a) The factors of* 12 *are* 1, 2, 3, 4, 6, 12, −1, −2, −3, −4, −6 *and* −12.

(b) The factors of −140 *are* 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140, −1, −2, −4, −5, −7, −10, −14, −20, −28, −35, −70 *and* −140.

(c) The factors of 25 *are* 1, 5, 25, −1, −5, *and* −25.

(d) The factors of 17 *are* 1, 17, −1 *and* −17. *(Notice* 17 *is a prime number.)*

In a similar way, **a factor of a polynomial is another polynomial which, when multiplied by a third polynomial, gives the original polynomial**. In the case that that the polynomial is already written as a product, some factors are easy to see. For example, the polynomial $(3x + 2)(x - 1)$ has two factors, being $3x + 2$ and $x - 1$. Whether it has any other factors will be investigated in the remainder of the chapter.

What happens if a polynomial is not written as a product?

So far, we have used the word "factor" as a noun. However, due to the importance in various contexts of seeing an expression written as a product, the word "factor" is also used as a verb.

Factoring

To *factor* means to write as a product of two (or more) factors.

For example, to factor the number 12, we could write $3 \cdot 4$ or $2 \cdot 6$ or $(-1)(-12)$. From this example, you can see that there is usually more than one way to factor an integer.[2]

Likewise, when we are asked to factor a polynomial, the answer should be a product of two (or more) polynomials. For example, to factor $x^2 + 3x + 2$, we would write $(x + 1)(x + 2)$. You can check that the answer is correct by multiplying the two polynomials $x + 1$ and $x + 2$—you should get $x^2 + 3x + 2$. By the end of this chapter, you will see how to obtain that answer, if it weren't given to you like it was here. But you should notice something right away: **Factoring is the opposite process as multiplying**. This will be our guide to presenting all the various methods of factoring below.

[2](The *Fundamental Theorem of Arithmetic* states, however, that there is only one way to factor a positive integer into factors which are powers of prime numbers, up to the order in which the factors are written. This is called the prime factorization of the integer. You might remember algorithms for producing the prime factorization of a number, like the so-called "factor tree.")

NOTICE: For the remainder of this chapter, all of our polynomials will have *integer* coefficients. In particular, when we are asked to factor a polynomial with integer coefficients, we will insist that the factors should also have integer coefficients.

7.2 "Factoring out" the greatest common factor

Let's start out with an example where we can "cheat."

Example 7.2.1. *Factor:* $6x^3 + 21x^2$.

Answer. *One answer is* $3x^2(2x + 7)$.

To see why, refer to Example 6.30 in the last chapter. We can "cheat" because we already multiplied two polynomials to obtain $6x^3 + 21x^2$*, so when we are now asked to factor the same polynomial—to write it as a product—we can just refer back to the original multiplication problem.*

The problem, of course, is that on many occasions we will not be able to refer back to a multiplication problem to find an answer. But let's look a little more carefully at the preceding example to try to find a strategy.

Our "answer" $3x^2(2x + 7)$ has two factors: $3x^2$ and $2x + 7$. How are the related to the original polynomial $6x^3 + 21x^2$?

Notice that the original polynomial had two terms: $6x^3$ and $21x^2$. The coefficients of these two terms, 6 and 21, have two positive common factors: 1 and 3. (When we look for common factors, we will keep in mind that 1 is always a common factor.) Of these, the *greatest* common factor is 3—which is the coefficient of the factor $3x^2$.

In addition to the common factor of 3, the terms $6x^3$ and $21x^2$ have a variable part in common—they both involve powers of x. How many factors of x are *common*? Both terms include a factor of x, since $x^3 = x \cdot x^2$ and $x^2 = x \cdot x$. Both terms also include a factor of x^2, since $x^3 = x^2 \cdot x$ and $x^2 = x^2 \cdot 1$. But only one of the terms includes a factor of x^3, since the second term includes only two factors of x. Summarizing, the *greatest* common factor of x^3 and x^2—the greatest number of factors of x that are in common to both—is x^2. Notice that x^2 is the variable part of our original factor $3x^2$.

To summarize the preceding two paragraphs: the factor $3x^2$ is the **greatest common factor** (often abbreviated as GCF) of the two original terms $6x^3$ and $21x^2$. We obtained it by separately considering the coefficients and the variable parts of the terms and multiplying the result.

What about the other factor $2x + 7$ from our answer? How is this factor related to the original polynomial?

Notice what happens when we divide our original polynomial $6x^3 + 21x^2$ by the common factor $3x^2$ that we just discussed:

$$\begin{gathered}\frac{6x^3 + 21x^2}{3x^2} \\ \frac{6x^3}{3x^2} + \frac{21x^2}{3x^2} \\ 2x + 7.\end{gathered}$$

In other words, the second factor $2x+7$ is the quotient of the original polynomial by the greatest common factor.

Let's summarize the method that we have taken out of the preceding discussion. The process of factoring by finding a greatest common factor is often referred to as "factoring out" the greatest common factor.

Factoring out the greatest common factor

To factor a polynomial whose terms have a common factor:

1. Find the greatest common factor of all the terms of the original polynomial, considering both the coefficients and the variable parts.

2. Divide the original polynomial by the GCF from Step 1 to obtain the second factor.

The answer is the product of the polynomials from Steps 1 and 2.

NOTICE: It is possible to factor polynomials using a common factor that is not the GCF. For example, we could have factored $6x^3 + 21x^2$ above as $x^2(6x + 21)$ or as $3x(2x^2 + 7x)$. (Check that these are all valid!) We have written the polynomial as a product of two factors, as required. However, these factorizations are not "complete," in the sense that one of the factors still has factors in common among its terms. From now one, we will always ask to factor *completely*, which in this context means to factor out not just any common factor, but the *greatest* common factor. We will have more to say about "factoring completely" below.

The following examples illustrate the procedure for factoring out the GCF, as well as a number of issues to watch out for.

Example 7.2.2. *Factor completely:* $12x^7 - 8x^5 + 16x^3$.

Answer. *The polynomial has three terms,* $12x^7$*,* $-8x^5$ *and* $16x^3$*. The coefficients have (positive) common factors* 1*,* 2 *and* 4*. The highest power of* x *that is common to all three terms is* x^3*. So the greatest common factor is* $4x^3$*.*

Dividing the original polynomial by the GCF of $4x^3$*:*

$$\frac{12x^7 - 8x^5 + 16x^3}{4x^3}$$
$$\frac{12x^7}{4x^3} + \frac{-8x^5}{4x^3} + \frac{16x^3}{4x^3}$$
$$3x^4 - 2x^2 + 4.$$

The answer is $4x^3(3x^4 - 2x^2 + 4)$.

Notice that the greatest common factor includes the *least* exponent appearing in any of the terms. Although this "rule" seems strange, keep in mind we are looking to what factors are in *common* to all terms.

The next example shows illustrates a basic feature of polynomials: Not every polynomial can be factored, at least in any way that will be considered in this text. (One should keep in mind the example of prime numbers from arithmetic.)

Example 7.2.3. *Factor completely:* $x^2 + 5$.

Answer. *The polynomial has two terms,* x^2 *and* 5*. The only positive common factor of the coefficients is* 1*. There is no common factor of* x*. So the GCF of these two terms is* 1*.*

Although we could conceivably divide the original polynomial by 1*, this will result in the same polynomial, and so as a factorization we would have to write* $(1)(x^2+5)$*. However, we have gained nothing in the sense that the new "factor" is the same as the original polynomial.*

The answer is: The polynomial cannot be factored.

In particular, from now on, we will be more precise about what we mean by the verb "to factor." To factor will mean: Write as a product of two or more factors, *none of which are 1.* (There will be one exception to this when we discuss factoring by grouping in Sections 7.5 and 7.6 below.)

CAUTION: We will see many examples in later sections of polynomials whose terms have no common factor, but that can be factored using other techniques. (For the record, the polynomial x^2+5 in the example above cannot be factored using *any* of the methods we will discuss.)

Example 7.2.4. *Factor completely:* $10x - 25$.

Answer. *The polynomial has two terms,* $10x$ *and* -25*. The coefficients have positive common factors* 1 *or* 5*. They do not have a common variable factor, since the second term does not involve* x*. So the greatest common factor is* 5*.*

Dividing the original polynomial by the GCF of 5*, we obtain*

$$\frac{10x - 25}{5}$$
$$\frac{10x}{5} + \frac{-25}{5}$$
$$2x - 5.$$

The answer is $5(2x - 5)$.

Example 7.2.5. *Factor completely:* $x^3 - 4x^2 - 2x$.

Answer. *The polynomial has three terms,* x^3, $-4x^2$ *and* $-2x$. *The coefficients have only positive common factor* 1. *The highest power of* x *that is common to all three terms is* x. *So the greatest common factor is* x.

Dividing the original polynomial by the GCF of x:

$$\frac{x^3 - 4x^2 - 2x}{x}$$
$$\frac{x^3}{x} + \frac{-4x^2}{x} + \frac{-2x}{x}$$
$$x^2 - 4x - 2.$$

The answer is $x(x^2 - 4x - 2)$.

When the leading coefficient of a polynomial is negative, it is customary to "factor out" a negative number, so that the more complicated factor has positive leading coefficient. The next two examples in this section illustrate that point.

Example 7.2.6. *Factor completely:* $-4x^2 + 8x - 6$.

Answer. *The polynomial has three terms,* $-4x^2$, $8x$ *and* -6. *The positive common factors of the coefficients are* 1 *and* 2. *There is no common factor involving* x. *Since the leading coefficient is negative, we will use* -2 *as the GCF.*

Dividing the original polynomial by the GCF of -2:

$$\frac{-4x^2 + 8x - 6}{-2}$$
$$\frac{-4x^2}{-2} + \frac{8x}{-2} + \frac{-6}{-2}$$
$$2x^2 - 4x + 3.$$

The answer is $-2(2x^2 - 4x + 3)$. *Notice that the second, more complicated factor (the trinomial) has a positive leading coefficient of* 2.

Example 7.2.7. *Factor completely:* $-x^2 - 2x + 4$.

Answer. *The polynomial again has three terms,* $-x^2$, $-2x$ *and* 4. *The only positive common factor of the coefficients is 1. There is no common factor involving* x. *Even though normally we might say that this polynomial cannot be factored, we will go to the trouble of "factoring out" the common factor of* -1 *because the leading coefficient is negative.*

Dividing the original polynomial by the GCF of -1*:*

$$\frac{-x^2-2x+4}{-1}$$
$$\frac{-x^2}{-1}+\frac{-2x}{-1}+\frac{4}{-1}$$
$$x^2+2x-4.$$

The factorization, according to what we have written so far, is $(-1)(x^2+2x-4)$*. However, it is typical in this case to suppress the multiplication by* -1*, which has the effect of "the opposite of." So we will simply write* $-(x^2+2x-4)$*.*

The answer is $-(x^2+2x-4)$*.*

We conclude with an example illustrating the fact that the principles of "factoring out" a variable common factor extend to polynomials with more than one variable.

Example 7.2.8. *Factor completely:* $4x^2y^3z^5-12x^5y^8z^3+16x^3y^4$*.*

Answer. *The polynomial has three terms,* $4x^2y^3z^5$*,* $-12x^5y^8z^3$ *and* $16x^3y^4$*. The coefficients have greatest common factor* 4*. The highest power of* x *that is common to all three terms is* x^2*. The highest power of* y *that is common to all three terms is* y^3*. Since the third term has no factor of* z*,* z *will not appear in the greatest common factor. Combining all this information, the greatest common factor of the three terms is* $4x^2y^3$*.*

Dividing the original polynomial by the GCF of $4x^2y^3$*:*

$$\frac{4x^2y^3z^5-12x^5y^8z^3+16x^3y^4}{4x^2y^3}$$
$$\frac{4x^2y^3z^5}{4x^2y^3}+\frac{-12x^5y^8z^3}{4x^2y^3}+\frac{16x^3y^4}{4x^2y^3}$$
$$z^5-3x^3y^5z^3+4xy.$$

The answer is $4x^2y^3(z^5-3x^3y^5z^3+4xy)$*.*

7.2.1 Exercises

Factor the following polynomials completely.

1. $6x^3-2x$
2. $4x^5-12x^3-8x^2$
3. $18x-9$
4. $-3x^4+15x^3-9x^2$

5. $6ab^3 - 12a^2b^2$

6. $-xy^4 - 2x^2y^3 - 15x^5y$

7.3 Differences of squares

Let's look at the polynomial

$$x^2 - 16.$$

Based on the previous section, we might be tempted to say that this polynomial cannot be factored. After all, the only positive factor x^2 and -16 have in common is 1.

But not so fast! Looking back at Example 6.34, we've seen this polynomial before—it was the result of the product $(x + 4)(x - 4)$. So once again, we can "cheat:" $x^2 - 16$ can be factored as $(x + 4)(x - 4)$!

We never would have noticed that $x^2 - 16$ could have been factored just by looking for common factors. The question is now: What made this polynomial so special, and is there a pattern that we can use?

The first thing to notice about this polynomial is that it has only two terms—it is a binomial. More important, though, each of the two terms (ignoring for a moment the signs) are *perfect squares*: x^2 is $(x)^2$ ("x squared") and 16 is $(4)^2$ ("four squared.") Finally, the two perfect squares are *subtracted.* For that reason, this example and those having these common features are called "differences of squares."

Notice, by the way, that the two quantities which are being squared—in this example, the x and the 4, play a key role in the factorization: $(x + 4)(x - 4)$. This pattern is at the heart of factoring a difference of squares.

Factoring a difference of squares

The factorization of a polynomial having the special form $a^2 - b^2$ is

$$(a + b)(a - b).$$

(Notice that since the answer is a product, the order that we write the factors is not important, thanks to the commutative property of multiplication.)

Exercise 7.3.1. *Show by multiplying $(a + b)(a - b)$ that this product is really the same as $a^2 - b^2$. You can look back at Example 6.34 if you need a hint.*

What this "formula" says is that once you see that you have a difference of squares, you are almost done. Just figure out what quantities are being squared (playing the roles of a and b in the formula), and fill them into the pattern

$$(\_\_ + \_\_)(\_\_ - \_\_).$$

The only thing that requires some care is recognizing a difference of squares. (If you aren't familiar with the perfect square numbers, write a list of the first 10 or 12: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, and so on.)

Example 7.3.2. *Factor completely:* $x^2 - 81$.

Answer. *Notice first that the two terms* x^2 *and* -81 *have no factors in common (except* 1*).*

However, x^2 *is a perfect square (x squared) and* 81 *is also a perfect square (*9 *squared). The perfect squares are being subtracted. It is a difference of squares!*

The answer is (applying the pattern) $(x + 9)(x - 9)$.

Example 7.3.3. *Factor completely:* $9x^2 - 25$.

Answer. *Again, the terms have no common factor (other than* 1*).*

The term $9x^2$ *is a perfect square:* $9x^2 = (3x)^2$*, using Property (E4) of exponents. Also,* 25 *(*$= (5)^2$*) is a perfect square. Since the perfect squares are subtracted, this is a difference of squares.*

The answer is $(3x + 5)(3x - 5)$.

Example 7.3.4. *Factor completely:* $x^2 + 4$.

Answer. *The two terms have no common factor (other than* 1*).*

Both terms x^2 *and* 4 *are again perfect squares:* $x^2 = (x)^2$ *and* $4 = (2)^2$*. However, the terms are not subtracted! This example is not a difference of squares, and so the "formula" approach we have been using cannot be applied. In fact, there is no simple factorization for* $x^2 + 4$*, at least with polynomials with integer coefficients.*

The answer is: $x^2 + 4$ *cannot be factored.*

The previous example is an example of a general fact: **a sum of squares** $a^2 + b^2$ **cannot be factored** using polynomials with integer coefficients. (The reader refer to Exercise 11 to see that $(a + b)(a + b)$ is not in general the same as $a^2 + b^2$.)

The next example shows that sometimes, differences of squares may appear "in disguise."

Example 7.3.5. *Factor completely:* $x^6 - 25y^4$.

Answer. *The two terms have no common factor other than* 1*.*

On the surface, the exponents for the variables are not 2*, so this may not appear to be a difference of squares. However, because both exponents are* ***even,*** *we can use Property (E4) and (E5) of exponents (Section 6.3) to express them as perfect squares. In particular,* $x^6 = (x^3)^2$ *and* $25y^4 = (5y^2)^2$*. Since the terms are subtracted, this is a difference of squares!*

The answer is $(x^3 + 5y^2)(x^3 - 5y^2)$*.*

In the previous examples, we have never had to worry about common factors. In fact, the reader might wonder why bother making note in all those examples that there were no common factors other than one. The next example shows that a difference of squares might not be apparent until common factors are "factored out."

Example 7.3.6. *Factor completely:* $3x^3 - 27x$*.*

Answer. *In this example, the two terms* $3x^3$ *and* $-27x$ *have a common factor of* $3x$*. Our first step will be to factor out the common factor, as in the previous section:*

$$3x(x^2 - 9).$$

We have obtained a factorization of $3x^3 - 27x$*, in the sense that we have written it as a product of two factors. However, the factor* $x^2 - 9$ *is a difference of squares, and so can itself be factored. In other words, we have not factored* completely*. The factor* $x^2 - 9$ *factors as* $(x + 3)(x - 3)$*.*

The answer is $3x(x + 3)(x - 3)$*. Notice that the common factor that we factored out first must appear in the final factorization.*

We close this section with another reminder about factoring *completely.*

Example 7.3.7. *Factor completely:* $16x^4 - 1$*.*

Answer. *The two terms have no common factor (other than 1).*

Both terms $16x^4$ *and* 1 *are perfect squares:* $16x^4 = (4x^2)^2$ *and* $1 = (1)^2$*. Since they are being subtracted, we have a difference of squares, and we write*

$$(4x^2 + 1)(4x^2 - 1).$$

We need to make sure that we have factored completely*. Looking carefully at the two remaining factors, we see that the first factor* $4x^2 + 1$ *is a sum of squares, and as mentioned above, cannot be factored. However, the second factor* $4x^2 - 1$ *is again a difference of squares:* $4x^2 = (2x)^2$ *and* $1 = (1)^2$*. In other words,* $4x^2 - 1$ *can be factored as* $(2x + 1)(2x - 1)$*.*

The answer is $(4x^2 + 1)(2x + 1)(2x - 1)$*.*

7.3.1 Exercises

1. $a^2 - 9$
2. $x^2 - 25$

3. $x^2 + 64$
4. $4x^2 - 36$
5. $25x^4 - 81y^6$
6. $x^4 - 4x^2$
7. $3x^3 - 75x$
8. (*) (Other special products) Use the formula $a^2+2ab+b^2 = (a+b)^2$ from Exercise 11 to factor the following polynomials:
   (a) $x^2 + 2xy + y^2$
   (b) $x^2 + 10x + 25$
   (c) $x^2 + 12x + 36$
9. (*) (Other special products) Show that $a^2 - 2ab + b^2 = (a - b)^2$ by multiplying the right side. In words, this is a "formula" which says that if we have a sum of squares *with an additional term that is the opposite of twice the product of the two quantities being squared*, it can be factored as the square of the difference of the two quantities. (You will notice that this is actually just a version of the formula in the previous exercises, replacing $-b$ for b.)
10. (*) Use the formula in the previous exercise to factor the following polynomials:
    (a) $x^2 - 2x + 1$
    (b) $x^2 - 18x + 81$
    (c) $x^2 - 8x + 16$
11. (*) (Difference of cubes) Show that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ by multiplying the right side. This is a "formula" which says that if we have a difference of perfect cubes (quantities raised to the third power), the expression can be factored as the product of the difference of the two quantities and the sum of the squares of the two quantities and the product of the two quantities.
12. (*) Use the formula in the previous exercise to factor the following polynomials completely:
    (a) $x^3 - 8$
    (b) $x^3y - 125y^4$
13. (*) (Sum of cubes) Show that $a^3+b^3 = (a+b)(a^2-ab+b^2)$ by multiplying the right side. Unlike the case of squares, sums of perfect cubes can be factored! The "formula" says that if we have a sum of perfect cubes, the expression can be factored as the product of the sum of the two quantities and the sum of the squares of the two quantities minus the product of the two quantities.

14. (*) Use the formula in the previous exercise to factor the following polynomials:

    (a) $x^3 + 64$

    (b) $8x^3 + 27$

7.4 Quadratic trinomials I. Monic trinomials

Many times, we will encounter polynomials having the form $ax^2 + bx + c$, where a, b, and c represent numerical coefficients. When a (the coefficient of x^2) is not zero, this polynomial is **quadratic**, meaning that it is a polynomial of degree 2, or, what is the same, that the highest degree of any term is 2. We will call polynomials of the form $ax^2 + bx + c$ **quadratic trinomials**, given that there are in general three terms whose highest degree is 2. (Notice though that if any of the coefficients are zero, there may be less than three terms.)

We will follow the custom from now on of always using the letter a to represent the coefficient of the degree 2 term (the *quadratic term*), b for the the coefficient of the degree 1 term (the *linear term*), and c for the degree 0 term (the *constant term*).

In the next two sections, we will discuss methods to factor quadratic trinomials. To make the presentation easier, we will first consider an easier case, when $a = 1$. (Polynomials whose leading coefficient is 1 are called *monic* polynomials.) Then, in the next section, we will take up the general case.

As we have in the past sections, let's start with an example. We will try to factor the quadratic trinomial

$$x^2 + 6x + 8.$$

(In the notation of the previous paragraphs, $a = 1$, $b = 6$ and $c = 8$.)

We first check that this polynomial cannot be factored using any of our two prior methods. There is no factor (except 1) common to all three terms. Also, it is clearly not a difference of squares—it has three terms, after all. So both of our methods so far fail.

Let's cheat! Looking back, in Example 6.31, $x^2 + 6x + 8$ happened to have been the result of the multiplication $(x+4)(x+2)$. In other words, $(x+4)(x+2)$ is the factorization for $x^2 + 6x + 8$.

As usual, we can't always hope that every polynomial we want to factor will have been the result of some multiplication problem we had previously done. However, as has been our pattern, let's see if we can find some key features of this example to help us find a general method for factoring (monic) quadratic trinomials.

Let's first set a goal of factoring a quadratic trinomial as a product of two linear (degree 1) polynomials. In fact, if we are factoring a *monic* quadratic trinomial, we will attempt to factor our quadratic trinomial $x^2 + bx + c$ into a product of the form

$$(x + \underline{\quad})(x + \underline{\quad}),$$

where the blanks will represent some numbers that we have to "fill in." (In our example, these numbers were 4 and 2.) Notice that this special form will guarantee that the result will be a monic polynomial, since the only degree 2 term from distributing will be $(x)(x) = x^2$.

How can we find numbers to "fill in the blanks" so that, when we multiply them, we obtain the correct product? Let's try to use our example above for clues. Was there any relationship between the numbers 4 and 2 in the factorization, on the one hand, and the coefficients 6 and 8 in the original trinomial? Actually, there are two relationships that you could notice: first, $6 = 4 + 2$, and second, $8 = (4)(2)$. Our method for factoring monic quadratic trinomials is based on these two important relationships: If a quadratic trinomial $x^2 + bx + c$ can be factored as $(x + p)(x + q)$ for some numbers p and q, then $p \cdot q = c$ and $p + q = b$.

These relationships are the key to the following method.

Factoring a monic quadratic trinomial

If a monic quadratic polynomial $x^2 + bx + c$ with integer coefficients b and c can be factored, the factorization has the form

$$(x + p)(x + q),$$

where p and q are integers satisfying $p \cdot q = c$ and $p + q = b$. To find p and q:

1. List all integer factors of c, positive and negative, in pairs;
2. From this list, find a pair of factors whose sum is b.

If no such integers exist, then the quadratic trinomial cannot be factored.

This technique, like the technique involving difference of squares, amounts to a kind of "fill in the blank"-type formula, where the p and q in this technique are exactly the numbers to "fill in the blanks" in the formula

$$(x + \_\_\_)(x + \_\_\_).$$

Although it may not be obvious from the description, *the signs of p and q are crucial to the method.* The remaining examples of the section will illustrate this point.

Example 7.4.1. *Factor completely:* $x^2 + 7x + 12$.

Answer. *The terms have no factor in common (other than 1). It is not a difference of squares. It is, however, a monic quadratic trinomial, with $b = 7$*

and $c = 12$*. According to the strategy, we will look for factors of* 12 *that add to* 7*.*

Below we list the factors of 12 *in pairs, along with the sums.*

$1, 12$	$(1 + 12 = 13)$	$-1, -12$	$((-1) + (-12) = -13)$
$2, 6$	$(2 + 6 = 8)$	$-2, -6$	$((-2) + (-6) = -8)$
$3, 4$	$(3 + 4 = 7)$	$-3, -4$	$((-3) + (-4) = -7)$

The pair we are looking for is 3 *and* 4*, since their product* $(3)(4)$ *is* 12 *and their sum* $3 + 4$ *is* 7*. These will be the values we will use to "fill in the blanks."*

The answer is $(x + 3)(x + 4)$*. (Again, we remind the reader that the orders of the factors is not important, thanks to the commutative property of multiplication. We could have also written the answer as* $(x + 4)(x + 3)$*.)*

Now that we have illustrated our method with an example, we turn to an example involving negative coefficients.

Example 7.4.2. *Factor completely:* $x^2 - 8x + 7$*.*

Answer. *We check whether we can factor out a common factor or apply the difference of squares formula; neither apply. The polynomial is a quadratic trinomial. In this case* $b = -8$ *and* $c = 7$*. Notice, as always with polynomials, we are considering the polynomial as* $x^2 + (-8x) + 7$*, and the coefficient of the* x*-term is negative.*

Listing the factors of 7 *(there are less this time, since* 7 *is prime!):*

$1, 7$	$(1 + 7 = 8)$	$-1, -7$	$((-1) + (-7) = -8)$

We see that the pair of factors whose product is 7 *and whose sum is* -8 *is* -1 *and* -7*. When we use these numbers to "fill in the blanks," we obtain* $(x + (-1))(x + (-7))$*. Normally, however, we will rewrite the "adding the opposite" as subtraction.*

The answer is $(x - 1)(x - 7)$*.*

Example 7.4.3. *Factor completely:* $x^2 - 2x - 8$*.*

Answer. *The terms of the polynomial have no common factor other than* 1*, and it is not a difference of squares. It is a quadratic trinomial, with* $b = -2$ *and* $c = -8$*.*

Listing the pairs of factors of -8*:*

$1, -8$	$(1 + (-8) = -7)$	$-1, 8$	$((-1) + 8 = 7)$
$2, -4$	$(2 + (-4) = -2)$	$-2, 4$	$((-2) + 4 = 2)$

(Notice that when c *is negative, we should choose our pairs of factors with opposite signs.)*

The pair of factors of -8 *we are looking for is* 2 *and* -4*, since their sum is* -2*.*

The answer is $(x+2)(x-4)$*. (This is NOT the same as* $(x-2)(x+4)$*, as you can see by multiplying!)*

Example 7.4.4. *Factor completely:* x^2+2x+6*.*

Answer. *The terms of the polynomial have no common factor other than* 1*, and it is not a difference of squares. It is a quadratic trinomial, with* $b=2$ *and* $c=6$*.*

Listing the pairs of factors of 6*:*

$1, 6$	$(1+6=7)$	$-1,-6$	$((-1)+(-6)=-7)$
$2, 3$	$(2+3=5)$	$-2,-3$	$((-2)+(-3)=-5)$

In this example, none of the pairs add up to the value of b*, which was* 2*.*

The polynomial cannot be factored.

Example 7.4.5. *Factor completely:* $2x^3-10x^2-48x$*.*

Example 7.4.6. *In this example, the three terms do have a common factor: the GCF is* $2x$*. The first step then will be to factor out the GCF:*

$$2x(x^2-5x-24).$$

However, we cannot yet say that the polynomial is factored completely*, since the second factor is a quadratic trinomial (with* $b=-5$ *and* $c=-24$*). The next step will be to attempt to factor* $x^2-5x-24$*.*

Listing the pairs of factors of -24*:*

$1,-24$	$(1+(-24)=-23)$	$-1,24$	$((-1)+24=23)$
$2,-12$	$(2+(-12)=-10)$	$-2,12$	$((-2)+12=10)$
$3,-8$	$(3+(-8)=-5)$	$-3,8$	$((-3)+8=5)$
$4,-6$	$(4+(-6)=-2)$	$-4,6$	$((-4)+6=2)$

We see that the pair we are looking for is 3 *and* -8*, since their sum is* -5 *. We can use these to "fill in the blank" and factor the quadratic trinomial, not forgetting about the factor of* $2x$ *we already found.*

The answer is $2x(x+3)(x-8)$*.*

7.4.1 Exercises

Factor the following quadratic trinomials.

1. $x^2-4x-32$
2. $y^2+3y-18$

3. $x^2 + 8y + 12$
4. $x^2 - 12x + 20$
5. $x^2 + 3x + 15$
6. $y^2 - y - 6$
7. $t^2 + 6t + 9$
8. $x^2 - 6x + 5$

Factor the following polynomials using any technique.

9. $3x^3 - 9x^2 - 12x$
10. $5x^2 - 80$
11. $6x^3 - 2x$
12. $2x^2 + 8$
13. $4x^4 - 12x^3 - 8x^2$
14. $3x^3 - 12x$
15. (*) Apply the method of this section to factor the following polynomials that have "quadratic form." In each case, an appropriate substitution will help. Don't forget to factor completely!
    (a) $x^4 + 5x^2 + 6$ (Hint: Substitute $u = x^2$).
    (b) $x^4 - x^2 - 12$
    (c) $x^6 + 2x^3 - 8$ (Hint: Substitute $u = x^3$).
    (d) $x^{64} - 10x^{32} + 9$

7.5 Quadratic trinomials II. The *ac*-method

The previous section showed that, at least for monic quadratic trinomials, the coefficients of the terms give important information as to how to factor the trinomial into a product of linear factors.

Why was it so important in the previous section that the quadratic trinomial be monic? Since the leading coefficient was 1, the coefficients of x in the linear factors were also forced to be 1, and so there were only two numbers left to find in the "formula" $(x + \underline{\quad})(x + \underline{\quad})$.

In the case of a non-monic quadratic trinomial $ax^2 + bx + c$ $(a \neq 1)$, we have no guarantee about the coefficients of x in the linear factors. One way to proceed would be to try to make a more elaborate "fill-in-the-blank" strategy, now having the form

$$(\underline{\quad}\ x + \underline{\quad})(\underline{\quad}\ x + \underline{\quad}).$$

In fact, this would be a reasonable approach. Of course, it would have to involve a, b and c, not just b and c like in the monic case.

We will follow a different strategy, however. Instead of the brute-force guessing and checking that the fill-in-the-blank approach would involve, we will follow a strategy that is designed to genuinely reverse the distributive law involved in multiplying two binomials. This method will be a little longer, but it involves no guessing at all. It is called the ac-method for reasons that will be clear shortly.

We will illustrate this method with an example. Let's try to factor the quadratic trinomial

$$6x^2 + 19x + 10.$$

The terms have no factors in common, and it is clearly not a difference of squares. And while the polynomial is a quadratic trinomial, it is not monic, since $a = 6$. A quick check will reveal that the fill-in-the-blank procedure of the previous section will lead nowhere in this case.

We will present the steps in the context of this example, then summarize the steps at the end.

Example 7.5.1. *Factor completely:* $6x^2 + 19x + 10$.

Answer. *We follow a four-step approach.*

Step 1. Form the product ac***.*** *In this case,* $a = 6$ *and* $c = 10$*, so the product* ac *is* $(6)(10) = 60$.

Step 2. Find a pair of factors of ac ***whose sum is*** b***.*** *We are looking for factors of* 60 *(from Step 1) whose sum is* $b = 19$*. This is exactly the process we used in the previous section; a little work will show that the pair of numbers we are looking for is* 4 *and* 15 *(since* $(4)(15) = 60$ *and* $4 + 15 = 19$*).*

Step 3. Use the pair of factors from Step 2 to "split" the x***-term.*** *We are going to rewrite the middle term using the two numbers we found in Step 2:*

$$6x^2 + 4x + 15x + 10.$$

Notice we have not changed the polynomial in any way, since $15x + 4x$ *is* $19x$*. We have only changed the way the polynomial is written.*

Step 4. Factor by grouping. *The heart of the ac-method is the following procedure. First, we will group the four terms from Step 3 into two groups:*

$$(6x^2 + 4x) + (15x + 10).$$

We are going to try to factor each group separately*. For example, the first group* $6x^2 + 4x$ *has a common factor of* $2x$*, which we can factor out:* $2x(3x + 2)$*. Likewise, the second group has a common factor of* 5*, which can be factored out to obtain* $5(3x + 2)$*. In other words, our polynomial now has the form*

$$2x(3x + 2) + 5(3x + 2).$$

Written in this way, the polynomial has two terms, one from each group. Notice that these two terms have a common factor of $(3x+2)$*! Even though this common* *factor looks more complicated that our usual monomial common factors, we treat it the same way: we factor it out. We will write the common factor* $3x + 2$ *outside, and we will be left with a factor of* $2x$ *(from the first term) and* 5 *(from the second term):*

$$(3x + 2)(2x + 5).$$

The answer is $(3x + 2)(2x + 5)$*.*

At this point, the reader should look back at Example 6.32, where we performed the multiplication $(2x + 5)(3x + 2)$ (the same as our final answer with the order of factors reversed) to obtain $6x^2 + 19x + 10$. The steps involved in that multiplication example were exactly the same as the steps of this method of factorization, but in reverse! The ac-method is designed to mimic (in reverse) the process of distributing in the case of multiplying a binomial by a binomial.

One thing should be pointed out right away. The order of the pair in splitting the middle term in Step 3 does not matter. The reader can verify this fact in the previous example, writing $6x^2 + 15x + 4x + 10$ instead of $6x^2 + 4x + 15x + 10$. The result should be the same, with the order of factors reversed. (We will see, however, that sometimes one way of splitting the middle term will give an easier result than the other.)

For the reader's reference, we repeat the four-step ac-method here.

Factoring a quadratic trinomial: The ac-method

If a quadratic polynomial $ax^2 + bx + c$ with integer coefficients a, b and c can be factored as a product of linear factors having integer coefficients, then the following procedure will give the factorization:

1. Form the product ac;
2. Find a pair of factors of ac whose sum is b;
3. Use the pair from Step 2 to "split" the x-term into a sum of two terms having the pair of numbers as coefficients;
4. Factor the resulting polynomial by grouping.

If there is no pair of factors of ac whose sum is b, then the quadratic trinomial cannot be factored into a product of linear factors with integer coefficients.

For the rest of this section, we will write our quadratic polynomials in descending order (as we usually do anyway). For this reason, we will sometimes refer to the x-term as the "middle term."

We will now illustrate the *ac*-method with several examples. Along the way, we will point out three "tips" to make using the *ac*-method easier.

Example 7.5.2. *Factor completely:* $3x^2 - 8x + 4$.

Answer. *First, notice that the three terms have no common factor other than* 1*, and that the polynomial is not a difference of squares. It is a quadratic trinomial, and it is not monic, since* $a = 3$*. We will use the ac-method.*

The product ac in this example is 12*. So we need to find a pair of factor of* 12 *whose sum is* -8*. The pair is* -6 *and* -2*.*

We use this pair to split the middle term:

$$3x^2 - 6x - 2x + 4.$$

There is a small but important difference in this example from the previous one: the coefficient of the second x-term is negative (of course, so is the coefficient of the first x-term, but that matters less). In this case, we are going to factor out a negative number.

We group the polynomials:

$$(3x^2 - 6x) + (-2x + 4).$$

The first group has a common factor of $3x$*. Factoring out we obtain* $3x(x-2)$*. The second group has a common factor of* 2*. However, as we mentioned, we will factor out* -2 *instead to obtain* $-2(x-2)$*. (Be careful of the signs when factoring out a negative number!) In other words, we obtain:*

$$3x(x-2) - 2(x-2).$$

Because we made the effort to factor out a negative number from the second group, we see the factor $(x-2)$ *in common to the two groups, giving a factorization of* $(x-2)(3x-2)$*.*

The answer is $(x-2)(3x-2)$*.*

The previous example contains an important lesson:

Helpful hint # 1: When factoring a polynomial whose leading coefficient is negative, it is usually a good idea to factor out a negative common factor.

The next example shows that this hint also leads to another tactic to make factoring simpler.

Example 7.5.3. *Factor completely:* $2x^2 - x - 10$.

Answer. *We check to see that the three terms have no common factor, and that the polynomial is not a difference of squares. It is a quadratic trinomial which is not monic (since* $a = 2$*), suggesting the ac-method.*

We first form the product ac, with $a = 2$ *and* $c = -10$*, so* $ac = -20$*.*

We now try to find a pair of factors of -20 *whose sum is* -1*. Listing the factors if necessary, we find that* 4 *and* -5 *are factors of* -20 *whose sum is* -1*, as required.*

As mentioned earlier, the order of this pair does not matter when splitting the middle term. This time, though, the factors have different signs. As we saw in the previous example, if we write the term with the negative coefficient second (to obtain $2x^2 + 4x - 5x - 10$*), we should aim to factor out a negative factor. Since this requires special care about the sign of the other term, we will instead write the term with the negative coefficient first:*

$$2x^2 - 5x + 4x - 10.$$

Grouping the terms as $(2x^2 - 5x) + (4x - 10)$*, we see that the first group has a common factor of* x*, while the second group has a common factor of* 2*. Factoring the two groups separately, we obtain*

$$x(2x - 5) + 2(2x - 5).$$

As we expect in the ac-method, we see that the two resulting terms have a common factor of $2x - 5$*. Factoring it out, we obtain*

$$(2x - 5)(x + 2).$$

The answer is $(2x - 5)(x + 2)$*.*

Exercise 7.5.4. *For practice, re-do the previous example, splitting the middle term as*

$$2x^2 + 4x - 5x - 10.$$

The lesson of the previous example can be summarized in the following tip.

> **Helpful hint # 2**: If the pair of factors used to split the middle term in the *ac*-method have different signs, it is usually more convenient to write the term with the negative coefficient *first*.

Example 7.5.5. *Factor completely:* $4x^3 + 4x^2 + 2x$*.*

Answer. *Notice first that this polynomial is* not *a quadratic trinomial. It is a trinomial, of course, but it is not quadratic since the leading term has degree* 3*.*

However, the three terms have a common factor of $2x$*. So we immediately factor out the (greatest) common factor to obtain*

$$2x(2x^2 + 2x + 1).$$

Although we now have a factorization, since we have written the polynomial as a product of two factors, we need to decide whether the polynomial is factored completely*. In particular, since the second factor* $2x^2 + 2x + 1$ *is a quadratic* *trinomial which is not monic (since $a = 2$), we should try to apply the ac-method to determine whether it can be factored further.*

To apply the ac-method to factor $2x^2 + 2x + 1$, we see that the product ac is 2, since $a = 2$ and $c = 1$. So we need to find factors of 2 whose sum is 2 (since $b = 2$). It shouldn't take long to check that there is no pair of factors that satisfy this property. In this case, the ac-method determines that the polynomial $2x^2 + 2x + 1$ cannot be factored into a product of linear factors.

The answer is $2x(2x^2 + 2x + 1)$.

The final example of this section will lead to one last hint to keep in mind when applying the *ac*-method.

Example 7.5.6. *Factor completely: $12x^2 - 33x - 9$.*

Answer. *Looking at the three terms, there is a common factor of 3. The first step will be to factor it out:*

$$3(4x^2 - 11x - 3).$$

As in the previous example, we need to determine whether the remaining quadratic trinomial $4x^2 - 11x - 3$ can be factored further as a product of linear factors. We will apply the ac-method, using $a = 4$, $b = -11$ and $c = -3$. (Notice that to do this, the common factor of 3 no longer needs to be considered, although it will remain in the final factorization.)

The product ac in this case is -12. We will look for a pair of factors of -12 whose sum is -11; such a pair is -12 and 1. Using this pair to split the middle term (writing the factor with the negative coefficient first), we obtain

$$4x^2 - 12x + x - 3.$$

Grouping the factors as $(4x^2 - 12x) + (x - 3)$, we see that the first group has a common factor of $4x$. The second group, however, normally would not be factored, since the only common factor of x and -3 is 1. However, to make the factorization more clear, we are going to factor out the common factor of 1! In other words, factoring the two groups separately we obtain

$$4x(x - 3) + 1(x - 3).$$

Written in this way, we see that the two terms have a common factor of $x - 3$. Factoring out this common factor, we obtain $(x - 3)(4x + 1)$. In other words, the factor $4x^2 - 11x - 3$ can be factored as a product of linear factors.

The answer is $3(x - 3)(4x + 1)$.

There are two things to notice about the previous example. First, taking the time to factor out the common factor first, apart from being good general practice, made the *ac*-method much smoother. After all, if we had applied the

ac-method to the quadratic trinomial $12x^2 - 33x - 9$ without factoring out the 3 first, we would need to find factors of -108 whose sum is -33. That can be done, but who wants to go through the list of 12 pairs of factors of -108? Even if we did that, we would still see that one of the linear (binomial) factors would have a common factor of 3 that we would still have to factor out in order to factor completely. Always look for common factors first!

The second thing to notice is more particular to the ac-method.

> **Helpful hint # 3**: When factoring by grouping using the ac-method, if one of the groups has no common factor other than 1, take the time to factor out the common factor of 1.

To close this section, we point out that even though we have developed the ac-method as a method of factoring non-monic quadratic trinomials, the method also works for the monic trinomials in the previous section. (We don't normally use the method for monic trinomials, though, since the "shortcut" presented in the previous section is so much faster.) In the challenge exercises, we give some other examples of situations where the ac-method can help.

7.5.1 Exercises

Factor the following polynomials completely.

1. $2x^2 - x - 55$
2. $3x^2 + 4x + 1$
3. $6x^2 + x - 2$
4. $15x^2 + x - 2$
5. $5x^2 - 3x - 1$
6. $2x^2 - x - 10$
7. $6x^2 - 22x + 20$
8. $2x^4 - 4x^3 - 16x^2$
9. (*) Factor completely the following polynomials that have "quadratic form." In each case, an appropriate substitution will help.
   (a) $2x^4 + 5x^2 + 3$ (Hint: Substitute $x^2 = u$, so $x^4 = u^2$).
   (b) $4x^4 + x^2 - 5$
   (c) $3x^6 - 10x^3 + 3$ (Hint: Substitute $u = x^3$).
   (d) $2x^6 + 5x^3 - 7$
   (e) $4x^{1000} - 9x^{500} - 9$

10. (*) Use the *ac*-method to factor the following quadratic trinomials *in two variables.*

    (a) $x^2 - xy - 12y^2$

    (b) $x^2 + 3xy + 2y^2$

    (c) $2x^2 - 5xy - 3y^2$

    (d) $3x^2 - 2xy - y^2$

7.6 Factoring by grouping

We will end our discussion of factoring by noticing that the technique of factoring by grouping, which we used as a key component of the *ac*-method, can be applied in a wider setting.

In each of the following examples, you will notice that none of the methods we have discussed so far can be applied: they have no factors (other than 1) common to all terms, they are not differences of squares, and they are not quadratic trinomials. However, because they have four terms involving pairs of variables, there is hope that they may be factored into a product of two binomials.

Example 7.6.1. *Factor completely:* $3ax + 2ay + 12bx + 8by$.

Answer. *The reader should first check that none of the preceding factoring techniques can be applied to this polynomial (in four variables!).*

Let's try to factor by grouping. Write the polynomial in two groups:

$$(3ax + 2ay) + (12bx + 8by).$$

We notice that the first group has a common factor of a*, while the second group has a common factor of* $4b$*. So factoring the two groups separately, we obtain*

$$a(3x + 2y) + 4b(3x + 2y).$$

Since the two terms now have a common factor of $3x + 2y$*, we can factor it out to obtain* $(3x + 2y)(a + 4b)$*.*

The answer is $(3x + 2y)(a + 4b)$*.*

The reader should notice that the procedure in this example is exactly the same as the one we encountered every time we apply the *ac*-method. There is an important difference, though. In applying the *ac*-method, the two terms obtained by factoring a common factor from the two groups separately *will always have a common factor* (as long as the middle term is split by using the factors of ac whose sum is b). For arbitrary polynomials like the ones we are looking at now, the two terms might not have a common factor *even if the two groups can be factored separately.*

Example 7.6.2. *Factor completely:* $6np + 4nq - 15mp - 10mq$.

Answer. *First check that none of the preceding factoring techniques can be applied to this polynomial.*

Again, we group the terms:

$$(6np + 4nq) + (-15mp - 10mq).$$

We see that the first group $6np+4nq$ *has a common factor of* $2n$. *In the second group* $-15mp-10mq$, *notice that the first term is negative. As in Example 7.5.2 above, we will factor out a common factor with a negative coefficient—in this case* $-5m$. *So we obtain*

$$2n(3p + 2q) - 5m(3p + 2q).$$

(Notice again that factoring out the -5 *in the second group changes the signs of both terms in the group.)*

Like last time, the two terms have a common factor of $3p + 2q$. *Factoring out, we obtain*

$$(3p + 2q)(2n - 5m).$$

The answer is $(3p + 2q)(2n - 5m)$.

There is one thing worth mentioning about the preceding example. Instead of grouping the first two terms and the last two terms, we could have rearranged the terms first as $6np - 15mp + 4nq - 10mq$. The reader should check that the result after factoring by grouping is the same as the result above (with the order of the factors possibly different). The difference is that written in this different order, there is no need to factor out a negative factor in the second group, which might eliminate some difficulty with signs.

Reordering the terms in the last example might have been helpful, but it was optional. The next example shows that sometimes reordering the terms is essential to apply the method of factoring by grouping we have described.

Example 7.6.3. *Factor completely:* $3sx + 2ty - 3tx - 2sy$.

Answer. *As usual, we check to see that none of the preceding factoring techniques can be applied to this polynomial.*

This time, if we try to group in the most obvious way, as $(3sx + 2ty) + (-3tx - 2sy)$, *neither term has any common factor at all (except* 1 *of course). However, before giving up, let's try to rearrange the terms. For example, let's try rewriting the polynomial as*

$$3sx - 3tx + 2ty - 2sy.$$

Now, grouping as usual, we get

$$(3sx - 3tx) + (2ty - 2sy).$$

Written this way, we see that the first group has a common factor of $3x$ while the second group has a common factor of $2y$. Factoring the two groups separately, we get

$$3x(s-t)+2y(s-t).$$

These two terms have a common factor of $s-t$. Factoring out this common factor, we get

$$(s-t)(3x+2y).$$

The answer is $(s-t)(3x+2y)$.

We end this section on a pessimistic note.

Example 7.6.4. *Factor completely: $6ax+2ay+5bx-3by$.*

Answer. *This polynomial has no factor common to all four terms. It is not a difference of squares or a quadratic trinomial. Since there are four terms with pairs of variables, we will try to factor by grouping.*

Grouped in the obvious way

$$(6ax+2ay)+(5bx-3by),$$

we see that the first group has a common factor of $2a$ while the second group has a common factor of b. Factoring the groups separately, we get

$$2a(3x+y)+b(5x-3y).$$

However, the two resulting terms have no factor in common! This obstacle is serious, since even though the groups have been factored, we still have not written the whole original polynomial as a product of two factors, since there are still two terms.

Before we give up, we remember from the last example that sometimes reordering the terms can be helpful. So let's try rewriting the original polynomial as $6ax+5bx+2ay-3by$. Now, grouping as

$$(6ax+5bx)+(2ay-3by),$$

we see that the first group has a common factor of x and the second group has a common factor of y. Factoring the two groups separately, we obtain

$$x(6a+5b)+y(2a-3b).$$

Again, these two terms have no common factor.

The reader should try other ways to reorder the terms to convince themself that in no case can we obtain two terms with a common factor, as we have above.

The polynomial cannot be factored.

7.6.1 Exercises

Factor by grouping, if possible.

1. $6xz + 9xw - 8yz - 12yw$
2. $45cw + 63cz - 20dw - 28dz$
3. $20ax - 24ay + 15bx + 18by$
4. $4ac - 9bd - 3ad + 12bc$
5. $20ax - 15ay - 8bx + 6by$
6. $3mx + 6my - 2nx - 4ny$

7.7 Chapter summary

- To factor a polynomial means to write it as a product of two or more factors, none of which are 1.
- Not every polynomial can be factored.
- In order to factor a polynomial, we have the following checklist to apply:

Factoring checklist

To factor a polynomial, answer the following questions in the given order:

1. Do the terms have any factors in common?
   - If so, "factor out" the greatest common factor.
2. Is the polynomial a difference of squares?
   - If so, apply the "formula" $a^2 - b^2 = (a + b)(a - b)$.
3. Is the polynomial a quadratic trinomial of the form $ax^2 + bx + c$?
   - If so, and $a = 1$, find factors of c whose sum is b to "fill in the blanks"
   $$(x + \_\_\_)(x + \_\_\_).$$
   - If so, and $a \neq 1$, apply the ac-method.
4. Can the method of factoring by grouping be applied?

If the above list gives a factorization of the polynomial, make sure to apply the checklist to the each of the factors to make sure the polynomial is factored *completely*.

Part IV

Quadratic equations and radical expressions

Chapter 8

Radical expressions

Vocabulary

- Quadratic equation
- Rational numbers
- Irrational numbers
- Radicand
- Rationalizing (a denominator)
- Imaginary unit
- Real numbers and complex numbers

8.1 Introduction: Quadratic equations and number systems

For the last two chapters, we have worked with polynomials as algebraic objects on which we can perform "symbolic arithmetic." We will now return to the question of solving equations. Up to this point, we have developed an approach to solving *linear* equations: degree one polynomial equations in one or two variables.

In particular, we will attempt to solve quadratic equations, or polynomial equations of degree two. However, we will see in this introductory section that these equations—even simple ones—force us to face some fundamental problems not just about algebra, but about the number systems we have been working with. That will be the topic of this chapter.

Up to now, we have been working almost exclusively with **rational numbers**—numbers that can be expressed as a ratio of two integers. That means the "worst" numbers we have had to work with have been fractions or (repeating) decimals. We will see that the setting of rational numbers is not adequate to solve any but the simplest quadratic equations. In fact, solving quadratic equations will force us to face "numbers" that have properties that are quite different from any we have seen up to now.

Let's start our discussion of quadratic equations with a simple example of a quadratic equation in one variable:

$$x^2 = 9.$$

This equation (in one variable) is quadratic, since the highest power of x is 2. A little bit of trial and error quickly reveals that 3 is a solution to this equation: substituting, $(3)^2 = 9$ is a true equation. After all, 9 is a perfect square; said in a different way, 3 is a square root of 9.

However, remember that to solve the equation means to find *all* solutions. So far we have found one. Keeping in mind, though, that the square of a negative number is positive, we can see that -3 is also a solution to $x^2 = 9$, since $(-3)^2 = 9$ is a true equation. We can already see a major difference between quadratic equations and linear equations: *a typical quadratic equation may have more than one solution.* (Recall that if a linear equation in one variable has more than one solution, then every number is a solution to that equation.) It shouldn't be hard to convince yourself that no other number is a solution.

Our first simple example points to a crucial fact about quadratic equations that will be at the core of all of our strategies to solve quadratic equations in Chapter 9:

Basic fact about quadratic equations: A typical quadratic equation in one variable has no more than two solutions.

We will return to this fact shortly to state it in a more precise way. (It can be proved with more detailed knowledge of polynomials, usually discussed in a precalculus course.)

Let's look at another simple example, which on the surface looks no different from the last one:

$$x^2 = 5.$$

The problem with this equation is that when we try to guess a solution, like we did last time, no obvious solution appears. After all, unlike 9 that appeared in the last equation, 5 is not a perfect square.

We might try a more refined version of guessing. We see, for example, that $(2)^2 = 4$ is less than 5, while $(3)^2 = 9$ is greater than 5. So we might guess a number between 2 and 3—say 2.5. With a little bit more work calculating, we see that $(2.5)^2 = 6.25$, still larger than 5. So we'll guess again, this time between 2 and 2.5—maybe 2.25. We see that $(2.25)^2 = 5.0625$, close, but still a little too large! If we try 2.24, we see that $(2.24)^2 = 5.0176$—closer, but still too large. If we try 2.235, we see that $(2.235)^2 = 4.995225$, which is less than 5 but closer still. We might say that a solution to $x^2 = 5$ is "approximately 2.235." That seems a little unsatisfying, though, as mathematics aims to be a precise discipline.

At the core of this problem is a fundamental fact about whole numbers: *If a whole number is not a perfect square, than its square root is not a rational number.* Since rational numbers either have a terminating or a repeating decimal expansion, this means that if $x^2 = 5$ has a solution, its decimal expansion will never terminate and it will never repeat. This is somewhat depressing from the point of view of the last paragraph, where we tried to "guess" a solution of $x^2 = 5$. Having said that, our guessing attempts do seem to point to the fact that a solution does, in fact, exist, even though it might be hard to pin down exactly what that solution is.

Our discussion of the quadratic equation $x^2 = 5$ has led us to numbers that might be completely unnecessary from the point of view of linear equations. We can rephrase the statement of the previous paragraph: *The solutions to $x^2 = k$, where k represents a positive integer which is not a perfect square, are* ***irrational numbers****.*[1] The next section will be devoted to working with irrational numbers, which will play an essential role in solving quadratic equations.

One final example will reveal another basic problem in solving quadratic equations. Consider the quadratic equation

$$x^2 = -4.$$

We might have an initial glimmer of hope, seeing that 4 is a perfect square. But we are looking for a number which, when multiplied by itself, gives a negative number. This reveals a problem that is much more basic than the problem of irrational numbers in the previous problem: *No number, rational or irrational, when squared, will give a negative number.* This fact might tempt us to simply say that the equation has no solution. In the Section 8.3, we give another way to handle this problem by introducing a new kind of "number" called *complex numbers.*

Let's summarize what this short discussion of the most simple quadratic equations in one variable has revealed.

[1] The discovery of irrational numbers—numbers which cannot be written as the ratio of two integers—is usually credited to the ancient Greek school of thinkers known as the Pythagoreans nearly 2,500 years ago. It is an irony of history that this discovery, derived logically, should have come from the Pythagoreans, according to whose world view all things could be understood as a ratio of whole numbers.

Important features of quadratic equations in one variable

- A typical quadratic equation in one variable will have two solutions.
- A quadratic equation may have irrational solutions (even if its coefficients are rational numbers).
- A quadratic equation may have solutions which are complex numbers.

8.2 Radical expressions

In this section, we will establish some conventions about how we will treat the types of irrational numbers that arise in solving quadratic equations. In particular, we will develop a way of writing square roots symbolically.

We already saw in the previous section that solutions to the equation $x^2 = a$ are irrational numbers whenever a is a whole number that is not a perfect square. In that discussion, we saw that solutions to this simple type of quadratic equation is closely related to the notion of a square root, which we treated in Chapter 1 as an operation. One way to treat irrational square roots would be to agree, in advance, that we will estimate them to a given decimal accuracy. For example, if we agree to estimate square roots to 8 decimal places, a calculator might tell us that the square root of 5 is given by $\sqrt{5} \approx 2.23606798$. If, on the other hand, we agree to estimate to 12 decimal places, we would write $\sqrt{5} \approx 2.236067977500$.

We are going to handle irrational square roots in a different way. Instead of estimating (which depends in practice on using a calculator), we will adopt a symbolic approach.

The square root as a symbol

For any non-negative number k, the symbol $\sqrt{k}$ ("the square root of k") represents the non-negative solution to the equation $x^2 = k$.

Said in another way, the symbol $\sqrt{k}$ represents the non-negative number which satisfies

$$(\sqrt{k})^2 = k.$$

It is worthwhile pointing out what is new about this definition. In our previous understanding of square roots, $\sqrt{k}$ actually consisted of two separate symbols: k (representing a number) and the *radical sign* $\sqrt{}$, which represented an operation performed on the number k. In our new definition, by contrast, $\sqrt{k}$ represents *one symbol* with two parts: the radical sign and the *radicand*, as the quantity k inside the radical sign is known.

Keep in mind from the introductory section that the equation $x^2 = k$ typically has two solutions, one positive and one negative. In this case, $\sqrt{k}$ represents the *positive* solution.

From the point of view of solving quadratic equations, at least for simple ones, this definition in a certain sense "cheats." We have *defined* the symbol $\sqrt{k}$ to be the nonnegative solution of $x^2 = k$; the other solution will then be its opposite, written as $-\sqrt{k}$. The following examples illustrate this point.

In fact, we can write this as our first strategy to solve quadratic equations.

First strategy to solve quadratic equations of the form $x^2 = k$

An equation having the form $x^2 = k$ has two solutions, written symbolically as $\sqrt{k}$ and $-\sqrt{k}$.

For now, we will be applying this strategy when k represents a nonnegative number. In Section 8.3, we will consider what happens when k is negative.

Notice that in the special case $x^2 = 0$, the "two" solutions $\sqrt{0}$ and $-\sqrt{0}$ are the same—they are both 0. But rather than thinking of this as a special case of a quadratic equation with only one solution, it is more convenient to think of this as a quadratic equation with two solutions that just happen to be the same.

Example 8.2.1. *Solve the following quadratic equations:*

(a) $x^2 = 7$*;*

(b) $x^2 = 129$*;*

(c) $x^2 = 15$*.*

Answer. *(a) The positive solution to* $x^2 = 7$ *is* $\sqrt{7}$ *(by definition!). The negative solution is written* $-\sqrt{7}$*. So the solutions are* $\sqrt{7}$ *and* $-\sqrt{7}$*.*

(b) The solutions are $\sqrt{129}$ *and* $-\sqrt{129}$*.*

(c) The solutions are $\sqrt{15}$ *and* $-\sqrt{15}$*.*

The point of these simple examples is not really to show how to solve a quadratic equation—although we have technically done so. The point is to illustrate the fact that we have used a symbol to represent a solution to an equation. It has the advantage of needing no estimation. For example, $\sqrt{129}$ is a symbol representing the *exact value* of the positive solution to $x^2 = 129$. It has the disadvantage, however, of hiding the fact that this symbol $\sqrt{129}$ is a symbol for an actual (irrational) number, whose value is approximately $11.357816691600547221\ldots$.

There is another, more serious, disadvantage to this symbolic approach. If we get too excited about this new notation, we would be tempted to say that the solutions to the quadratic equation $x^2 = 9$ are $\sqrt{9}$ and $-\sqrt{9}$, and done! This seems like a wrong way to answer a question, when it is much more easily understood to say that the solutions are 3 and -3. Unlike the examples above, 9 is a perfect square, and so there is no need to have to have a special notation made specifically to address irrational numbers.

For these (and other related) reasons, we are going to agree to a series of rules about how we write square roots symbolically. These rules have evolved over the course of history and are generally accepted.

Simplified square root notation

The symbol $\sqrt{k}$ is called **simplified** if the following conditions hold:

1. The radicand k has no perfect square factors;
2. The radicand k contains no fractions.

In addition, any expression containing radicals must satisfy a third condition:

3. No radical expression shall appear in a denominator.

What happens if we encounter a radical expression which is not simplified? We will take the effort to *simplify* it, rewriting it in an equivalent form which is simplified according to the conditions above. In order to do this, we will rely on two basic properties of square roots.

Some properties of square roots

If a and b represent non-negative numbers, then:

(S1) $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$; (8.1)

(S2) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

These properties are really just versions of the properties of exponents we discussed in Chapter 6. For example, the first property really says that the number represented by $\sqrt{a} \cdot \sqrt{b}$ should be the (non-negative) solution to $x^2 = ab$. Is this true? Since $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$, $(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 = ab$, by property (E4) of exponents. So $\sqrt{a} \cdot \sqrt{b}$ really is a solution to $x^2 = ab$, and the equality of Property (S1) is valid.

We now show through a series of examples how these properties can help us to simplify square roots.

Example 8.2.2. *Simplify the following square roots:*

(a) $\sqrt{12}$*;*

(b) $\sqrt{72}$*;*

(c) $\sqrt{75}$.

Answer. *Before applying the properties of square roots, let's look at the three square roots we are being asked to simplify. Notice that all three represent irrational numbers, since neither* 12*,* 72 *nor* 75 *are perfect squares. However, all three have perfect square* factors. *For this reason, none of the three square roots are simplified; they all violate Rule 1 of our definition of a simplified square root.*

To simplify them, we will write the radicand as a product of a perfect square (preferably as large as possible, if there are more than one) with another number, and then apply Property (S1).

(a) 12 *has a perfect square factor of* 4*. So*

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.$$

The answer is $2\sqrt{3}$.

(b) 72 has several perfect square factors, but the largest is 36.

$$\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}.$$

The answer is $6\sqrt{2}$. (What would have happened if we had factored out the perfect square factor of 9?)

(c) 75 has a perfect square factor of 25.

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}.$$

The answer is $5\sqrt{3}$.

The next example illustrates how to cope with a radical expression with a fraction in the radicand.

Example 8.2.3. *Simplify the following radical expressions:*

(a) $\sqrt{\frac{3}{4}}$;

(b) $\sqrt{\frac{50}{9}}$;

Answer. *These radical expressions violate Rule 2 for simplified square roots. However, we can apply Property (S2) of square roots directly:*

(a) Applying Property (S2):

$$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}.$$

Notice we did not eliminate the fraction from the expression. However, the only remaining radicand (which is 3) does not involve fractions.

The answer is $\frac{\sqrt{3}}{2}$.(Now try this: Use a calculator to find an approximate numerical value for the expression $\sqrt{3}/2$ (by finding an estimate for $\sqrt{3}$ and dividing by 2). Square the result. What number do you obtain? Compare your answer to the original expression.)

(b) Applying Property (S2):

$$\sqrt{\frac{50}{9}} = \frac{\sqrt{50}}{\sqrt{9}} = \frac{\sqrt{50}}{3}.$$

This time, although we have an expression that satisfies Rule 2 for simplified square roots, the remaining radicand of 50 still has a perfect square factor of 25. Hence

$$\frac{\sqrt{50}}{3} = \frac{\sqrt{25 \cdot 2}}{3} = \frac{\sqrt{25} \cdot \sqrt{2}}{3} = \frac{5\sqrt{2}}{3}.$$

The answer is $\frac{5\sqrt{2}}{3}$.

In the previous example, we were lucky to encounter fractions in the radicand whose denominators were perfect squares. The next example illustrates how to simplify when this is not the case. It involves a technique known as *rationalizing the denominator*. Here, the word "rationalize" implies making an irrational number rational by multiplying by an appropriate number.

Example 8.2.4. *Simplify the following square roots:*

(a) $\sqrt{\dfrac{1}{2}}$;

(b) $\sqrt{\dfrac{3}{8}}$.

Answer. *The first thing we notice about both examples is that we have a fraction in the radicand. As in the last example, we begin by using Property (S2) of square roots.*

(a) Applying Property (S2),

$$\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Unfortunately, this time, the radicand in the denominator is not a perfect square. According to Rule 3, this expression is not yet completely simplified, since there is a square root symbol remaining in the denominator.

Our strategy will be to multiply the numerator and denominator of the fraction by the same quantity (in other words, multiply the expression by 1*, which does not change the expression). We will choose the quantity in such a way that, after using Property (S1), the radicand in the denominator becomes a perfect square.*

In particular, we will ask: what perfect square has the given radicand as a divisor? In this example, the smallest perfect square that has 2 *as a divisor is* 4*. In order to obtain a radicand of* 4 *in the demoninator, we will multiply the numerator and denominator by* $\sqrt{2}$*:*

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}.$$

Notice that we chose what radical expression to multiply the denominator (and the numerator) specifically in order to obtain a perfect square as a radicand in the denominator, after applying Property (S1). Notice that this process does not eliminate the radical expression completely. It only changes the expression is written in such a way that the radical expression appears in the numerator and not in the denominator, in compliance with our Rule 3 of simplified radical expressions.

The answer is $\dfrac{\sqrt{2}}{2}$.

(b) Again, we begin by applying Property (S2):

$$\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}}.$$

As in Example (a), we are left with a radical expression in the denominator. Although we notice that $\sqrt{8}$ can be simplified, since 8 has a perfect square factor of 4, we will first address the more serious problem of the radical expression in the denominator.

Our goal is to obtain a radicand in the denominator which is a perfect square. We look for a perfect square which has 8 as a factor; the smallest such perfect square is 16. In order to obtain the radicand of 16 in the denominator, we multiply the denominator (and numerator) by $\sqrt{2}$:

$$\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}.$$

The answer is $\frac{\sqrt{6}}{4}$.

(In the challenge exercises at the end of Section 8.4, we will come back to the question of rationalizing the denominator for more complicated cases.)

We close this section by incorporating our practice of simplifying radical expressions into solving quadratic equations, following the approach of Example 8.2.1.

Example 8.2.5. *Solve the following quadratic equations.*

(a) $x^2 = 16$.

(b) $x^2 = 98$.

Answer. *In each case, we will follow the approach of Example 8.2.1, finding the positive solution and simplifying if necessary.*

(a) By definition, the positive solution of $x^2 = 16$ is $\sqrt{16}$. Since 16 is a perfect square, we simplify $\sqrt{16}$ as 4. Since 4 is a solution, -4 is a solution as well.

The solutions are 4 and -4.

(b) The positive solution to $x^2 = 98$ is $\sqrt{98}$. While 98 is not a perfect square, it does have a perfect square factor of 49. Simplifying,

$$\sqrt{98} = \sqrt{49 \cdot 2} = \sqrt{49} \cdot \sqrt{2} = 7\sqrt{2}.$$

Since $7\sqrt{2}$ is a solution, so is its opposite $-7\sqrt{2}$.

The solutions are $7\sqrt{2}$ and $-7\sqrt{2}$.

8.2.1 Exercises

Simplify the radical expressions below.

1. $\sqrt{32}$
2. $\sqrt{500}$
3. $\sqrt{98}$
4. $\sqrt{192}$
5. $\sqrt{50}$
6. $\sqrt{\dfrac{3}{16}}$
7. $\sqrt{\dfrac{8}{25}}$
8. $\sqrt{\dfrac{5}{6}}$
9. $\sqrt{\dfrac{1}{12}}$
10. $\sqrt{\dfrac{9}{32}}$
11. (*) Radical notation is also used to handle more general algebraic equations. The n^{th} *root* of a (positive[2]) number k, written $\sqrt[n]{k}$ is defined to be the (positive) solution to the equation $x^n = k$. For example, $\sqrt[3]{64} = 4$, since $(4)^3 = 64$, and $\sqrt[5]{32} = 2$, since $(2)^5 = 32$.
    (a) Evaluate $\sqrt[4]{81}$ and $\sqrt[3]{125}$.
    (b) Using the rules for simplifying square roots as a guide, write the corresponding rules for "simplified n^{th} roots."
    (c) Simplify: $\sqrt[3]{24}$.
    (d) Simplify: $\sqrt[3]{\dfrac{1}{2}}$.
    (e) Find a positive solution to $x^3 = 120$.

Solve the following quadratic equations.

12. $x^2 = 100$
13. $x^2 = 12$
14. $x^2 = 150$
15. $x^2 = \dfrac{3}{4}$

[2]In this example, we will not discuss roots of negative numbers.

8.3 Introduction to complex numbers

We already saw in the chapter introduction that certain quadratic equations might have no "real" number as a solution. For example, for the quadratic equation $x^2 = -1$, no "real" number, when multiplied by itself, can result in any negative number, and in particular cannot be -1.

However, rather than simply ending satisfied with the equation having no solution, we will instead adopt the approach of the previous section: we will suppose there is a solution, and denote this solution with a symbol.

> **The imaginary unit i**
>
> The symbol i will be used to denote one solution to the equation $x^2 = -1$.
>
> Stated differently, i is a symbolic "number" having the property that $i^2 = -1$.

Because of the similarity of this definition with the symbolic definition of the square root $\sqrt{a}$ as a solution to $x^2 = a$, we sometimes write

$$i = \sqrt{-1}.$$

Why is i called an "imaginary unit?" The word "unit" (in the sense of "one") is due to the fact that i^2 $(= -1)$, by definition, has *magnitude* one. The word "imaginary" (due to the mathematician–philosopher René Descartes) is meant to emphasize that this symbol has different properties than the "real" numbers we are used to working with, and in particular its square is negative. In fact, the widely-used phrase "*real number*" arose (also due to Descartes) to distinguish these numbers from "imaginary numbers." We will adopt this usage[3]. From now on, we will use the term **real number** to be any number that does not involve the imaginary unit i. By contrast, we call a **complex number** a "number" that may involve the imaginary unit i. (This terminology is meant to be less judgmental than the phrase, "imaginary numbers.")

We emphasize again that complex numbers (involving i) have some very different properties from real numbers (not involving i). We already have seen that it is possible for the square of a complex number to be negative. Another difference, which is not so obvious, is that there is no way to order complex numbers with the usual comparison relations of "less than" or "greater than."

[3] The precise definition of a real number is quite technical. In fact, a rigorous definition of a real number that included all the properties commonly accepted as "real" was only stated toward the end of the 19th Century, some 250 years after Descartes first used the term. Most students (all except some math majors) will never encounter the "real" definition of a real number.

In particular, it makes no sense to call a complex number positive ("greater than zero") or negative ("less than zero").

Once we have made the definition of the imaginary unit, we will treat it (symbolically) exactly as we have treated radical expressions up to now. In this section, we will see how the imaginary unit arises in solving quadratic equations. In the next section, we will see how it is manipulated in the most basic cases.

Notice first that once we have defined i to be a solution to the equation $x^2 = -1$, then we should also admit the symbol $-i$ as another solution:

$$(-i)^2 = (-1 \cdot i)^2 = (-1)^2 \cdot i^2 = 1 \cdot (-1) = -1,$$

assuming that the symbol i should behave in accordance with the properties of exponents. In other words, once we allow for complex numbers, then the equation $x^2 = -1$ has *two* (complex) solutions, i and $-i$. (Remember, complex numbers involving the imaginary are neither positive nor negative. The best we can say is that $i(= 1 \cdot i)$ has a positive *coefficient* of 1, while $-i(= -1 \cdot i)$ has a negative coefficient.)

The following examples show how the imaginary unit arises in a variety of settings, once we introduce the rules of radical expressions that we have seen so far.

Example 8.3.1. *Simplify the following radical expressions.*

(a) $\sqrt{-4}$

(b) $\sqrt{-7}$

(c) $\sqrt{-50}$

Answer. *We adopt exactly the same approach to simplifying square roots with negative radicands as we did in the previous section. The only extra ingredient will be that we will use the symbol i to represent $\sqrt{-1}$.*

(a) Separating the factor of -1 in anticipation of complex number notation,

$$\sqrt{-4} = \sqrt{(-1) \cdot 4} = \sqrt{(-1)} \cdot \sqrt{4} = i \cdot 2 = 2i.$$

The answer is $2i$. (Notice that we will write the integer part as a "coefficient," in the same way that we customarily write $2x$ instead of $x \cdot 2$.)

(b)

$$\sqrt{-7} = \sqrt{(-1) \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}.$$

The answer is $i\sqrt{7}$. (In this case, it is customary to write the radical expression second, *even though $\sqrt{7}$ represents a real number "coefficient." This avoids writing $\sqrt{7}i$, where it might be misunderstood to indicate that the i is part of the radicand.)*

(c) *In this case, in addition to the presence of a complex number, we see that the radicand contains a (real!) perfect square factor.*

$$\begin{aligned}\sqrt{-50} &= \sqrt{(-1)\cdot 50}\\ &= \sqrt{-1}\cdot\sqrt{50}\\ &= i\sqrt{25\cdot 2}\\ &= i\cdot\sqrt{25}\cdot\sqrt{2}\\ &= i\cdot 5\cdot\sqrt{2}\\ &= 5i\sqrt{2}.\end{aligned}$$

The answer is $5i\sqrt{2}$. *(This notation, with the integer factor first, then the imaginary unit, then the irrational radical symbol, is customary. However,* $(5\sqrt{2})i$ *might be more in keeping with using a real number coefficient for the imaginary unit.)*

Complex numbers, really by definition, appear as solutions to quadratic equations. In the following examples, we proceed exactly as in Example 8.2.1, keeping in mind our convention of writing complex numbers using the imaginary unit i.

Example 8.3.2. *Solve the following quadratic equations.*

(a) $x^2 = -15$

(b) $x^2 + 18 = 0$.

Answer. *We will follow the same approach as Examples 8.2.1 and 8.2.5.*

(a) *One solution to* $x^2 = -15$ *is, by definition,* $\sqrt{-15}$. *Simplifying to indicate the imaginary unit,*

$$\sqrt{-15} = \sqrt{(-1)\cdot 15} = \sqrt{-1}\cdot\sqrt{15} = i\sqrt{15}.$$

The other solution will be $-i\sqrt{15}$.

The solutions are $i\sqrt{15}$ *and* $-i\sqrt{15}$.

(b) *The main thing to notice about this equation is that it does not have the special form* $x^2 = a$ *that we have been relying on so far. Fortunately, that is easy to fix in this case:*

$$\begin{array}{cccccc} x^2 & + & 18 & = & 0 \\ & & -18 & \vdots & -18 \\ \hline x^2 & & & = & -18. \end{array}$$

Now that the x^2*-term is by itself on one side of the equation, we can apply our strategy.*

One solution to $x^2 = -18$ *is* $\sqrt{-18}$. *We see that in addition to being a complex number, the radicand has a perfect square factor of* 9. *Simplifying,*

$$\sqrt{-18} = \sqrt{(-1) \cdot 9 \cdot 2} = \sqrt{-1} \cdot \sqrt{9} \cdot \sqrt{2} = i \cdot 3 \cdot \sqrt{2} = 3i\sqrt{2}.$$

(Notice we performed several simplifications at once.)

One solution is $3i\sqrt{2}$. *Hence the other solution is* $-3i\sqrt{2}$.

The solutions are $3i\sqrt{2}$ *and* $-3i\sqrt{2}$.

8.3.1 Exercises

Simplify the following radical expressions using the imaginary unit i.

1. $\sqrt{-16}$
2. $\sqrt{-45}$
3. $\sqrt{-\frac{5}{8}}$
4. (*) Assuming the rules of exponents apply to complex numbers, compute the first 10 powers of i: i^1, i^2, i^3, ..., i^{10}. (Hint: $i^3 = i^2 \cdot i^1$.)

Solve the following quadratic equations.

5. $x^2 = -36$
6. $x^2 + 24 = 0$

8.4 Arithmetic of radical expressions

We have seen so far that even simple quadratic equations in one variable may have "complicated" solutions—they may be irrational numbers, for example, or even complex numbers that have unusual properties compared to the real numbers we grew up with. So far, we have emphasized a *symbolic* approach to these numbers. In other words, we have used a symbol (a radical expression or an expression involving i) to represent a solution to an equation of a particular form (primarily of the form $x^2 = a$). This has the advantage of sidestepping the exact value of these solutions, but it carries the price of adhering to a set of customary rules about how such symbols will be written.

In this section, we will discuss how to perform arithmetic with these symbols—how to add them, subtract them, multiply them and divide them. In many ways, this will be exactly like how we approached the arithmetic of polynomials, and the reader will notice many similarities to how we approach the arithmetic of radical expressions. That shouldn't be a big surprise: polynomial arithmetic is really a kind of symbolic arithmetic, where the symbols are the variables.

The main difference between the arithmetic of radical expressions and the arithmetic of polynomials is that polynomials involve *indefinite* symbols: the variables are meant to represent an unknown or changing quantity. The symbols we have been using for irrational and complex expressions, like $\sqrt{2}$ and i, are *definite* symbols. They have a specific value or meaning (although it may be hard to write down exactly what they are), and this value is fixed and unchanging. This difference will show up repeatedly in the examples below.

Adding and subtracting radical expressions, like adding or subtracting polynomials, is based on the principle of *combining like terms*. Two expressions involving square roots are considered like terms if their radicands are the same. Like terms are added by adding their coefficients (and leaving the radical symbol the same). For example, the expression $5\sqrt{3}+3\sqrt{3}$ consists of two like terms, since their symbolic part is the same $\sqrt{3}$. We can write

$$5\sqrt{3}+3\sqrt{3}=8\sqrt{3}.$$

(Compare this to $5x+3x$.) However, $4\sqrt{2}+6\sqrt{5}$ involves two terms which are not like terms, since the radicands are different, and so cannot be added or further simplified. (Compare this to $4x+6y$.)

The only thing that needs to be mentioned is that it is important to simplify radical expressions *before* adding or subtracting, as the following examples illustrate.

Example 8.4.1. *Perform the indicated operations:*

(a) $\sqrt{24}-3\sqrt{150}+2\sqrt{3}+15$.

(b) $5\sqrt{18}-2\sqrt{2}$.

(c) $(3+4i)-(2-3i)$.

Answer. *(a) None of the radicands appearing in the expression*

$$\sqrt{24}-3\sqrt{150}+2\sqrt{3}+15$$

are the same, and so there do not appear to be like terms. However, the first two (24 *and* 150*) have perfect square factors. Simplifying,*

$$\begin{aligned}
&\sqrt{24}-3\sqrt{150}+2\sqrt{3}+15\\
&\sqrt{4\cdot 6}-3\sqrt{25\cdot 6}+2\sqrt{3}+15\\
&\sqrt{4}\cdot\sqrt{6}-3\left(\sqrt{25}\cdot\sqrt{6}\right)+2\sqrt{3}+15\\
&2\sqrt{6}-3(5\sqrt{6})+2\sqrt{3}+15\\
&2\sqrt{6}-15\sqrt{6}+2\sqrt{3}+15
\end{aligned}$$

Notice that at the last step, we multiplied the coefficient in the term containing $5\sqrt{6}$ *by* -3.

In any case, now there are like terms, namely, the $\sqrt{6}$*-terms. None of the other terms are like terms. Adding the coefficients for the* $\sqrt{6}$*-terms, we obtain*

$$-13\sqrt{6} + 2\sqrt{3} + 15.$$

The answer is $-13\sqrt{6} + 2\sqrt{3} + 15$*.*

(b) As in the previous example, the two radicands appearing in $5\sqrt{18} - 2\sqrt{2}$ *are not the same, and so do not appear to be like terms. However,* 18 *contains a perfect square factor, and so can be simplified:*

$$\begin{aligned} 5\sqrt{18} - 2\sqrt{2} \\ 5\sqrt{9 \cdot 2} - 2\sqrt{2} \\ 5\left(\sqrt{9} \cdot \sqrt{2}\right) - 2\sqrt{2} \\ 5(3\sqrt{2}) - 2\sqrt{2} \\ 15\sqrt{2} - 2\sqrt{2}. \end{aligned}$$

After simplifying, the two remaining terms are like terms, and so they can be combined to obtain $13\sqrt{2}$*.*

The answer is $13\sqrt{2}$*.*

(c) Adding complex numbers, the real number parts are like terms and the imaginary parts (the terms containing $i = \sqrt{-1}$*) are like terms. Proceeding exactly like subtracting polynomials, we will change the problem to one of "adding the opposite" and combine like terms:*

$$\begin{aligned} (3 + 4i) - (2 - 3i) \\ (3 + 4i) + (-2 + 3i) \\ (3 + (-2)) + (4i + 3i) \\ 1 + 7i. \end{aligned}$$

The answer is $1 + 7i$*.*

The complex numbers in part (c) of the last example are typical of how complex numbers are written. In fact, any complex number can be written in the form

$$a + bi,$$

where a and b are real numbers.

Multiplying expressions involving radicals will typically involve the distributive law, exactly like multiplying polynomials. However, instead of relying on the rules of exponents (which we needed to multiply powers of a variable), we will use Property (S1) of roots.

Example 8.4.2. *Multiply:* $(8\sqrt{3} + 2\sqrt{5})(\sqrt{2} - 4\sqrt{5})$*.*

Answer. *Distributing, we obtain*

$$(8\sqrt{3}+2\sqrt{5})(\sqrt{2}-4\sqrt{5})$$
$$(8\sqrt{3})(\sqrt{2})+(8\sqrt{3})(-4\sqrt{5})+(2\sqrt{5})(\sqrt{2})+(2\sqrt{5})(-4\sqrt{5}).$$

According to Property (S1) of square roots, which states that $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$*, we will multiply the coefficients* and *the radicands of each term:*

$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-8\sqrt{25}.$$

All the radicands are different. However, one of the terms can be simplified, since 25 *is a perfect square:*

$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-8(5)$$
$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-40.$$

After simplifying, there are no like terms.

The answer is $8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-40$.

In the preceding example, the factor of 5 in the last term (which first appeared at the second-last step) arose after multiplying $(\sqrt{5})(\sqrt{5})$. We chose to apply Property (S1) to obtain $\sqrt{25}$, then simplified. Notice, though, that by definition

$$(\sqrt{5})(\sqrt{5})=(\sqrt{5})^2=5.$$

Example 8.4.3. *Multiply:* $(5-3\sqrt{2})(4+\sqrt{2})$.

Answer. *We begin by distributing:*

$$(5-3\sqrt{2})(4+\sqrt{2})$$
$$(5)(4)+(5)(\sqrt{2})+(-3\sqrt{2})(4)+(-3\sqrt{2})(\sqrt{2})$$
$$20+5\sqrt{2}-12\sqrt{2}-3\sqrt{4}$$
$$20+5\sqrt{2}-12\sqrt{2}-3(2) \quad \textit{Simplifying } \sqrt{4}$$
$$20+5\sqrt{2}-12\sqrt{2}-6$$
$$14-7\sqrt{2} \quad \textit{Combining like terms}$$

The answer is $14-7\sqrt{2}$.

The next example involves multiplying two complex numbers. We will use the fact that $i^2=-1$.

Example 8.4.4. *Multiply:* $(3-2i)(-7+5i)$.

Answer. *Since we are using the symbol i for the radical expression* $\sqrt{-1}$*, our multiplication of complex numbers will look very much like multiplication of two binomials involving one variable—until the last steps.*

$$\begin{aligned}
&(3-2i)(-7+5i) && \\
&(3)(-7)+(3)(5i)+(-2i)(-7)+(-2i)(5i) && \textit{Distributing} \\
&-21+15i+14i-10i^2 && \textit{Multiplying term-by-term} \\
&-21+29i-10i^2 && \textit{Combining like terms} \\
&-21+29i-10(-1) && \textit{Since } i^2=-1 \\
&-21+29i+10 && \\
&-11+29i. && \textit{Combining like terms}
\end{aligned}$$

The answer is $-11+29i$.

The reader should take a moment to compare Examples 8.4.3 and 8.4.4. Both involve multiplying radical expressions with two terms, but one uses radical notation while the other uses imaginary i notation instead of radical notation $\sqrt{-1}$.

We will only consider the simplest examples of division of radical expressions. Some more complicated examples will appear as challenge exercises at the end of the section.

Example 8.4.5. *Simplify:* $\dfrac{\sqrt{3}\cdot\sqrt{66}}{\sqrt{2}}$.

Answer. *In this context, the word "simplify" means to perform all operations, and then simplify according to the rules of radical notation.*

Since the only operations involved are multiplication and division, we can rely on Properties (S1) and (S2) of square roots. In particular, combining the two properties, we can perform all the operations inside *the radicand:*

$$\frac{\sqrt{3}\cdot\sqrt{66}}{\sqrt{2}}=\sqrt{\frac{3\cdot 66}{2}}.$$

For convenience, in performing the operations within the radicand, we will take advantage of the fact that the 2 *in the denominator divides the larger factor* 66 *in the numerator:*

$$\sqrt{\frac{3\cdot 66}{2}}=\sqrt{3\cdot 33}=\sqrt{99}.$$

All that remains is to simplify the result, noticing that 99 *has a perfect square factor of* 9*:*

$$\sqrt{99}=\sqrt{9\cdot 11}=\sqrt{9}\cdot\sqrt{11}=3\sqrt{11}.$$

The answer is $3\sqrt{11}$.

For the record, there are several other approaches to the previous example, due to some flexibility with the order of operations.

8.4.1 Exercises

Perform the indicated operations. Simplify all radical expressions.

1. $2\sqrt{50} - 4\sqrt{8} + 6\sqrt{12}$
2. $-\sqrt{24} + 4\sqrt{3} - \sqrt{27}$
3. $3\sqrt{20} + 2\sqrt{45}$
4. $(3 - 2i) - (5 + 7i)$
5. $\sqrt{5}(2\sqrt{10} - 1)$
6. $\dfrac{\sqrt{5} \cdot \sqrt{30}}{\sqrt{3}}$.
7. $\sqrt{6}(4\sqrt{3} - 5\sqrt{2})$
8. $(1 + 5i)(3 - 2i)$
9. $(1 + \sqrt{2})(1 - \sqrt{2})$
10. $(2 + 3i)(2 - 3i)$
11. $(\sqrt{3} - \sqrt{2})^2$
12. $(1 + i)^2$
13. (*) Use corresponding properties of n^{th} roots (see Exercise 11 of the Section 8.2.1) to simplify
$$5\sqrt[3]{16} - 2\sqrt[3]{54} + 6\sqrt[3]{24}.$$
14. (*) This exercise gives and indication of how to simplify expressions with a binomial involving a radical in the denominator.
    (a) Perform the following multiplication using properties of radicals:
    $$\frac{3 - \sqrt{2}}{5 + 2\sqrt{3}} \cdot \frac{5 - 2\sqrt{3}}{5 - 2\sqrt{3}}.$$
    (Notice that we have really multiplied the expression $\dfrac{3 - \sqrt{2}}{5 + 2\sqrt{3}}$ by 1, so have not changed the value of the expression.)
    (b) Use the idea of the previous exercise to simplify the radical expression
    $$\frac{\sqrt{2}}{4 + \sqrt{5}}.$$
    The technique hinted at in this exercise is known as *rationalizing the denominator*.

15. (*) The technique in the previous exercise can be used to divide complex numbers.

    (a) Perform the following multiplication:

    $$\frac{4+i}{2+3i} \cdot \frac{2-3i}{2-3i}.$$

    (Notice that just like in the last exercise, we are multiplying the expression $\frac{4+i}{2+3i}$ by 1.)

    (b) Use the idea in the previous exercise to write the quotient

    $$\frac{3-2i}{1+6i}$$

    in standard complex form $a + bi$.

16. (*) What is wrong with the following "proof" that $-1 = 1$?

$$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = i^2 = -1.$$

8.5 Chapter summary

- A typical quadratic equation in one variable will have two solutions.
- The solutions to a quadratic equation in one variable may be rational, irrational, or complex (even when the coefficients of the equation are integers).
- Irrational and complex numbers are generally treated symbolically, according to historically-evolved rules for what is considered "simplified."

Chapter 9

Quadratic equations

Vocabulary

- Plus-or-minus ($\pm$) notation
- Hypotenuse
- Pythagorean theorem
- Completing the square
- Quadratic formula
- Discriminant
- Zero product property
- Parabola

9.1 Solving quadratic equations I. A first strategy

In the last chapter, we introduced the notation and manipulation of symbols representing irrational square roots and complex numbers. Both of these types of numbers arise from the simplest quadratic equations, and we will meet them over and over again for the remainder of this chapter.

Now we are ready to return to the main theme: solving quadratic equations in one variable. So far, we have seen one strategy to solve the simplest quadratic equations. Let us recall that strategy, which we repeat again here for emphasis:

First strategy to solve quadratic equations of the form $x^2 = k$

An equation having the form $x^2 = k$ has two solutions, written symbolically as $\sqrt{k}$ and $-\sqrt{k}$.

This first strategy only applies to quadratic equations in a very special form. In particular, the x^2 term is by itself on one side of the equation, and the other side has no variable terms. The main point in this section is to see that the first strategy can in fact be modified to solve other quadratic equations in one variable, as long as there is any perfect square involving the variable on one side of the equation and no variable terms on the other.

Recall that two algebraic statements are *equivalent* when they have the same solutions. Most of the statements we have seen up to now have been *simple* statements, in the sense that they involve just one equation or inequality. A *compound statement* is formed by several simple statements, joined by the words AND, OR or NOT. For example, the system of linear equations $\begin{cases} x + y = 1 \\ x - y = 5 \end{cases}$ is a compound statement of the form $x + y = 1$ AND $x - y = 5$; a solution to the system (a compound statement) must be a solution BOTH $x + y = 1$ AND $x - y = 5$. As another example, a solution to the statement "Either $x = 5$ or $x = 2$" must be EITHER a solution to the simple statement $x = 5$ OR it must be a solution to $x = 2$.

With this is mind, we will introduce a new notation. The symbol $\pm$ (which is unfortunately[1] read "plus or minus") will be used to indicate an either/or statement:

Plus-or-minus ($\pm$) notation

$\pm k$ means the same as "either k or $-k$."

So, for example, ± 3 means "either 3 or -3." Similarly, $\pm\sqrt{5}$ means the same as "either $\sqrt{5}$ or $-\sqrt{5}$.

The "plus-or-minus" notation is often seen in the context of equations.

[1] It should really be read as "positive or negative."

Compound plus-or-minus statements

The statement $x = \pm k$ means the same as the compound statement "either $x = k$ or $x = -k$."

With this new notation, we can rephrase our basic strategy to solve simple quadratic equations in one variable:

First strategy to solve quadratic equations of the form $x^2 = k$ (revisited)

The equation $x^2 = k$ is equivalent to the statement $x = \pm\sqrt{k}$.

In particular, it has two solutions, $\sqrt{k}$ and $-\sqrt{k}$. (When $k = 0$, the two solutions are the same.)

The reader may refer to Example 8.2.1, where this strategy was illustrated in the simplest cases.

The rephrasing of our basic strategy for solving quadratic equation is useful because it can be applied to any equation having the form $\square^2 = k$: one side of the equation (represented by the box) is a perfect square, while the other side is a constant. The next example shows this principle in action, in a sightly more complicated setting.

Example 9.1.1. *Solve:* $16x^2 = 25$.

Answer. *We will show two ways that our "first strategy" can be used to solve this equation, which is should be noticed involves more than just x^2 on the left side of the equation.*

Method 1: "Solve for x^2"

Although the x^2 is not "by itself," we can use the multiplication principle to write an equivalent equation in the special form to apply the basic strategy.

$$\begin{aligned} 16x^2 &= 25 \\ \tfrac{16x^2}{16} &= \tfrac{25}{16} \\ x^2 &= \tfrac{25}{16} \end{aligned}$$

The resulting equivalent equation is in the special form to apply our basic strategy. The equation $x^2 = 25/16$ is equivalent to the compound statement $x = \sqrt{\dfrac{25}{16}}$ or $x = -\sqrt{\dfrac{25}{16}}$.

The solutions are (after simplifying the radical expressions) $5/4$ *and* $-5/4$.

Method 2: Perfect square form

Notice that even though original equation $16x^2 = 25$ *is not in the simplest form (with* x^2 *by itself on one side of the equation), the left hand side involving the variable is, in fact, a perfect square!*

$$\begin{aligned} 16x^2 &= 25 \\ (4x)^2 &= 25 \qquad \textit{Emphasizing the perfect square} \end{aligned}$$

We can apply our basic strategy. The quadratic equation $(4x)^2 = 25$ *is equivalent to the compound statement*

$$4x = \sqrt{25} \qquad OR \qquad 4x = -\sqrt{25}.$$

After simplifying the radicals, we see that we have a compound statement involving two linear *equations:* $4x = 5$ *or* $4x = -5$. *Each of them is easily solved.*

The solutions are $5/4$ *and* $-5/4$.

The next example gives another illustration of Method 2, where we apply our basic strategy to a perfect square more complicated than just x^2.

Example 9.1.2. *Solve:* $(x+4)^2 = 7$.

Answer. *The left side of the equation is a perfect square, so we may apply the same basic strategy we have seen in the simpler case. Namely, the equation* $(x+4)^2 = 7$ *is equivalent to the statement*

$$x + 4 = \pm\sqrt{7},$$

or, what is the same, to the compound statement

$$x + 4 = \sqrt{7} \qquad OR \qquad x + 4 = -\sqrt{7}.$$

Each of the two equations are linear, and can be solved using our standard techniques for solving linear equations:

$$\begin{array}{cccccc} x & + & 4 & = & & \sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & +\sqrt{7} \end{array}$$

The solution to $x + 4 = \sqrt{7}$ *is* $-4 + \sqrt{7}$. *Similarly, solving the other equation in the compound statement:*

$$\begin{array}{cccccc} x & + & 4 & = & & -\sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & -\sqrt{7} \end{array}$$

The solution to $x + 4 = -\sqrt{7}$ *is* $-4 - \sqrt{7}$.

Combining to obtain the solutions to the compound equation, the solutions to $(x+4)^2 = 7$ *are* $-4 + \sqrt{7}$ *and* $-4 - \sqrt{7}$.

It is common, although a little lazy, to combine the two solving steps in the previous example as follows:

$$\begin{array}{cccccc} x & + & 4 & = & & \pm\sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & \pm\sqrt{7}, \end{array}$$

and then to say that the solutions are $-4 \pm \sqrt{7}$. Keep in mind these are two different solutions!

9.1.1 The Pythagorean theorem

Even though we have only been dealing with quadratic equations of the simplest form, these simple equations occur naturally in the context of using the Pythagorean theorem. Given the importance of this theorem in a wide variety of contexts (it is used to obtain a formula for measuring distances, for example), we illustrate the theorem and several examples of its application here.

The Pythagorean[2] theorem is a theorem—a mathematical statement which can be proved, or deduced, from definitions and previously-proved statements—about right triangles. Recall that a right triangle is a triangle which includes one right angle. The side opposite to the right angle is called the *hypotenuse*. The other two sides are referred to as the *legs* of the right triangle. See Figure 9.1.

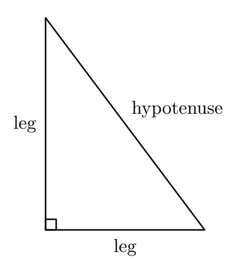


Figure 9.1: A right triangle. The right angle is indicated with the small square symbol.

[2] Although the theorem is named after Pythagorus and his followers, the relationship it describes had been noticed by the earliest civilizations in Egypt, Mesopotamia, India and China. The oldest proof of the statement is from Euclid, around 300 BCE.

The Pythagorean theorem

Given a right triangle whose hypotenuse has length h and whose legs have lengths a and b, the following relationship holds:

$$h^2 = a^2 + b^2.$$

In fact, if the lengths of the three sides of a triangle satisfy this equation, then the triangle must be a right triangle.

Notice that in the equation relating the side lengths in the theorem, the square of the length of the special side, the hypotenuse, is by itself on one side. The squares of the lengths of the other two sides that are not the hypotenuse are added (in either order) on the other side of the equation.

As a consequence of the Pythagorean theorem, the hypotenuse must be the longest side of a right triangle. (In fact, sometimes the hypotenuse is *defined* to be the longest side of a right triangle. In that case, the Pythagorean theorem would emphasize that the hypotenuse must be opposite the right angle.)

For our purposes, the main consequence of the Pythagorean theorem is that if the lengths of two sides of a right triangle are known, then the length of the third side can be found as well. We illustrate this fact with several examples.

Example 9.1.3. *In a right triangle, the length of the hypotenuse is* 13 *inches, while the length of one of the legs is* 5 *inches. Find the length of the remaining side.*

Answer. *In this example, we use the values* $h = 13$ *and* $a = 5$ *in the Pythagorean theorem and attempt to solve for* b.

$$\begin{aligned} (13)^2 &= (5)^2 + b^2 \\ 169 &= 25 + b^2 \end{aligned}$$

The resulting equation (in the variable b*) can be written in the special form we have been considering by subtracting* 25 *from both sides:*

$$\begin{array}{rcrcl} 169 & = & 25 & + & b^2 \\ -25 & \vdots & -25 & & \\ \hline 144 & = & & & b^2. \end{array}$$

The solutions to this equation (in which b^2 *is by itself on one side of the equation) are* $\sqrt{144}$ *and* $-\sqrt{144}$*, or* 12 *and* -12*. However, since* b *is to represent a length, we ignore the negative solution as being meaningless in the given problem.*

The length of the remaining side is 12 *inches.*

Example 9.1.4. *In a right triangle, one leg has length 4 cm and the other has length 6 cm. Find the length of the hypotenuse.*

Answer. *We substitute the values $a = 4$ and $b = 6$ into the equation $h^2 = a^2+b^2$ in order to solve for the value of h.*

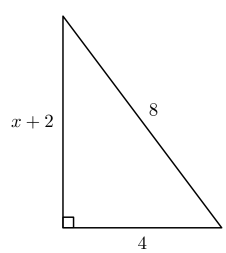
$$\begin{aligned} h^2 &= (4)^2 + (6)^2 \\ h^2 &= 16 + 36 \\ h^2 &= 52. \end{aligned}$$

According to our basic strategy, the solutions to this equation are $\sqrt{52}$ and $-\sqrt{52}$; we ignore the negative solution since our answer is to represent a length, which is always a positive quantity. We will simplify our result:

$$\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}.$$

The length of the hypotenuse is $2\sqrt{13}$ cm. (This is approximately 7.21 cm.)

Example 9.1.5. *In the figure below, find the value of x so that the triangle is a right triangle.*



Answer. *Since we require the triangle to be a right triangle, the Pythagorean theorem must hold:*

$$\begin{aligned} (8)^2 &= (4)^2 + (x+2)^2 \\ 64 &= 16 + (x+2)^2. \end{aligned}$$

We will rewrite the equation so that the perfect square involving x, $(x+2)^2$, is by itself on one side of the equation, and then proceed as in Example 9.1.2.

$$\begin{array}{ccccc} 64 & = & 16 & + & (x+2)^2 \\ -16 & \vdots & -16 & & \\ \hline 48 & = & & & (x+2)^2. \end{array}$$

According to our basic strategy, this equation is equivalent to the compound statement $x + 2 = \pm\sqrt{48}$*, or* $x = -2 \pm \sqrt{48}$*. Simplifying the radical expression* $\sqrt{48} = \sqrt{16 \cdot 3}$*, this is equivalent to* $x = -2 \pm 4\sqrt{3}$*, which has two solutions,* $-2 + 4\sqrt{3}$ *and* $-2 - 4\sqrt{3}$*. However, the second solution would make one side of the triangle have a length of* $-4\sqrt{3}$*, which makes no geometric sense.*

The value of x *which makes the triangle a right triangle is* $-2+4\sqrt{3}$*. (Check on your calculator that this is approximately* 4.93*.)*

9.1.2 Exercises

Solve the following quadratic equations.

1. $x^2 = 150$
2. $4x^2 = 3$
3. $x^2 = -49$
4. $x^2 + 18 = 0$
5. $12x^2 = 75$
6. $(x + 3)^2 = 12$
7. $(2x - 1)^2 = 144$
8. $(x - 5)^2 = -18$

Find the length of the third of the right triangle whose other two side lengths are given. (Here h represents the length of the hypotenuse and a and b represent the lengths of the other two sides, as shown in Figure 9.2.)

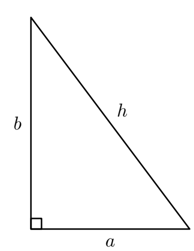


Figure 9.2: Diagram for Exercises 8–10. (Not drawn to scale)

9. $a = 4$, $b = 6$.
10. $h = 10$, $b = 5$.
11. $a = 12$, $b = 4$.

9.2 Solving quadratic equations II. Completing the square

We saw in the previous section that even if a quadratic equation was not as simple as $x^2 = k$, we can still apply the same basic strategy as long as we have a perfect square on one side of the equation and a constant on the other. But what about equations like

$$x^2 + 6x - 4 = 0?$$

Because of the $6x$ term, there is little hope that we can obtain a perfect square on one side and a constant on the other by the same kind of simple manipulations as in the previous section.

We will now develop a procedure where *any* quadratic equation of the form $ax^2 + bx + c = 0$ can be written in the special form $\boxed{}^{\,2} = k$, where one side is a perfect square and the other side is a constant.

Consider the following identities (which you can confirm my multiplication):

$$\begin{aligned}
(x+1)^2 &= x^2 + 2x + 1 \\
(x+2)^2 &= x^2 + 4x + 4 \\
(x+3)^2 &= x^2 + 6x + 9 \\
(x+4)^2 &= x^2 + 8x + 16 \\
(x+5)^2 &= x^2 + 10x + 25 \\
(x+6)^2 &= x^2 + 12x + 36 \\
&\vdots
\end{aligned}$$

Do you see any patterns on the right hand side? There are several: the constant terms are all perfect squares, and the coefficient of x is always an even number. To be more specific,

- The constant term in the expansion of $(x + n)^2$ is n^2;
- The coefficient of x in the expansion of $(x + n)^2$ is $2n$ (twice the value of n).

Can you use these two facts to write down the next five expansions in the list, without actually multiplying polynomials?

Now, using the patterns we noticed above, let's try to answer the following question: How can I fill in the blank with a number in the expression

$$x^2 + 56x + \underline{\quad}$$

is a perfect square of the form $(x + n)^2$? We see the coefficient of x is 56, which according to the pattern above, should be twice the value of n. In other words, n should be $56/2 = 28$. But that tells us what to fill in as the constant, according to the other pattern: the constant terms should be $(28)^2 = 784$. In other words, the polynomial

$$x^2 + 56x + 784$$

is a perfect square: it is $(x + 28)^2$. No other constant would have worked other than 784; our choice was determined by the coefficient of x. This process of determining a constant to add to a quadratic expression to obtain a perfect square is known as *completing the square*.

The next example shows how this technique, combined with the strategy we have already developed, can be used to solve a quadratic equation which is not initially written in the special form $(x + n)^2 = k$.

Example 9.2.1. *Solve:* $x^2 - 6x + 4 = 0$.

Answer. *If the goal is to write the equation in the form $(x + n)^2 = k$, the first step will be to arrange to write the constant term by itself on one side of the equation. This is not hard to do:*

$$\begin{array}{ccccccc} x^2 & - & 6x+ & 4 & = & 0 \\ & & & -4 & \vdots & -4 \\ \hline x^2 & - & 6x & & = & -4 \end{array}$$

We are now in a position to complete the square on the left side. To illustrate what we are about to do, let's write

$$x^2 \quad - \quad 6x \quad + \quad \_\_\_ \quad = \quad -4 \quad + \quad \_\_\_$$

to indicate that we will "fill in the blanks" to complete the square on the left side, and add the same quantity to the right side to guarantee that the new equation will be equivalent. The coefficient of x on the left side is -6. As before, we find half of -6:

$$\frac{-6}{2} = -3.$$

We square the result:

$$(-3)^2 = 9.$$

In other words, we will add 9 to both sides (to "fill in the blank"):

$$x^2 \quad - \quad 6x \quad + \quad 9 \quad = \quad -4 \quad + \quad 9.$$

The left-hand side is now a perfect square of the form $(x+n)^2$, where $n = -3$. The right-hand side can be combined to obtain:

$$(x - 3)^2 = 5.$$

We are now in a position to apply our basic strategy. The equation $(x-3)^2 = 5$ is equivalent to the compound statement

$$x - 3 = \pm\sqrt{5}.$$

Using the shorthand notation discussed above, we can solve this compound statement:

$$\begin{array}{cccccc} x & - & 3 & = & & \pm\sqrt{5} \\ & & +3 & \vdots & +3 & \\ \hline x & & & = & 3 & \pm\sqrt{5} \end{array}$$

The two solutions are $3+\sqrt{5}$ *and* $3-\sqrt{5}$.

Exercise 9.2.2. *Before continuing, apply the completing the square method to solve the following quadratic equations. Be sure to simplify square roots when possible.*

(a) $x^2 - 8x + 12 = 0$

(b) $x^2 + 4x + 5 = 0$

(c) $x^2 - 2x - 6 = 0$

With a little thinking, we can see that here was nothing really special about the equation that we started with in the previous example. It was written at the outset in general form $ax^2+bx+c=0$. The technique of completing the square was applied to write it in the general form $(x+n)^2 = k$. From there we applied our basic strategy, which had started off as a special strategy only for equations involving x^2 by itself on one side of the equation. **In fact, the technique of completely the square can be used to solve any quadratic equation in one variable.**

The next two examples show that the calculations involved in completing the square can become tedious.

Example 9.2.3. *Solve:* $x^2 + 5x + 7 = 0$.

Answer. *As before, we begin by arranging the equation so that the variable terms are on one side of the equation and the constant term is on the other:*

$$\begin{array}{cccccc} x^2 & + & 5x+ & 7 & = & 0 \\ & & & -7 & \vdots & -7 \\ \hline x^2 & + & 5x & & = & -7 \end{array}$$

Write

$$x^2 \quad + \quad 5x+ \quad \underline{\quad} \quad = \quad -7 \quad +\underline{\quad}.$$

The coefficient of x *this time is an odd number, which means that half of it will be a fraction. This doesn't stop us, though: we find half of* 5*:*

$$\frac{1}{2}\cdot 5 = \frac{5}{2}.$$

We square the result:

$$\left(\frac{5}{2}\right)^2 = \frac{25}{4}.$$

We add this quantity to both sides to both sides (to "fill in the blank"):

$$x^2 \quad + \quad 5x \quad + \quad \tfrac{25}{4} \quad = \quad -7 \quad + \tfrac{25}{4}.$$

The left-hand side is by design a perfect square of the form $(x+n)^2$*, where now* $n = 5/2$ *(notice this was the result of the first calculation, finding half of the coefficient of* x*). The right-hand side can be combined by finding a common denominator:*

$$-7 + \frac{25}{4} = -\frac{28}{4} + \frac{25}{4} = -\frac{3}{4}.$$

Hence our original equation is equivalent to

$$\left(x + \frac{5}{2}\right)^2 = -\frac{3}{4}.$$

Applying our basic strategy, this equation is equivalent to the compound statement

$$x + \frac{5}{2} = \pm\sqrt{\frac{-3}{4}},$$

which, after simplifying the right side, becomes

$$x + \frac{5}{2} = \pm\frac{i\sqrt{3}}{2}.$$

Solving the compound statement:

$$\begin{array}{cccccc} x & + & \frac{5}{2} & = & & \pm\frac{i\sqrt{3}}{2} \\ & & -\frac{5}{2} & \vdots & -\frac{5}{2} & \\ \hline x & & & = & -\frac{5}{2} & \pm\frac{i\sqrt{3}}{2}. \end{array}$$

The right hand side, while a little complicated, at least has the virtue of having a common denominator. It is customary to combine this sum into a single fraction:

$$x = \frac{-5 \pm i\sqrt{3}}{2}.$$

The two solutions (which are complex numbers!) are $\dfrac{-5+i\sqrt{3}}{2}$ *and* $\dfrac{-5-i\sqrt{3}}{2}$*.*

Example 9.2.4. *Solve* $2x^2 - 5x = 3$*.*

Answer. *The first thing to notice is that the coefficient of* x^2 *is not* 1*. Although we have not made a point of it until now, the fact that we are aiming for an equation involving an expression of the form* $(x+n)^2$ *makes it essential that the coefficient of the* x^2 *term is* 1*. In order to address this problem, we will simply divide both sides of the equation by the coefficient* 2*:*

$$\frac{2x^2 - 5x}{2} = \frac{3}{2}$$
$$x^2 - \frac{5}{2}x = \frac{3}{2}.$$

(Despite the fractions, notice that we already have the constant on one side of the equation with the variable terms on the other.)

We are aiming to complete the square on the left side:

$$x^2 \quad - \quad \tfrac{5}{2}x \quad + \quad \_\_\_ \quad = \quad \tfrac{3}{2} \quad + \_\_\_.$$

The coefficient of x is now $-\dfrac{5}{2}$. We will still aim to find half of this coefficient:

$$\frac{1}{2}\left(-\frac{5}{2}\right) = -\frac{5}{4}.$$

When we square the result we obtain

$$\left(-\frac{5}{4}\right)^2 = \frac{25}{16}.$$

This is the number we add to both sides to obtain a perfect square:

$$x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16}.$$

The left side is a perfect square by design: it is $\left(x - \dfrac{5}{4}\right)^2$. The right side can be combined:

$$\frac{3}{2} + \frac{25}{16} = \frac{24}{16} + \frac{25}{16} = \frac{49}{16}.$$

Hence our equation now has the form

$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}.$$

According to our basic strategy, this is equivalent to the statement

$$x - \frac{5}{4} = \pm\sqrt{\frac{49}{16}}.$$

Notice that the square root on the right side can be simplified:

$$x - \frac{5}{4} = \pm\frac{7}{4}.$$

Solving the compound statement

$$\begin{array}{ccccccc} x & - & \frac{5}{4} & = & & \pm\frac{7}{4} \\ & & +\frac{5}{4} & \vdots & +\frac{5}{4} & \\ \hline x & & & = & \frac{5}{4} & \pm\frac{7}{4}. \end{array}$$

In this case, though, we can do better; since no radicals appear in the expression, we can look at the two solutions separately to write them in simpler form:

$$\frac{5}{4} + \frac{7}{4} = \frac{12}{4} = 3,$$

and

$$\frac{5}{4} - \frac{7}{4} = -\frac{2}{4} = -\frac{1}{2}.$$

The two solutions are 3 *and* $-1/2$*.*

We end this section by summarizing the method of completing the square.

Solving quadratic equations by completing the square

To apply the technique of completing the square to solve a quadratic equation of the form

$$ax^2 + bx + c = 0:$$

1. Divide both side of the equation by the coefficient of x^2 to obtain an equivalent equation whose leading coefficient is 1;
2. Use the addition principle to obtain an equivalent equation having all variable terms on one side and the constant term on the other;
3. Complete the square based on the new coefficient of the x-term: Add the square of half the coefficient of x to both sides;
4. Apply the basic strategy of solving equations of the form $x^2 = k$ to obtain two solutions.

9.2.1 Exercises

Solve the following quadratic equations by completing the square.

1. $x^2 - 4x - 5 = 0$
2. $x^2 + 2x = 24$
3. $x^2 - 6x = 6$
4. $x^2 + 10x + 21 = 0$
5. $x^2 + 5x + 6 = 0$
6. $x^2 - 6x + 3 = 0$
7. $x^2 + 3x = 0$
8. $2x^2 + x = 6$
9. $x^2 + 4x + 6 = 0$
10. $3x^2 - 4x = 1$

9.3 Solving quadratic equations III. The quadratic formula

Completing the square is a powerful tool for solving quadratic equations in one variable. In fact, if our only goal was to solve quadratic equations, we could stop here: completing the square will always work. However, because quadratic equations occur relatively frequently, both in algebra and its applications, several other techniques have been developed to solve them.

One of the most well-known techniques arises directly from completing the square. The *quadratic formula* is a formula which expresses the solutions to a quadratic equation completely in terms of the equations' coefficients. To derive this formula, let's apply the technique of completing the square to the quadratic equation

$$ax^2 + bx + c = 0, \qquad (a \neq 0). \tag{9.1}$$

We begin by ensuring a leading coefficient of 1;

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \left(\frac{b}{a}\right) x + \frac{c}{a} = 0.$$

Arranging the constant on the right side and the variable terms on the left:

$$\begin{array}{ccccccc} x^2 & + & \left(\frac{b}{a}\right) x & + & \frac{c}{a} & = & 0 \\ & & & & -\left(\frac{c}{a}\right) & \vdots & -\frac{c}{a} \\ \hline x^2 & + & \left(\frac{b}{a}\right) x & + & \_\_ & = & -\frac{c}{a} + \_\_ \end{array}$$

To complete the square, we find half the coefficient of x and square the result:

$$\frac{1}{2} \cdot \left(\frac{b}{a}\right) = \frac{b}{2a},$$

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}.$$

(We used properties of exponents in the squaring operation.)

Completing the square:

$$\begin{array}{ccc} x^2 + \left(\frac{b}{a}\right) x + \frac{b^2}{4a^2} & = & -\frac{c}{a} + \frac{b^2}{4a^2} \\ \left(x + \frac{b}{2a}\right)^2 & = & \frac{b^2 - 4ac}{4a^2}. \end{array}$$

(On the right side of the last equation, we rewrote the first fraction using common denominator $4a^2$ by multiplying numerator and denominator by $4a$. We also reordered the terms in the numerator.)

According to our basic strategy, the resulting equation is equivalent to the compound statement

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

This can be simplified using properties of square roots:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$
$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

Solving the resulting (compound) linear equation:

$$\begin{array}{ccccc} x + \frac{b}{2a} & = & & \pm\frac{\sqrt{b^2-4ac}}{2a} \\ -\frac{b}{2a} & \vdots & -\frac{b}{2a} & \\ \hline x & = & -\frac{b}{2a} & \pm\frac{\sqrt{b^2-4ac}}{2a} \\ x & = & \frac{-b\pm\sqrt{b^2-4ac}}{2a}. & \end{array}$$

Let's summarize the result of this somewhat tedious calculation.

The quadratic formula

The quadratic equation

$$ax^2 + bx + c = 0$$

(where the coefficient $a \neq 0$) is equivalent to the compound statement

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular, the two solutions are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Notice: The quadratic equation was derived from a quadratic equation having a very particular form, the standard form Equation 9.1 above. In particular, to use the quadratic formula, *one side of the quadratic equation must be equal to 0*!

The quadratic equation is not particularly pleasant-looking. That is not surprising, given the fact that we began from a general quadratic equation with arbitrary coefficients. However, an enormous amount of material can be learned just from studying this formula:

- The solutions to a quadratic equation can be expressed completely as algebraic expressions involving its coefficients a, b and c.
- The nature of the solutions of a quadratic equation depend on its *discriminant* $b^2 - 4ac$:
  - If the discriminant is positive, the two solutions will be *real numbers*.
  - If the discriminant is negative, the two solutions will be *complex numbers*.
  - If the discriminant is zero, the two solutions will coincide, so that there is only one distinct solution.
  - If the discriminant is a (non-negative) perfect square, the solutions will be rational; otherwise, they will be irrational.

We illustrate the quadratic formula by solving the same three examples we solved in the last section by completing the square. Keep in mind that the quadratic formula involves *nine* separate operations. A look back at Chapter 1 should remind you that some of these operations can be performed at the same step.

Example 9.3.1. *Solve:* $x^2 - 6x + 4 = 0$.

Answer. *We see that the equation is in standard form, since one side is equal to zero, with* $a = 1$, $b = -6$ *and* $c = 4$.

Substituting into the quadratic formula and evaluating:

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 16}}{2} \\ &= \frac{6 \pm \sqrt{20}}{2} \\ &= \frac{6 \pm \sqrt{4 \cdot 5}}{2} && \textit{Simplifying the square root} \\ &= \frac{6 \pm 2\sqrt{5}}{2} \\ &= \frac{6}{2} \pm \frac{2\sqrt{5}}{2} \\ &= 3 \pm \sqrt{5}. \end{aligned}$$

The solutions are $3 + \sqrt{5}$ *and* $3 - \sqrt{5}$.

Example 9.3.2. *Solve:* $x^2 + 5x + 7 = 0$.

Answer. *The equation is in standard form, since one side is equal to zero, with* $a = 1$, $b = 5$ *and* $c = 7$.

Substituting into the quadratic formula and evaluating:

$$\begin{aligned} x &= \frac{-(5) \pm \sqrt{(5)^2 - 4(1)(7)}}{2(1)} \\ &= \frac{-5 \pm \sqrt{25-28}}{2} \\ &= \frac{-5 \pm \sqrt{-3}}{2} \\ &= \frac{-5 \pm \sqrt{-1 \cdot 3}}{2} && \textit{Simplifying the square root} \\ &= \frac{-5 \pm i\sqrt{3}}{2} \end{aligned}$$

(Notice that compared to the previous example, the numerator and denominator have no factors in common, so there is no need to take an extra step to divide.

The solutions are $\frac{-5+i\sqrt{3}}{2}$ *and* $\frac{-5-i\sqrt{3}}{2}$.

Example 9.3.3. *Solve:* $2x^2 - 5x = 3$.

Answer. *In this example, the quadratic equation is* not *in standard form, since neither side is equal to zero. This is not hard to fix, though, using the addition principle:*

$$\begin{array}{cccccc} 2x^2 & - & 5x & & = & 3 \\ & & & -3 & \vdots & -3 \\ \hline 2x^2 & - & 5x & - \;\; 3 & = & 0. \end{array}$$

We can now apply the quadratic formula with $a = 2$, $b = -5$ *and* $c = -3$.

Substituting and evaluating:

$$\begin{aligned} x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} \\ &= \frac{5 \pm \sqrt{25+24}}{4} \\ &= \frac{5 \pm \sqrt{49}}{4} \\ &= \frac{5 \pm 7}{4} \end{aligned}$$

In the case when the discriminant is a perfect square, as in this example, it *is useful to write the compound statement more explicitly:*

$$x = \frac{5+7}{4} \quad OR \quad x = \frac{5-7}{4}$$

$$x = \frac{12}{4} \quad OR \quad x = \frac{-2}{4}$$

$$x = 3 \quad OR \quad x = -\frac{1}{2}.$$

The solutions are 3 *and* $-1/2$.

9.3.1 Exercises

Solve each of the following quadratic equations by using the quadratic formula. (These are the same equations as in Exercises 9.2.1. Compare your answers obtained by using the quadratic formula with those you obtained by completing the square.)

1. $x^2 - 4x - 5 = 0$
2. $x^2 + 2x = 24$
3. $x^2 - 6x = 6$
4. $x^2 + 10x + 21 = 0$
5. $x^2 + 5x + 6 = 0$
6. $x^2 - 6x + 3 = 0$
7. $x^2 + 3x = 0$
8. $2x^2 + x = 6$
9. $x^2 + 4x + 6 = 0$
10. $3x^2 - 4x = 1$

9.4 Solving quadratic equations IV. Factoring

In this section, we describe a completely different approach to solving quadratic equations by using the techniques of factoring developed in Chapter 7. The factoring technique has the advantage of being less tedious (in many cases) than using either the quadratic formula or completing the square. In addition, unlike the methods we have seen so far, the new method can be applied to solve polynomial equation of higher degree (a few examples are given in the challenge exercises at the end of the section). It has the disadvantage, however, that it cannot be applied to solve *every* quadratic equation.

Solving quadratic equations by factoring is based on the following property of numbers, sometimes called the *zero product property*: *If a and b are two numbers having the property that $a \cdot b = 0$, then either a or b is* 0. Said in another way, the only way that the product of two numbers can be zero is if one of the two numbers is zero.

This property of numbers leads immediately to the following strategy for solving polynomial equations, including quadratic equations. We will state this strategy using function notation (see Chapter 1).

Solving polynomial equations by factoring

Suppose $P(x)$ is a polynomial which can be factored as a product of two polynomials, $P(x) = F(x) \cdot G(x)$. Then then polynomial equation

$$P(x) = 0$$

is equivalent to the compound polynomial statement

$$F(x) = 0 \quad \text{OR} \quad G(x) = 0.$$

In other words, the solutions to $P(x) = 0$ are exactly the solutions to $F(x) = 0$ and the solutions to $G(x) = 0$.

The idea behind this strategy is that instead of solving a quadratic polynomial equation (having degree 2), we will try to factor the polynomial and then solve two *linear* equations (of degree 1).

Notice: As in the situation when we solved a quadratic equation using the quadratic formula, it is essential that the polynomial be set equal to zero! After all, the strategy is based on the zero product property.

The following examples illustrate the technique of factoring to solve a quadratic equation.

Example 9.4.1. *Solve:* $x^2 - 3x - 10 = 0$.

Answer. *The quadratic equation does have zero on one side of the equation. So we will decide whether it is possible to factor the polynomial* $x^2 - 3x - 10$. *In fact, factoring the monic quadratic trinomial yields* $(x + 2)(x - 5)$.

Hence, using our factoring strategy, the equation $x^2-3x-10 = 0$ *is equivalent to the compound statement*

$$x + 2 = 0 \quad OR \quad x - 5 = 0.$$

Each of the two linear equations is quite simple to solve:

$$\begin{array}{cccccc} x & + & 2 & = & 0 \\ & & -2 & \vdots & -2 \\ \hline x & & & = & -2 \end{array}$$

and

$$\begin{array}{cccccc} x & - & 5 & = & 0 \\ & & +5 & \vdots & +5 \\ \hline x & & & = & 5. \end{array}$$

The solutions of $x^2 - 3x - 10 = 0$ *are* -2 *and* 5.

Example 9.4.2. *Solve:* $16x^2 = 25$.

Answer. *We solved this equation (two different ways!) in Example 9.1.1. However, we will now see that it can also be solved by factoring.*

First, the equation is not in the required form, having zero on one side of the equation. We apply the addition principle:

$$\begin{array}{ccccc} 16x^2 & & & = & 25 \\ & & -25 & \vdots & -25 \\ \hline 16x^2 & - & 25 & = & 0. \end{array}$$

Notice that the left side of the equivalent equation $16x-25 = 0$ *can be factored as a difference of squares:*

$$(4x + 5)(4x - 5) = 0.$$

So, using the zero product property, the equation is equivalent to the compound statement

$$4x + 5 = 0 \quad OR \quad 4x - 5 = 0.$$

The first equation has solution $-5/4$, *while the second has solution* $5/4$.

The solutions are $5/4$ *and* $-5/4$.

Example 9.4.3. *Solve:* $4x^2 + 12x = 0$

Answer. *The quadratic equation does have zero on one side, so we can ask whether it is possible to factor the polynomial* $4x^2 + 12x$. *In fact, the terms of this polynomial have a common factor of* $4x$, *giving a factorization of* $4x(x+3)$.

So the solutions of $4x^2 + 12x = 0$ *are the same as those of*

$$4x = 0 \quad OR \quad x + 3 = 0.$$

The first equation has solution 0*; the second equation has solution* -3.

The solutions are 0 *and* -3.

Example 9.4.4. *Solve:* $2x^2 + 5x = 3$.

Answer. *In this case, the quadratic equation is not in standard form, since neither side of the equation is zero. However, we can rewrite the equation by subtracting* 3 *from both sides:*

$$2x^2 + 5x - 3 = 0.$$

We attempt to factor the polynomial $2x^2 + 5x - 3$*. This is a non-monic quadratic trinomial; we will need to rely on the ac-method. We look for factors of* $(2)(-3) = -6$ *whose sum is* 5*. Such a pair is* 6 *and* -1*. Splitting the middle term we obtain*

$$2x^2 + 6x - 1x - 3.$$

Factoring by groups yields

$$\begin{aligned} 2x(x+3) - 1(x+3) \\ (x+3)(2x-1). \end{aligned}$$

All this, combined with our strategy to solve by factoring, means that the solutions of $2x^2 + 5x - 3 = 0$ *are the same as those of the compound statement*

$$x + 3 = 0 \quad OR \quad 2x - 1 = 0.$$

The solutions are -3 *and* $1/2$*.*

The reader might compare this example with Example 9.3.3, where we solved the same equation by using the quadratic formula, to see which method is more comfortable.

Example 9.4.5. *Solve:* $x^2 - 6x + 4 = 0$*.*

Answer. *The quadratic equation is in standard form, with one side being zero. We try to factor the polynomial* $x^2 - 6x + 4$*. Its terms have no factor in common, but it is a monic quadratic trinomial. However, there are no factors of* 4 *whose sum is* -6*. This polynomial cannot be factored.*

In this case, our quadratic equation cannot be solved by factoring. We are forced to rely on either completing the square or the quadratic formula. Fortunately, we already did that in Example 9.2.1.

The solutions (which did not come from factoring!) are $3 + \sqrt{5}$ *and* $3 - \sqrt{5}$*.*

As the reader might guess from the last example, when solving quadratic equations with integer coefficients, the factoring technique is only helpful when the solutions are rational numbers.

9.4.1 Exercises

Solve the following quadratic equations by factoring and using the zero product property.

1. $x^2 - 4x + 3 = 0$

2. $x^2 - x - 12 = 0$

3. $x^2 + 7x = 0$

4. $x^2 + 9x = -20$

5. $2x^2 - 3x - 2 = 0$

6. $4x^2 + 12x + 8 = 0$

7. $4x^2 + 4x = 15$

8. $9x^2 - 25 = 0$

The following exercises explore the zero product property further.

9. (*) The zero product property may be applied to polynomials in any degree. Use it to solve the following cubic (degree 3) polynomial equations.

   (a) $x^3 - 6x^2 + 8x = 0$
   (b) $(x + 3)(x - 1)(2x + 7) = 0$

10. (*) The zero product property is crucial to solving polynomial equations by factoring. To see what happens without it, consider the "clock number system." In this number system, the only numbers allowed are the integers from 0 (represented by 12 on the clock) to 11. We can add and multiply these numbers normally, except if the result is greater than or equal to 12, we use instead the remainder when the result is divided by 12. For example, in the clock number system, $9 + 8 = 5$, since 17 divided by 12 has remainder 5. Likewise, $5 \cdot 8 = 4$, since $40 \div 12 = 3\ R\ 4$. Note that in the clock number system, the zero product property does not hold. For example, $3 \cdot 4 = 0$, but neither 3 nor 4 is zero.

    Solve the following quadratic equations by guessing and checking all possible solutions. Remember, only numbers from 0 to 11 are allowed!

    (a) $x^2 - 6x = 0$
    (b) $x^2 + 4x + 3 = 0$
    (c) $x^2 + x + 1 = 0$.

9.5 Summary and applications

In the previous sections, we have seen a number of different strategies to solve quadratic equations in one variable. Broadly speaking, these strategies fall into two categories. On the one hand, there are the strategies based on the basic principle that equations of the form $x^2 = k$ have two solutions, $\pm\sqrt{k}$. Under this broad principle are included the strategies of completing the square and the quadratic formula, both of which apply to any quadratic equation in one variable. On the other hand, there is the strategy of factoring, based on the zero product property of numbers. While this strategy cannot be applied to every quadratic equation in one variable, it is generally more efficient in the cases where it can be applied.

Only practice can guide the student to an intuition as to which strategy best applies in solving a given equation. With that said, the following can be used as a rough guide toward solving quadratic equations in one variable.

Guide to solving quadratic equations in one variable

To solve a quadratic equation in one variable x, first write the equation in the standard form

$$ax^2 + bx + c = 0,$$

that is, with one side of the equation being zero. Consider the following questions:

1. *Can the equation be easily written in the form $x^2 = k$?* This is always the case when $b = 0$.
   - If so, apply the basic strategy to write the solutions $\sqrt{k}$ and $-\sqrt{k}$.
2. *Can the quadratic expression be easily factored?* Look especially for common factors and monic trinomials with integer coefficients.
   - If so, solve the equation by factoring.
3. *If neither of the two preceding cases hold, either apply the completing the square technique, or use the quadratic formula.*
   - Completing the square is most practical when $a = 1$ and b is an even integer; otherwise the quadratic formula is more efficient.

Always make sure that radical expressions are simplified!

At the end of this section, there are more exercises with which to practice deciding which technique to apply to solve quadratic equations.

We conclude our discussion of quadratic equations in one variable with some examples of how these equations arise in the context of word problems. The reader should review the four-step strategy for handling word problems in Section 5.3.5.

Example 9.5.1. *One number is three more than twice another number. Their product is seven more than their sum. Find the two numbers.*

Answer. ***Step 1: Create a dictionary.*** *The problem involves two unknown* *quantities. As usual, one of them will be denoted x, in this case the quantity referred to as "another number" in the first sentence. In that case, the first sentence describes the other quantity as three more than twice x, or $2x+3$. The dictionary can be written as*

One number	$2x+3$
Another number	x

Step 2: Write an equation. *We have used the first sentence to create the dictionary. The second sentence describes a relationship between them as an equation ("is"):*

$$(2x+3)(x) = [(2x+3)+(x)]+7.$$

Step 3: Solve the equation. *We will first simplify both sides of the equation separately, distributing and combining like terms where possible:*

$$\begin{aligned}(2x+3)(x) &= (2x+3)+(x)+7\\ (2x)(x)+(3)(x) &= 2x+3+x+7\\ 2x^2+3x &= 3x+10.\end{aligned}$$

It is clear now we are working with a quadratic equation. Let's rewrite it in standard form, so that one side is zero.

$$\begin{array}{ccccccccc} 2x^2 & + & 3x & & & = & 3x & + & 10 \\ & & -3x & & -10 & \vdots & -3x & & -10 \\ \hline 2x^2 & & & - & 10 & = & 0. & & \end{array}$$

Since the x-term does not appear, we can rewrite this equation again in the special form $x^2 = k$:

$$\begin{array}{ccccc} 2x^2 & - & 10 & = & 0 \\ & & +10 & \vdots & +10 \\ \hline 2x^2 & & & = & 10 \\ \frac{2x^2}{2} & & & = & \frac{10}{2} \\ x^2 & & & = & 5. \end{array}$$

(The reader may have noticed that the work we did to rewrite the equation in the standard form $2x^2 - 10 = 0$ was really unnecessary in this problem. We could have proceeded to isolate the x^2 term as soon as we noticed that there was no linear term, in other words no term involving x^1.)

Now that the equation is written in the special form $x^2 = 5$, we can apply the basic strategy directly.

The solutions are $\sqrt{5}$ and $-\sqrt{5}$.

Step 4: Answer the question. *Notice right away: The two solutions to the equation in Step 3 are not the two numbers we are looking for! The solutions represented the value of x, which, according to our dictionary, represent* *"another number." In other words, each solution to the quadratic equation will correspond to a* pair *of numbers as an answer.*

To find the other number in each pair, we rely on the dictionary: the "one number" is given by $2x+3$*. So for the solution* $\sqrt{5}$*, the corresponding "one number" would be* $2(\sqrt{5})+3$*, or* $3+2\sqrt{5}$*. For the solution* $-\sqrt{5}$*, the corresponding "one number" would be* $2(-\sqrt{5})+3$*, or* $3-2\sqrt{5}$*.*

The problem has two possible answers. Either one number is $3+2\sqrt{5}$ *and another number is* $\sqrt{5}$*, or one number is* $3-2\sqrt{5}$ *and another number is* $-\sqrt{5}$*.*

Example 9.5.2. *The length of a rectangle is four feet less than twice the width. The area of the rectangle is* 70 *square feet. Find the dimensions of the rectangle.*

Answer. ***Step 1: Create a dictionary.*** *The problem involves two unknown quantities, the length and the width. The first sentence relates the two. We will denote the width by* x*, so that according to the first sentence, the length will be* $2x-4$*. The dictionary can be written as*

Length	$2x-4$
Width	x

Step 2: Write an equation. *We recall from geometry that the area of a rectangle is given by the formula*

$$Area = (Length)(Width).$$

Using the dictionary and the second sentence, we substitute the information into this formula to obtain the equation

$$70 = (2x-4)(x).$$

Step 3: Solve the equation. *Let's simplify the right-hand side first by distributing:*

$$\begin{aligned} 70 &= (2x-4)(x) \\ 70 &= (2x)(x) + (-4)(x) \\ 70 &= 2x^2 - 4x. \end{aligned}$$

We see that this is a quadratic equation. We write it in standard form by subtracting 70 *from both sides:*

$$\begin{array}{rcrcrcr} 70 & = & 2x^2 & - & 4x & & \\ -70 & \vdots & & & & & -70 \\ \hline 0 & = & 2x^2 & - & 4x & - & 70. \end{array}$$

The right-hand side can be factored:

$$\begin{aligned} 0 &= 2x^2 - 4x - 70 \\ 0 &= 2(x^2 - 2x - 35) \\ 0 &= 2(x+5)(x-7). \end{aligned}$$

Applying the zero product property, this equation is equivalent to the compound statement

$$x + 5 = 0 \quad OR \quad x - 7 = 0.$$

(Notice the factor of 2 does not contribute any solutions to the equation.) This statement has solutions -5 and 7.

The solutions are -5 and 7.

Step 4: Answer the question. Recall from our dictionary that x represents the width of the rectangle. For that reason, we will ignore the negative solution to the equation as being physically meaningless, and only consider the solution 7. From the dictionary, we know that the length is expressed by $2x - 4$. Substituting the solution 7, we find that the length is $2(7) - 4 = 14 - 4 = 10$.

The width is 7 and the length is 10.

Example 9.5.3. An object is thrown straight upwards from the ground with an initial velocity of 60 ft/sec. It's height h above the ground is related to the number t of seconds that pass by the equation $h = -16t^2 + 60t$. When will the object be 12 feet above the ground?

Answer. ***Step 1: Create a dictionary.*** In this case, the only unknown quantity is the time t. (The height is also unknown, but the problem specifically asks for the time ("when").) Writing the dictionary would simply mean writing

$$\text{Time} \quad t$$

Step 2: Write an equation. The problem gives a relationship between the height h and the time t. Using the fact that the problem asks specifically about the object when the height is 12 feet, we substitute to obtain the quadratic equation

$$12 = -16t^2 + 60t.$$

Step 3: Solve the equation. We begin by writing the equation in standard form with one side being zero. For the sake of having the leading coefficient positive, we will add $16t^2 - 60t$ to both sides:

$$\begin{array}{ccccccccc} & & & & 12 & = & -16t^2 & + & 60t \\ +16t^2 & & -60t & & & \vdots & +16t^2 & & -60t \\ \hline 16t^2 & - & 60t & + & 12 & = & 0. & & \end{array}$$

The terms on the left side have a common factor of 4:

$$4(4t^2 - 15t + 3) = 0.$$

By the zero product property (or by dividing both sides of the equation by 4), this is equivalent to the equation

$$4t^2 - 15t + 3 = 0.$$

Since the quadratic trinomial cannot be factored any further, we will apply the quadratic formula with $a = 4$, $b = -15$ *and* $c = 3$*:*

$$\begin{aligned} t &= \frac{-(-15) \pm \sqrt{(-15)^2 - 4(4)(3)}}{2(4)} \\ &= \frac{15 \pm \sqrt{225 - 48}}{8} \\ &= \frac{15 \pm \sqrt{177}}{8}. \end{aligned}$$

The solutions are $\dfrac{15+\sqrt{177}}{8}$ *and* $\dfrac{15-\sqrt{177}}{8}$

Step 4: Answer the question. *Since* $\sqrt{177} \approx 13.304$, *both solutions are positive numbers and so are physically meaningful in the problem. Using the approximate value (again rounding to three decimal places), substituting, and evaluating, we find that both after* 0.212 *seconds and again after* 3.538 *seconds, the object is* 12 *feet above ground. (This corresponds to the object "going up" and then "coming down.")*

9.5.1 Exercises

Solve the following quadratic equations by any method.

1. $x^2 - 4x + 4 = 0$
2. $3x^2 + 5x - 2 = 0$
3. $6x^2 - 6x = 6$
4. $2x^2 + 6x + 2 = 0$
5. $4b^2 + 8b = 0$
6. $(x + 6)(x - 4) = -9$
7. $x^2 = -16$
8. $2x^2 - 3x = 2$
9. $6x^2 + 15x - 9 = 0$
10. $(x - 2)^2 = 3$
11. $3x^2 + 24 = 0$
12. $4x^2 - 3x = 1$

In each of the following, solve a quadratic equation to answer the question.

13. A rectangular apartment is designed so that the length is 10 feet more than three times the width. If the area enclosed is 800 square feet, find the dimensions of the apartment.

14. In an isosceles triangle, two of the three sides of the triangle have equal length. If the hypotenuse of an isosceles right triangle measures 10 units more than the length of either side, find the lengths of all three sides.

15. The product of the first two of three consecutive integers is 16 more than 10 times the third. Find the three integers.

16. The length of a rectangle measures 6 inches longer than twice the width. Find the perimeter of the rectangle if the length of the diagonal is 25 inches.

17. An object is tossed straight upwards from the ground with an initial velocity of 25 ft/sec. It's height h above the ground is related to the number t of seconds that pass by the formula $h = -16t^2 + 25t$. How long will it take for the object to fall back to the ground?

18. Taissha is standing on top of a building 100 feet tall. She drops her cell phone, whose height from the ground after t seconds is given by the formula $h = -16t^2 + 100$. How long does it take for her phone to hit the ground?

19. (*) If you want to divide a line segment into two parts (not necessarily equal), what is the perfect way to do it? For centuries, many artists and mathematicians thought the answer was the related to "golden ratio:" the ratio of the length of the larger part to the smaller part should be the same as the ratio of the length of the whole segment to the length of the larger part. If a segment of length 10 is divided according to the golden ratio, find the length of the two parts.

9.6 Introduction to quadratic equations in two variables

In this section, we will consider the simplest type of quadratic equations in two variables. As might be expected after Chapter 5, these equations will generally have infinitely many solutions, which we will graph in the xy-plane. The main feature of these equations that we want to point out in this section is that their graphs have a distinct shape *which is not a line*.

The simplest case of such an equation already illustrates the essential features of these graphs. Consider the equation

$$y = x^2.$$

This equation is quadratic (degree 2) in the variable x, but linear (degree 1) in the variable y. Nevertheless, the equation is a quadratic equation since the highest degree of *any* term is 2.

We will start off exactly as we did in the case of linear equations in two variables. We will begin finding solutions by choosing values for x (since the equation is written with y by itself on one side of the equation) and finding the corresponding value of y. By considering the resulting ordered pairs, we will plot the solutions and see if we can determine a pattern to "connect the dots."

Choosing 0 for x, we substitute:

$$\begin{aligned} y &= (0)^2 \\ y &= 0. \end{aligned}$$

Choosing 1 for x, we obtain

$$\begin{aligned} y &= (1)^2 \\ y &= 1. \end{aligned}$$

Choosing 2 for x,

$$\begin{aligned} y &= (2)^2 \\ y &= 4. \end{aligned}$$

So far, we can summarize our results in the following chart:

x	y	Solution
$\boxed{0}$	0	$(0, 0)$
$\boxed{1}$	1	$(1, 1)$
$\boxed{2}$	4	$(2, 4)$

(Recall we write a box around the values we chose, as opposed to those we obtained by solving the equation for the chosen values.)

In Figure 9.3, we plot these three solutions.

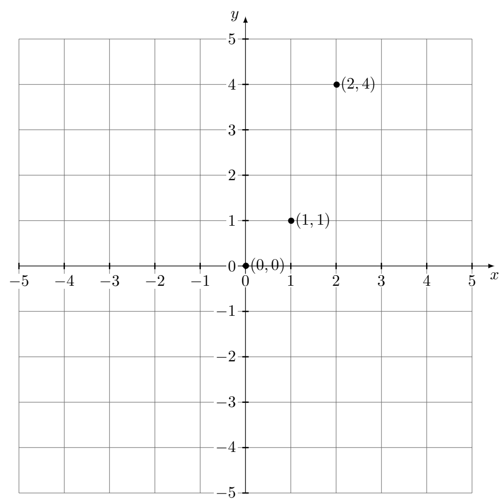


Figure 9.3: Three solutions of $y = x^2$.

We can notice one thing right away about Figure 9.3: no line will pass through all three of these points on the graph! Since it is not clear what the relationship is between these three solutions, we will continue to find more solutions until a pattern begins to emerge.

So far, we have chosen positive values for x. Let's see what happens when we chose some negative values. Choosing -1 for x, for example, we obtain

$$\begin{aligned} y &= (-1)^2 \\ y &= 1. \end{aligned}$$

Choosing -2 for x,

$$\begin{aligned} y &= (-2)^2 \\ y &= 4. \end{aligned}$$

Our table of solutions now appears as

x	y	Solution
$\boxed{0}$	0	$(0, 0)$
$\boxed{1}$	1	$(1, 1)$
$\boxed{2}$	4	$(2, 4)$
$\boxed{-1}$	1	$(-1, 1)$
$\boxed{-2}$	1	$(-2, 4)$

The graph of these five solutions is given in Figure 9.4.

At this point, some patterns start to take shape. First, there is some symmetry to the solutions: the y-axis appears to be a "mirror," with pairs of solutions appearing at equal distances from this axis. Furthermore, the origin $(0, 0)$, which is a solution to the equation, appears to be a "turning point" for the graph of the

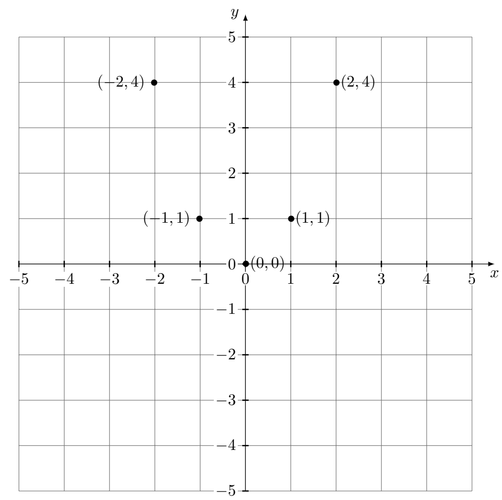


Figure 9.4: Five solutions of $y = x^2$.

solutions: the y-values seem to be decreasing as the x-values increase (through negative values) to 0, then begin to increase as the x-values increase past zero (through positive values).

Based on this information, we might "connect the dots" as in Figure 9.5.

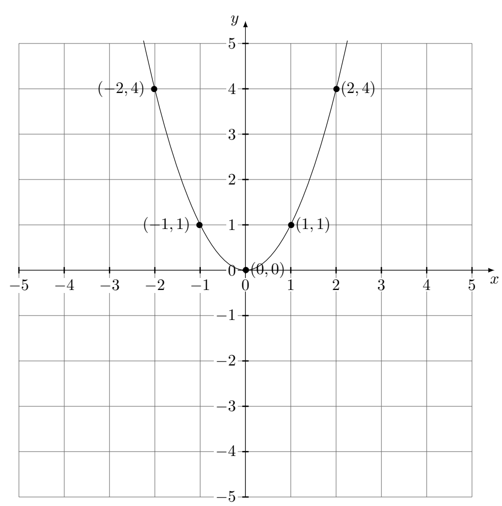


Figure 9.5: All solutions of $y = x^2$.

The shape of this graph is what is known as a *parabola*. For our purposes, the important features of a parabola is that it has one turning point (called the *vertex*) and a line (or *axis*) of symmetry. It is worth pointing out that the parabola's "U" shape does not make it a parabola; there are graphs that also have a "U" shape but are not parabolas.

Parabolas

The graph of solutions to a quadratic equation of the form

$$y = ax^2 + bx + c \quad (a \neq 0)$$

is a parabola.

In a more detailed study of these equations, we could ask whether it is possible, given such an equation, we could find the essential features of its graph, namely its vertex and axis of symmetry. For our limited purposes, though, our strategy will be to simply find enough solutions until the vertex and axis of symmetry are apparent, then connect the solutions by drawing a parabola. In general, we will choose at least five solutions, although if a pattern does not emerge, we may even choose to find more.

Example 9.6.1. *Graph the solutions:* $y = -x^2 + 2x$.

Answer. *We begin by finding solutions by choosing values of* x *close to* 0, *keeping the previous example in mind.*

Choosing -2 *for* x*:*

$$\begin{aligned} y &= -(-2)^2 + 2(-2) \\ y &= -4 + (-4) \\ y &= -8. \end{aligned}$$

Choosing -1 *for* x*:*

$$\begin{aligned} y &= -(-1)^2 + 2(-1) \\ y &= -1 + (-2) \\ y &= -3. \end{aligned}$$

Choosing 0 *for* x*:*

$$\begin{aligned} y &= -(0)^2 + 2(0) \\ y &= 0. \end{aligned}$$

Choosing 1 *for* x*:*

$$\begin{aligned} y &= -(1)^2 + 2(1) \\ y &= -1 + (2) \\ y &= 1. \end{aligned}$$

Choosing 2 *for* x*:*

$$\begin{aligned} y &= -(2)^2 + 2(2) \\ y &= -4 + (4) \\ y &= 0. \end{aligned}$$

So for our first five choices of values of x, we have obtained the following table:

x	y	*Solution*
$\boxed{-2}$	-8	$(-2,-8)$
$\boxed{-1}$	-3	$(-1,-3)$
$\boxed{0}$	0	$(0,0)$
$\boxed{1}$	1	$(1,1)$
$\boxed{2}$	0	$(2,0)$

We plot these five solutions in Figure 9.6.

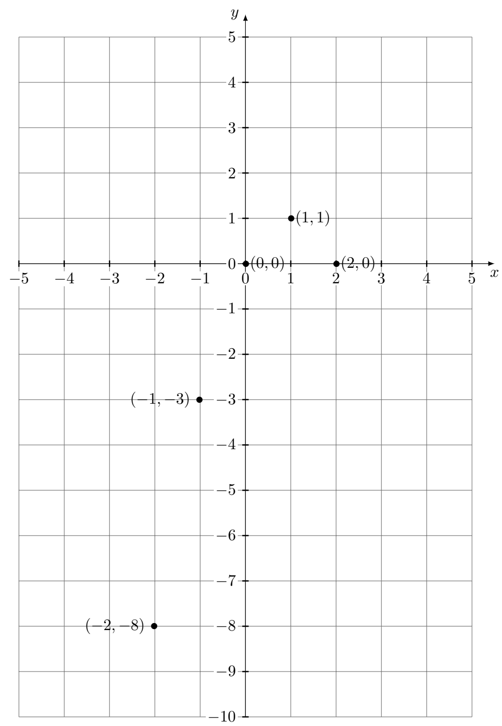


Figure 9.6: Five solutions of $y = -x^2 + 2x$.

Judging from the solutions plotted so far, we might guess that the vertex is the point corresponding to $(1, 1)$. We might confirm that by checking that $(3, -3)$ and $(4, -8)$ are also solutions. We also see that unlike the previous example, a parabola joining these solutions will face downwards.

We conclude by drawing the parabola as in Figure 9.7.

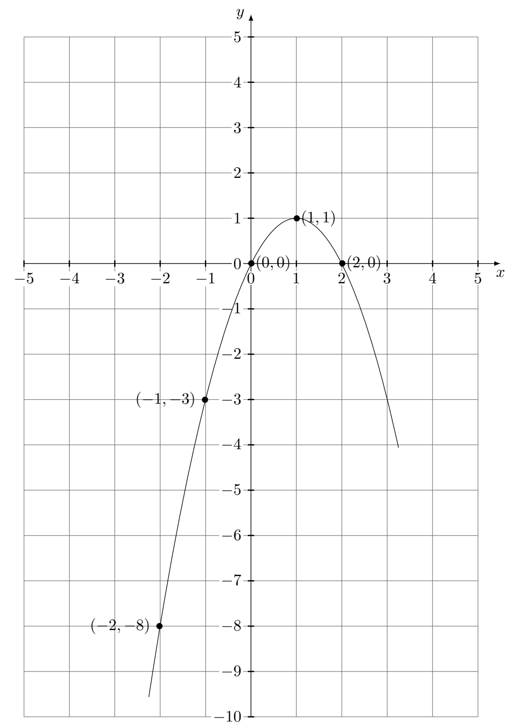


Figure 9.7: All solutions of $y = -x^2 + 2x$.

9.6.1 Exercises

Graph the solutions to the following equations.

1. $y = x^2 + 2$
2. $y = x^2 - 1$
3. $y = 2x^2$
4. $y = -x^2$
5. $y = x^2 - 4x$
6. $y = 1 - x^2$

9.7 Chapter summary

- A typical quadratic equation in one variable will have two solutions.
- Quadratic equations of the form $x^2 = k$ have two solutions, represented symbolically as $\sqrt{k}$ and $-\sqrt{k}$. (The two solutions are the same when $k = 0$.)

- Any quadratic equation in one variable can be solved by either the technique of completing the square or by using the quadratic formula.
- Some quadratic equations (but not all!) can be solved by the alternate technique of factoring and using the zero product property of numbers.
- The graph of an equation of the form $y = ax^2 + bx + c$ in the xy-plane (with $a \neq 0$) will be a parabola.

Glossary

Absolute value The *absolute value* of a number is its magnitude. It is represented symbolically by two bars around a number $| \cdot |$.

Addition principle The *addition principle* expresses the fact that when the same quantity is added to (or subtracted from) each side of an algebraic statement, the resulting statement is equivalent to (has the same solutions as) the original one.

Algebraic expression An *algebraic expression* is an expression formed by combining numbers and variables using the operations of addition, subtraction, multiplication, division, (numerical) exponents, and roots.

Algebraic statement An *algebraic statement* is a comparison of two algebraic expressions using the relations of equality, greater than, or less than. An algebraic statement may be true or false.

Associative law The *associative law* (of addition or of multiplication) expresses the fact that when performing the same operation several times, the way in which quantities are grouped, and hence the order that the operations are performed, does not affect the outcome of performing the operations. It is usually expressed symbolically as $(a+b)+c = a+(b+c)$ (for addition) or $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (for multiplication).

Base (of an exponential expression) In an exponential expression, the *base* is the quantity which is written immediately to the left of the exponent (which is written as a superscript). For natural number exponents, the base is the quantity which is multiplied a repeated number of times.

Coefficient The *coefficient* of a term is the "number part" of the term, or the constant by which the variable part is multiplied.

Commutative law The *commutative law* (of addition or of multiplication) expresses the fact that the order in which the quantities are expressed does not affect the outcome of performing the operation. It is usually expressed symbolically as $a + b = b + a$ (for addition) or $a \cdot b = b \cdot a$ (for multiplication).

Completing the square *Completing the square* is a process by which an appropriate quantity is added to an algebraic expression in such a way that the result is a perfect square. (The quantity added must be compensated for in such a way that the new expression or statement is equivalent to the old one, for example by adding the same quantity to both sides of an equation.)

Complex numbers A *complex number* is a symbolic expression involving (real) numbers as well as the imaginary unit i. Symbolically, a complex number has the form $a + bi$, where a and b are (real) numbers.

Conditional statement An algebraic statement is *conditional* if it may be true or false, depending on the values assigned to the variables involved.

Constants *Constants* are numbers, or symbols that represent a definite value.

Contradiction A *contradiction* is an algebraic statement which is false for all values of the variables involved. A contradiction therefore has no solution.

Denominator The *denominator* of a fraction is the number written below the bar, and represents the divisor of the quotient.

Descending order A polynomial is written in *descending order* if terms having higher degree are written to the left of terms having lower degree.

Difference of squares A *difference of squares* is a binomial involving the difference of two terms which are each perfect squares. Symbolically, a difference of squares has the form $a^2 - b^2$, where a and b represent any algebraic expression.

Discriminant The *discriminant* of a quadratic formula of the form $ax^2 + bx + c = 0$ is the quantity $b^2 - 4ac$. The discriminant allows one to determine if the solutions of the given equation are rational or irrational, real or complex, and how many distinct solutions the equation has.

Equivalent fractions Two fractions are *equivalent* if they represent the same quotient.

Fraction in reduced form A fraction is written in *reduced form* if the numerator and denominator are integers that have no factor in common (other than 1).

Integer An *integer* is either a whole number or the opposite of a whole number. The set of all integers can be expressed as $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Degree (of a polynomial) The *degree* of a polynomial is the highest degree of any of its terms.

Degree (of a term) The *degree* of a term in a polynomial in one variable is the exponent of the variable part.

Distributive law The *distributive law* expresses a relationship between the operations of addition and multiplication. It states that the product of a quantity with a sum is the same as the sum of the product of the quantity with each of the summands. It is usually expressed symbolically as $a \cdot (b + c) = a \cdot b + a \cdot c$.

Equation An *equation* is an algebraic statement involving the relation of equality (=). An equation is true if both sides of the statement have the same value.

Equivalent equations Two equations are *equivalent* if they have the same solutions.

Evaluate To *evaluate* an expression means to find its value, generally after performing all indicated operations.

Factor (verb) To *factor* an expression means to write it as a product of two or more factors (usually different than 1).

Factor (of an integer) A *factor* of an integer is another integer which, when multiplied by a third integer, is equal to the original integer. (This is sometimes stated as, "A number which divides the original number evenly.")

Factor (of a polynomial) A *factor* of a polynomial is another polynomial which, when multiplied by a third polynomial, gives the original polynomial.

Factor completely To factor a polynomial completely means to write it as a product of two or more factors, none of which can be factored and further.

Fraction A *fraction* is a symbolic way of writing a quotient that involves two numbers separated by a bar representing the operation of division.

Function notation *Function notation* gives a symbolic way of referring to algebraic expressions. It involves naming the expression with a single letter (like f, g, or P), and indicating the variables on which the expression depends. The notation $f(x)$ (read "f of x") is meant to indicate the expression named f with variable x; $f(3)$ indicates the value of $f(x)$ when x is assigned the value 3.

Graph To *graph* an algebraic statement means to plot all solutions, either on a number line (statements involving one variable) or on an xy-plane (statements involving two variables).

Greatest common divisor The *greatest common factor* (GCF) of monomials (with integer coefficients) is the product of the GCF of the coefficients with the lowest power of each variable appearing in any of the terms.

Horizontal lines A line in an xy-plane is called *horizontal* if it is parallel to the x-axis. Horizontal lines have slope 0.

Hypotenuse In a right triangle, the *hypotenuse* is the side opposite to the right angle. Due to the Pythagorean theorem, it is also the longest side of a right triangle.

Identity An *identity* is an algebraic statement which is true for all values of the variables involved.

Imaginary unit The *imaginary unit*, denoted i, is the symbol defined to have the property that $i^2 = -1$.

Inequality An *inequality* is an algebraic statement involving the relations of "greater than" ($>$) or "less than" ($<$). An inequality is true when the inequality symbol "points" in the direction of the side with the smaller value. Inequalities include the compound inequalities "greater than or equal to" ($\geq$) and "less than or equal to" ($\leq$), which are true when either the inequality is true or when the sides have the same value.

Intercepts (x– and y–intercept of a line in an xy-plane) The *intercepts* of a line drawn in an xy-plane are the points where the line intersects the x-axis (the x-intercept) and where the line intersects the y-axis (the y-intercept). The ordered pair corresponding to the x-intercept will have 0 as a y-coordinate, while the ordered pair corresponding to the y-intercept will have 0 as an x-coordinate.

Irrational numbers A real number is *irrational* if it is impossible to represent it as a ratio of two integers. The decimal expansion of an irrational number never terminates and never repeats.

Leading term The *leading term* of a polynomial is the term with the highest degree. (Hence, when a polynomial is written in descending order, the leading term is the first term written.)

Like terms *Like terms* have the same variable part. For a linear equation in one variable, two terms are like terms if they either both involve the variable or both do not involve the variable. For a polynomial in one variable, like terms are those having the same degree.

Linear equation An algebraic statement is *linear* if the only operations performed on a variable are addition, subtraction, and multiplication by a constant. As a consequence, every term of a linear equation involves at most one variable, and the highest exponent of any variable is 1. In the language of polynomials, an equation is linear if it only involves polynomial expressions of degree 1.

Literal equation A *literal equation* (or *formula*) is an equation that relates several variables, usually meant to express relationships between measured quantities.

Magnitude The *magnitude* of a number is a non-negative number that represents how "big" a number is, regardless of sign. On a number line, the magnitude of a number is the distance from the point representing the number to the point representing 0.

Monic polynomial A polynomial is called *monic* if the coefficient of the leading term is 1.

Multiplication principle The *multiplication principle* expresses the fact that when the same non-zero quantity is multiplied by (or divided by) each side of an algebraic equation, the resulting equation is equivalent to (has the same solutions as) the original one. The multiplication principle also applies to inequalities when multiplying by *positive* quantities. When multiplying an algebraic inequality by a *negative*, the original inequality is equivalent to the new inequality with the opposite sense of the inequality.

Natural number A *natural number* is a number which occurs in the "natural" operation of counting, and are sometimes called "counting numbers." The set of all natural numbers can be expressed as $\{1, 2, 3, \dots\}$.

Negative number A *negative* number is a number which is less than zero. On a number line, negative numbers are represented by points to the left of the point representing zero.

Non-negative number A *non-negative* number is either positive or zero. Said differently, a non-negative number is a number greater than or equal to zero.

Numerator The *numerator* of a fraction is the number written above the bar, and represents the dividend of the quotient.

Opposite of a number The *opposite* of a number is the number which, when added to the original number, gives zero. The opposite of a number is also called the "additive inverse" of a number. The opposite of a number has the same magnitude, but opposite sign as the original number. Symbolically, the opposite of a number a is $-a$.

Order of operations The *order of operations* is the conventional order in which operations are performed when there is more than one operation involved.

Ordered pair An *ordered pair* is a notation that is well-suited to describe solutions of algebraic statements in two variables. It involves two numbers (called *coordinates*) separated by a comma and enclosed in parentheses, for example $(3, 2)$. The first number is traditionally called the x-coordinate while the second number is traditionally called the y-coordinate.

Parabola The shape of the graph of solutions to equations of the form $y = ax^2 + bx + c$ is called a *parabola*. A parabola has a vertex (or turning point) and an axis of symmetry.

Parallel lines Two lines in a plane are *parallel* if they do not intersect. In an xy-plane, parallel lines have the same slope.

Parentheses (and other grouping symbols) *Parentheses*, along with other symbols like brackets, are used in a mathematical expression to indicate grouping. In the order of operations, grouped operations are performed before other operations.

Perpendicular lines Two lines in a plane are *perpendicular* if they meet at a right angle. In an xy-plane, the product of the slopes of perpendicular lines is -1.

Plotting an ordered pair To *plot* an ordered pair is the process of representing an ordered pair with a point in an xy-plane. The coordinates represent distances to the axes, with the distance measured in a direction corresponding to whether the coordinate is positive or negative.

Plus-or-minus ($\pm$) notation The symbol $\pm k$ denoted the compound expression, "Either k or $-k$." An equation $x = \pm k$ is the same as the compound statement, "Either $x = k$ or $x = -k$.

Point-slope form of a linear equation in two variables An equation derived from the formula defining the slope of a line in an xy-plane, which makes explicit the slope of the line and the coordinates of one point. It is written

$$y - y_0 = m(x - x_0),$$

where m is the slope and (x_0, y_0) represent the (constant) coordinates of one point on the line. The point-slope form of a line is generally used as a "formula" to write the equation describing a line in an xy-plane with some given geometric data.

Polynomial A *polynomial* (in one variable) is an algebraic expression, each of whose terms have the form ax^n where a is a constant coefficient and n is a whole number.

Positive number A *positive* number is a number which is greater than zero. On a number line, positive numbers are represented by points to the right of the point representing zero.

Pythagorean theorem The *Pythagorean theorem* expresses a relationship between the lengths of the sides of a right triangle. Specifically, it states that a triangle with hypotenuse with length h and with the two other sides having lengths a and b is a right triangle if and only if $h^2 = a^2 + b^2$.

Quadratic equation A *quadratic equation* is an equation involving polynomials of degree 2. In particular, a polynomial in one variable x must have an x^2 term (with nonzero coefficient), and no term may have degree higher than 2.

Quadratic formula The *quadratic formula* expresses the solutions to a quadratic equation of the form $ax^2+bx+c=0$ in terms of the coefficients a, b and c. Specifically, it states that the equation $ax^2+bx+c=0$ is equivalent to the (compound) statement

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quadratic trinomial A *quadratic trinomial* is a polynomial with three terms with degree two. Symbolically, a quadratic trinomial in one variable x can be written ax^2+bx+c, where a, b and c represent constant coefficients.

Radicand The *radicand* of a radical symbol is the quantity enclosed by the radical sign $\sqrt{}$. Symbolically, the radicand of the symbol $\sqrt{k}$ is k.

Rational number A number is said to be *rational* if can be expressed as a ratio of two integers. The decimal expansion of a rational number either terminates or repeats.

Rationalizing (a denominator) The process of multiplying the numerator and denominator of a fraction by (the same) appropriate quantity so that the resulting denominator does not involve a radical symbol after simplifying is called *rationalizing the denominator*.

Real numbers Any number or numerical expression not involving the imaginary unit i is called a *real number*.

Reciprocal The *reciprocal* of a number is the number which, when multiplied by the original number, has product 1. The reciprocal is also called the "multiplicative inverse" or just "inverse." Symbolically, the reciprocal of a number a is $1/a$.

Scientific notation A number is written in *scientific notation* if it has the form $a \times 10^n$, where a is a number whose magnitude is greater than or equal to 1, but strictly less than 10, and where n is an integer.

Slope of a line The *slope* of a line is a number measuring the steepness of the line. In an xy-plane, it is given by the ratio of the change in the y-coordinates of any two points on the line to the change in the x-coordinates of the same two points.

Slope-intercept form of a linear equation in two variables A linear equation in two variables x and y is said to be written in *slope-intercept* *form* if the y variable is by itself on one side of the equation. An equation in slope-intercept form is often written $y = mx + b$, since the coefficient of the variable x represents the slope of the line obtained by plotting solutions to the equation, and b represents the y-coordinate of the y-intercept.

Solution A *solution* to a conditional statement is a value for each variable in the statement which, when substituted into the expressions involved, yield a true statement.

Solve To *solve* a (conditional) algebraic statement means to find all solutions of the statement.

Strict inequality An inequality is *strict* if it involves only the relations of $<$ or $>$, and not the compound inequalities $\leq$ and $\geq$.

System of linear equations A *system* of linear equations consists of two or more equations which are taken as part of a compound statement, usually indicated by writing a brace symbol ($\{$). A solution to a system must be a solution to all of the equations involved in the system.

Term A *term* is an algebraic expression which is not itself the sum of two or more expressions. A general algebraic expression can be written as a sum of terms.

Variables *Variables* are mathematical symbols, usually indicated by letters, that indicate an unknown number, or a number that changes with time.

Vertical lines A line in an xy-plane is called *vertical* if it is parallel to the y-axis. Vertical lines do not have a slope (or the slope is *undefined*).

Whole number A *whole number* is either a natural number or 0. The set of all whole numbers can be expressed as $\{0, 1, 2, 3, \ldots\}$. Whole numbers answer the question, "How many?"

xy**-plane** An xy-plane is a method for representing ordered pairs as points in a plane. It involves two fixed perpendicular lines (called *axes*) which intersect at one point (called the *origin*).

Zero product property The *zero product property* of numbers expresses the fact that for any two numbers a and b, $a \cdot b = 0$ implies that either a or b is 0.

Answers to Exercises

Chapter 1

Exercises 1.2.1

1. 0.75
2. $0.\overline{45}$
3. $0.\overline{285714}$
4. $\dfrac{41}{333}$
5. (a) $\dfrac{41152}{333333}$
   (b) $\dfrac{311}{99}$

Exercises 1.3.1

1. $3\frac{4}{5}$
2. $33\frac{1}{3}$
3. $\dfrac{33}{8}$
4. $\dfrac{23}{10}$

Exercises 1.5.3

1. $\dfrac{3}{4}$
2. $\dfrac{4}{5}$
3. 3
4. $\dfrac{2}{3}$
5. $\dfrac{7}{5}$

Exercises 1.6.3

1. $\dfrac{3}{10}$
2. $\dfrac{5}{6}$
3. $\dfrac{34}{3}$
4. 3
5. $\dfrac{1}{48}$
6. $\dfrac{1}{8}$

Exercises 1.7.1

1. $\dfrac{68}{35}$

2. $\frac{1}{6}$

3. $\frac{11}{10}$

4. $\frac{1}{12}$

5. $\frac{13}{3}$

6. $\frac{11}{12}$.

Chapter 2

Exercises 2.3.3

1. -7
2. -7
3. 1.45
4. -14
5. 0
6. $-\frac{13}{20}$
7. $\frac{1}{24}$
8. -3
9. -7
10. -8
11. -1.93
12. -1
13. $\frac{7}{12}$
14. $-\frac{3}{14}$
15. $-\frac{1}{8}$

Exercises 2.4.1

1. -36
2. 25
3. -3
4. 80
5. $-\frac{4}{7}$
6. $\frac{21}{16}$

Exercises 2.5.1

1. 81
2. -8
3. 32
4. -9
5. Not a real number

Chapter 3

Exercises 3.1.1

1. -12
2. 5
3. 2
4. -7
5. $27/16$
6. 5
7. $5/6$

8. 3

Exercises 3.3.2

1. -115
2. 13
3. 1
4. 1875
5. 53/9
6. (a) 2
   (b) -2
   (c) 2
   (d) -15
   (e) 13/5
7. (a) 3
   (b) 3
   (c) 2
   (d) 1
   (e) $-3/2$
8. (a) -40
   (b) 5
   (c) 50
   (d) 77
   (e) 448/5
9. (a) 24
   (b) 5/4
   (c) -1
10. (a) -12
    (b) -8
    (c) 10

Exercises 3.4.1

1. $2(x+8)$
2. $\frac{1}{2}x-7$
3. $\left(\frac{1}{4}\right)(x-12)$
4. Thirty less than seven times a number.
5. Three times the sum of a number and two.
6. The sum of three times a number and the square of five less than the same number
7. The quotient of one less than two times a number and three
8. The quotient of three less than seven times a number and three less than the same number

Chapter 4

Exercises 4.1.1

1. Yes.
2. Yes.
3. Yes.
4. No.
5. Yes.
6. No.
7. No.
8. Yes.
9. Yes.
10. No.
11. Yes.
12. Yes.
13. Yes.
14. Yes.

15. No.

Exercises 4.2.6

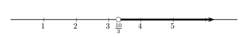
1. 2
2. 10/3
3. −1
4. 1/5
5. No solution
6. 15/8
7. All real numbers
8. 0
9. 15/8
10. 9
11. 14/3
12. 5/3
13. 30, 31, and 32
14. 131 and 133
15. 9/4 and 51/4
16. \$ 1,500 per week
17. −1, 1 and 3.
18. 17/4 and 43/4

Exercises 4.3.1

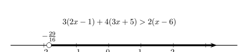
1. $y = \frac{3}{2}x - 3$
2. $y = -\frac{5}{4}x + \frac{5}{2}$
3. $r = \frac{I}{Pt}$
4. $C = \frac{5}{9}(F - 32)$
5. $m = \frac{y_0 - y}{x_0 - x}$

Exercises 4.4.3

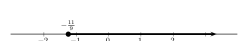
1. $3x - 4 > 6$



2. $2(x-3) + 4 \leq x - 5$

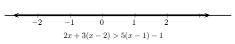
3. 

4. $x - 5(2x+1) \leq 6$



5. No solution

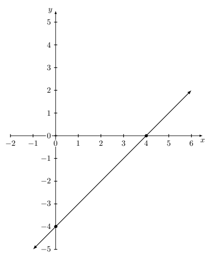
6. All real numbers



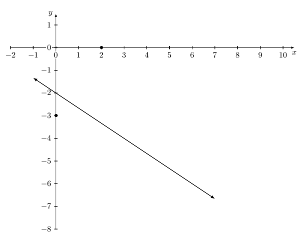
Chapter 5

Exercises 5.1.3

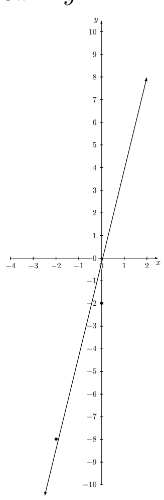
1. $x - y = 4$



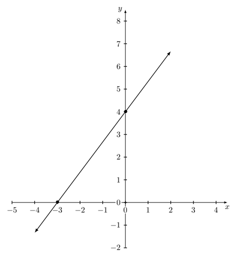
2. $2x + 3y = -6$



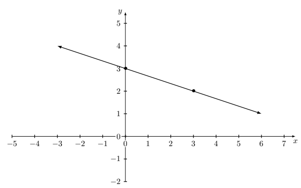
3. $5x - y = 2$



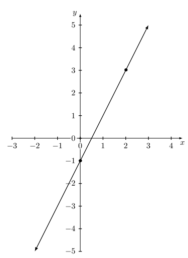
4. $-4x + 3y = 12$



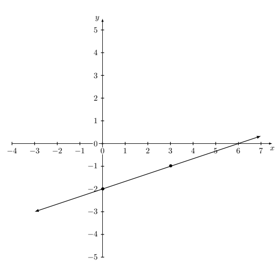
5. $-x + 3y = 9$



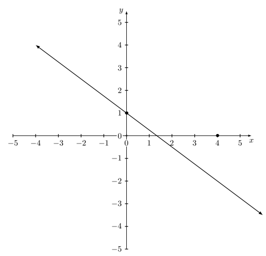
6. $y = 2x - 1$



7. $y = \frac{1}{3}x - 2$



8. $y = -\frac{3}{4}x + 1$

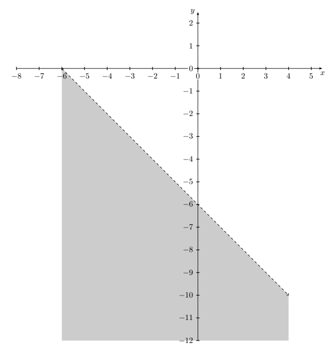


Exercises 5.2.4

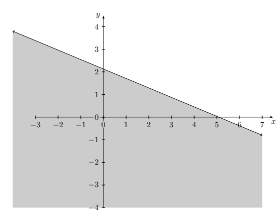
1. (a) -2
   (b) 2
   (c) Undefined
   (d) $1/3$
   (e) $-2/3$
   (f) 5
   (g) 2
   (h) $1/3$
   (i) $-3/4$
   (j) 0
   (k) -1
2. Slope $-3/4$, y-intercept $(0, 1)$
3. Slope 5, y-intercept $(0, -2)$
4. $y = \dfrac{3}{4}x - \dfrac{17}{4}$
5. $y = \dfrac{2}{3}x - \dfrac{7}{3}$
6. $y = \dfrac{3}{4}x - 5$
7. $y = 4x - 4$
8. Find the slopes and compare.
9. $3x + 5y = -9$
10. Find the slopes and use the slope interpretation of perpendicular.
11. $-5x - y = 0$

Exercises 5.3.1

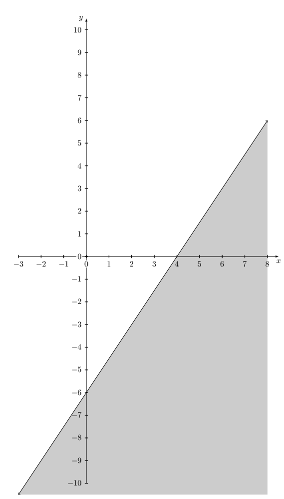
1. $-x - y > 6$



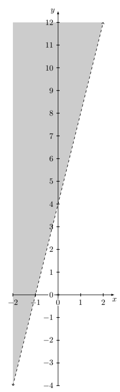
2. $2x + 5y \leq 10$



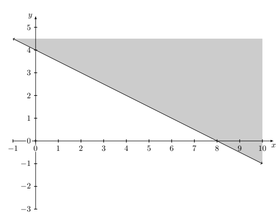
3. $3x - 2y \geq 12$



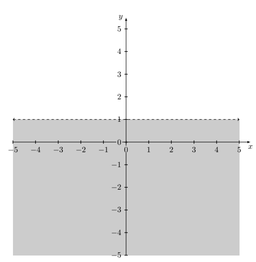
4. $-4x + y > 4$



5. $y \geq -\frac{1}{2}x + 4$



6. $y < 1$



Exercises 5.4.2

1. $(3, -5)$
2. $(-1, -1)$
3. $\left(\frac{15}{8}, -\frac{17}{16}\right)$
4. $\left(\frac{48}{11}, \frac{10}{11}\right)$
5. $(0, 2)$
6. $\left(\frac{13}{7}, \frac{10}{7}\right)$
7. No solution
8. $\left(\frac{14}{3}, \frac{13}{3}\right)$

Chapter 6

Exercises 6.1.1

1. Polynomial
2. Polynomial
3. Polynomial
4. Not polynomial
5. Not polynomial
6. (a)-(b) Terms: 3 (degree 0, coefficient 3), $-x$ (degree 1, coefficient -1); (c)-(d) $-x+3$ (degree 1)
7. (a)-(b) Terms: 5 (degree 0, coefficient 3), x^2 (degree 2, coefficient 1), $-3x$ (degree 1, coefficient -3); (c)-(d) $x^2 - 3x + 5$ (degree 2)
8. (a)-(b) Terms: 1 (degree 0, coefficient 1), $-x$ (degree 1, coefficient -1), x^2 (degree 2, coefficient 1), $-x^3$ (degree 3, coefficient -1), $x^4/2$ (degree 4, coefficient 1/2); (c)-(d) $\frac{x^4}{2} - x^3 + x^2 - x + 1$ (degree 4)

Exercises 6.2.3

1. $3x^2 - 3x - 2$
2. $4x^3 - x^2 - x + 7$
3. $4y^2 + 3y + 3$
4. $-3x^3 + 2x^2 - 3x - 3$
5. $4x^3 + 3x^2 - 4x - 3$

6. $-5x + 13$
7. $x^2 + 2x + 6$

Exercises 6.3.2

1. x^7
2. z^{12}
3. $x^{-6}/16$
4. ab
5. x^5y^{-4} or x^5/y^4
6. y
7. (a) 25 (b) 13
8. (a) 1 (b) 5

Exercises 6.4.2

1. 7.5×10^{18}
2. 2.75×10^{-10}
3. $602,200,000,000,000,000,000,000$
4. 0.001
5. 1.8×10^8
6. 2×10^{-2}
7. 9.1×10^{-14}
8. 5.1×10^3
9. 7.14×10^3
10. 5×10^{-9}
11. 1.2555×10^1
12. -2.5×10^9

Exercises 6.5.1

1. $x^2 - x - 6$
2. $6x^2 - 11x + 4$
3. $4x^2 + 12x + 9$
4. $x^3 - 1$
5. $x^4 + 6x^3 + 3x^2 - 11x - 6$
6. $x^4 - x^3 - 4x^2 - 11x - 3$
7. $9x^2 - 12x + 4$
8. $8x^3 + 6x^2 - 7x + 6$
9. $2x^3 + 3x^2 - 32x + 30$
10. $8x^3 - 12x^2 + 6x - 1$
11. Distribute.
12. $a = 0$ or $b = 0$.

Exercises 6.6.1

1. $-5x + 10$
2. $\dfrac{x}{3} - 1 + 3x^{-1}$
3. $x^4 - 4x^3 - 3x$
4. $x^4 - x - 4$
5. $-5x^2 + 7x - 1$

Chapter 7

Exercises 7.2.1

1. $2x(3x^2 - 1)$
2. $4x^2(x^3 - 3x - 2)$
3. $9(2x - 1)$
4. $-3x^2(x^2 - 5x + 3)$
5. $6ab^2(b - 2a)$
6. $-xy(y^3 + 2xy^2 + 15x^4)$

Exercises 7.3.1

1. $(a + 3)(a - 3)$
2. $(x + 5)(x - 5)$
3. Cannot be factored

4. $4(x+3)(x-3)$
5. $(5x^2+9y^3)(5x^2-9y^3)$
6. $x^2(x+2)(x-2)$
7. $3x(x+5)(x-5)$
8. (a) $(x+y)^2$
   (b) $(x+5)^2$
   (c) $(x+6)^2$
9. Remember $(a-b)^2 = (a-b)(a-b)$.
10. (a) $(x-1)^2$
    (b) $(x-9)^2$
    (c) $(x-4)^2$
11. Combine like terms!
12. (a) $(x-2)(x^2+2x+4)$
    (b) $y(x-5y)(x^2+5y+25y^2)$
13. Combine like terms!
14. (a) $(x+4)(x^2-4x+16)$
    (b) $(2x+3)(4x^2-6x+9)$

Exercises 7.4.1

1. $(x+4)(x-8)$
2. $(y+6)(y-3)$
3. $(x+2)(x+6)$
4. $(x-2)(x-10)$
5. Cannot be factored
6. $(y+2)(y-3)$
7. $(t+3)^2$
8. $(x-1)(x-5)$
9. $3x(x+1)(x-4)$
10. $5(x+4)(x-4)$
11. $2x(3x^2-1)$
12. $2(x^2+4)$
13. $4x(x^2-3x-2)$
14. $3x(x+2)(x-2)$
15. (a) $(x^2+2)(x^2+3)$
    (b) $(x^2+3)(x+2)(x-2)$
    (c) $(x^3+4)(x^3-2)$
    (d) $(x^{16}+3)(x^{16}-3)(x^{16}+1)(x^8+1)(x^4+1)(x^2+1)(x+1)(x-1)$

Exercises 7.5.1

1. $(x+5)(2x-11)$
2. $(3x+1)(x+1)$
3. $(3x+2)(2x-1)$
4. $(5x+2)(3x-1)$
5. Cannot be factored
6. $(x+2)(2x-5)$
7. $2(3x-5)(x-2)$
8. $2x^2(x-4)(x+2)$
9. (a) $(2x^2+1)(x^1+3)$
   (b) $(4x^2+5)(x+1)(x-1)$
   (c) $(3x^3-1)(x^3-3)$
   (d) $(2x^3+7)(x^3-1)$
   (e) $(4x^{500}+3)(x^{500}-3)$
10. (a) $(x+3y)(x-4y)$
    (b) $(x+2y)(x+y)$
    (c) $(2x+y)(x-3y)$
    (d) $(3x+y)(x-y)$

Exercises 7.6.1

1. $(2z+3w)(3x-4y)$
2. $(5w+7z)(9c-4d)$
3. Cannot be factored

4. $(a+3b)(4c-3d)$
5. $(5a-2b)(4x-3y)$
6. $(3m-2n)(x+2y)$

Chapter 8

Exercises 8.2.1

1. $4\sqrt{2}$
2. $10\sqrt{5}$
3. $7\sqrt{2}$
4. $8\sqrt{3}$
5. $5\sqrt{2}$
6. $\frac{\sqrt{3}}{4}$
7. $\frac{2\sqrt{2}}{5}$
8. $\frac{\sqrt{30}}{6}$
9. $\frac{\sqrt{3}}{6}$
10. $\frac{3\sqrt{2}}{8}$
11. (a) 3 and 5
    (b) "No n^{th} powers in the radicand."
    (c) $2\sqrt[3]{3}$
    (d) $\frac{\sqrt[3]{4}}{2}$
    (e) $2\sqrt[3]{15}$
12. 10 and -10
13. $2\sqrt{3}$ and $-2\sqrt{3}$
14. $5\sqrt{6}$ and $-5\sqrt{6}$
15. $\sqrt{3}/2$ and $-\sqrt{3}/2$

Exercises 8.3.1

1. $4i$
2. $3i\sqrt{5}$
3. $\frac{i\sqrt{10}}{4}$
4. $i,-1,-i,1,i,-1,-i,1,i,-1$
5. $6i$ and $-6i$
6. $2i\sqrt{6}$ and $-2i\sqrt{6}$

Exercises 8.4.1

1. $2\sqrt{2}+12\sqrt{3}$
2. $-2\sqrt{6}+\sqrt{3}$
3. $12\sqrt{5}$
4. $-2-9i$
5. $10\sqrt{2}-\sqrt{5}$
6. $12\sqrt{2}-10\sqrt{3}$
7. $5\sqrt{2}$
8. $13+13i$
9. -1
10. 13
11. $5-2\sqrt{6}$
12. $2i$
13. $4\sqrt[3]{2}+12\sqrt[3]{3}$
14. (a) $\frac{15-5\sqrt{2}+6\sqrt{3}-2\sqrt{6}}{13}$
    (b) $\frac{4\sqrt{2}-\sqrt{10}}{21}$
15. (a) $\frac{11-10i}{13}$
    (b) $-\frac{9}{37}-\frac{20}{37}i$

16. Read the fine print in the properties of square roots listed in Box 8.1.

Chapter 9

Exercises 9.1.2

1. $5\sqrt{6}$ and $-5\sqrt{6}$
2. $\sqrt{3}/2$ and $-\sqrt{3}/2$
3. $7i$ and $-7i$
4. $3i\sqrt{2}$ and $-3i\sqrt{2}$
5. $5/2$ and $-5/2$
6. $3+2\sqrt{3}$ and $3-2\sqrt{3}$
7. $13/2$ and $-11/2$
8. $5+3i\sqrt{2}$ and $5-3i\sqrt{2}$
9. $h = 2\sqrt{13}$
10. $a = 5\sqrt{3}$
11. $h = 4\sqrt{10}$

Exercises 9.2.1

1. -1 and 5
2. -6 and 4
3. $3+\sqrt{15}$ and $3-\sqrt{15}$
4. -7 and -3
5. -3 and -2
6. $3+\sqrt{6}$ and $3-\sqrt{6}$
7. -3 and 0
8. $3/2$ and -2
9. $-2+i\sqrt{2}$ and $-2-i\sqrt{2}$
10. $\dfrac{2+\sqrt{7}}{3}$ and $\dfrac{2-\sqrt{7}}{3}$

Exercises 9.3.1

Same as previous section.

Exercises 9.4.1

1. 1 and 3
2. -3 and 4
3. -7 and 0
4. -5 and -4
5. $-1/2$ and 2
6. -2 and -1
7. $-5/2$ and $3/2$
8. $-5/3$ and $5/3$
9. (a) 0, 2 and 4
   (b) -3, $-7/2$ and 1
10. (a) 0 and 6 (two solutions)
    (b) 3, 5, 9, and 11 (four solutions)
    (c) 3 (one solution)

Exercises 9.5.1

1. 2
2. -2 and $1/3$
3. -2 and 0
4. $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$
5. $\dfrac{-3+i\sqrt{3}}{2}$ and $\dfrac{-3-i\sqrt{3}}{2}$
6. -5 and 3
7. $4i$ and $-4i$
8. $-1/2$ and 2
9. $\dfrac{-5+\sqrt{71}}{6}$ and $\dfrac{-5-\sqrt{71}}{6}$

10. $2+\sqrt{3}$ and $2-\sqrt{3}$

11. $2i\sqrt{2}$ and $-2i\sqrt{2}$

12. $-1/4$ and 1

13. Width $\dfrac{-5+5\sqrt{97}}{3} \approx 14.748$ ft, length $5+5\sqrt{97} \approx 54.244$ ft.

14. Both legs have length $10+10\sqrt{2} \approx 24.142$ units, and the hypotenuse has length $20+10\sqrt{2} \approx 34.142$ units.

15. Either 12, 13 and 14, or -3, -2 and -1.

16. The perimeter is $\dfrac{-12+6\sqrt{3089}}{5} \approx 64.295$ inches.

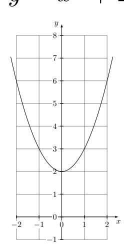
17. The object will hit the ground after $5/4 = 1.25$ seconds.

18. The phone will hit the ground after $5/2 = 2.5$ seconds.

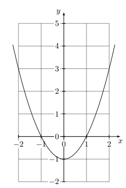
19. The segment should be divided into parts having length $-5+5\sqrt{5} \approx 6.18$ units and $15-5\sqrt{5} \approx 3.82$ units.

Exercises 9.6.1

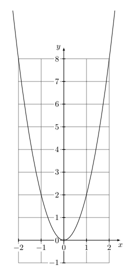
1. $y = x^2 + 2$



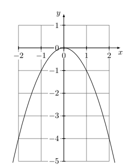
2. $y = x^2 - 1$



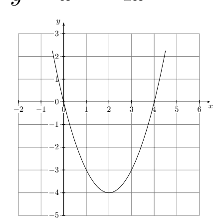
3. $y = 2x^2$



4. $y = -x^2$



5. $y = x^2 - 4x$



6. $y = 1 - x^2$