Take Home Exam for Thanksgiving

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Nikos Apostolakis

Due Date: Tuesday November 28, 2017

1. Borachio works in an automotive tire factory. The number x of sound but blemished tires that he produces on a random day has the probability distribution

x	2	3	4	5
$p(x)$	0.48	0.36	0.12	0.04

   (a) Find the probability that Borachio will produce more than three blemished tires tomorrow.
   (b) Find the probability that Borachio will produce at most two blemished tires tomorrow.
   (c) Compute the mean and standard deviation of x.

2. Pomegranates are sold in crates of 10. It is estimated that about 25% of the pomegranates in a crate are bad.
   (a) What's the expected number of good pomegranates in a crate?
   (b) What's the standard deviation of the number of good pomegranates in a crate?
   (c) What's the probability that there are at least 8 good pomegranates in a crate?

3. The amount x of beverage in a can labeled 12 ounces is normally distributed with mean 12.1 ounces and standard deviation 0.05 ounce. A can is selected at random.
   (a) Find the probability that the can contains at least 12 ounces.
   (b) Find the probability that the can contains between 11.9 and 12.1 ounces.

4. Suppose the mean number of days to germination of a variety of seed is 22, with standard deviation 2.3 days. Find the probability that the mean germination time of a sample of 160 seeds will be within 0.5 day of the population mean.

5. Suppose that in 20% of all traffic accidents involving an injury, driver distraction in some form (for example, changing a radio station or texting) is a factor. Find the probability that in a random sample of 275 such accidents between 15% and 25% involve driver distraction in some form. First verify that the sample is sufficiently large to use the normal distribution.

6. Tests of a new light bulb led to an estimated mean life of $1,321$ hours and standard deviation of 106 hours. The manufacturer will advertise the lifetime of the bulb using the largest value for which it is expected that 90% of the bulbs will last at least that long. Assuming bulb life is normally distributed, find that advertised value.

7. Four hundred randomly selected working adults in a certain state, including those who worked at home, were asked the distance from their home to their workplace. The average distance was $\bar{x} = 8.84$ miles. Assume that the standard deviation of the population of all working adults in that state is $\sigma = 2.70$ miles. Construct a 99% confidence interval for the mean distance from home to work for all residents of this state.

8. A software engineer wishes to estimate, to within 5 seconds, the mean time that a new application takes to start up, with 95% confidence. Estimate the minimum size sample required if the standard deviation of start up times for similar software is 12 seconds.

9. Emily works from home, selling health supplements to a network of clients. She wants to estimate how much money a client spends on her products. From a random sample of 25 receipts, she calculated a mean of $\bar{x} = \$350.69$ with a standard deviation of $s = \$70.30$. Find a 90% confidence interval for the mean amount μ spend by all her clients. Assume that x has a distribution that is approximately normal.

10. A study was conducted to determine how effective hypnotherapy is in increasing the number of hours of sleep subjects get each night. As part of the study the hours of sleep of 12 subjects was measured, with the following results:

8.2	9.1	7.7	8.6	6.9	11.2
10.1	9.9	8.9	9.2	7.5	10.5

    Assume that the population from which the data was drawn is normal.

    (a) Compute the sample mean $\bar{x}$ for the above data.

    (b) Compute the sample standard deviation s for the above data.

    (c) Construct 95% confidence interval for the mean hours of sleep each night for the entire population that these subjects were drawn from.

11. In a random sample of 900 adults, 42 defined themselves as vegetarians.

    (a) Give a point estimate of the proportion of all adults who would define themselves as vegetarians.

    (b) Verify that the sample is sufficiently large to use it to construct a confidence interval for that proportion.

    (c) Construct an 80% confidence interval for the proportion of all adults who would define themselves as vegetarians.

Answers

① a) $P(x>3) = P(x=4 \text{ OR } x=5)$

$= P(x=4) + P(x=5)$

$= 0.12 + 0.04$

$= \boxed{0.16}$

b) $P(x \le 2) = P(x=2)$ because 2 is the smallest value x takes

$= \boxed{0.48}$

c)

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
2	0.48	0.96	4	1.92
3	0.36	1.08	9	3.24
4	0.12	0.48	16	1.92
5	0.04	0.2	25	1.00
Σ	1	2.72		8.08

$\mu = \sum x p(x) = \boxed{2.72}$

$\sigma = \sqrt{\sum x^2 p(x) - \mu^2} = \sqrt{8.08 - 7.3894} = \sqrt{0.6816}$

$\approx \boxed{0.83}$

② About 75% of fruits in a crate are good.

Let r be the number of good pomegranates in a crate.

Then r follows a binomial distribution with $n=10$, $p=0.75$

a) We have $\mu = np$. So $\mu = 10 \cdot 0.25 = \boxed{2.5}$

b) We have $\sigma = \sqrt{n \cdot p \cdot q}$ with $q = 1 - 0.25 = 0.75$

So $\sigma = \sqrt{10 \cdot 0.25 \cdot 0.75} = \sqrt{1.875} \approx \boxed{1.37}$

c) Using the tables we get

$P(r \ge 8) = P(r=8 \text{ OR } r=9 \text{ OR } r=10)$

$\approx .282 + .188 + .056$

≈ 0.524

③ a) The z-score for $x=12$ is $z = \frac{x-\mu}{\sigma} = \frac{12-12.1}{0.05} = \frac{0.1}{0.05} = 2$

So $P(x \ge 12) = P(z \ge -2) = P(z \le 2) = \boxed{0.9772}$

b) The z-score for $x = 11.9$ is $z = \frac{11.9 - 12.1}{0.05} = -4$

The z-score for 12.1 is $z = \frac{12.1 - 12.1}{0.05} = 0$

So $P(11.9 \le x \le 12.1) = P(-4 \le z \le 0)$

$= P(z \le 0) - P(z \le -4)$

$\approx 0.5 - 0 = \boxed{0.5}$

④ Since $n = 160 \ge 30$ the sample mean $\bar{x}$ follows

a normal distribution with $\mu_{\bar{x}} = \mu = 22$ and

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{2.3}{\sqrt{160}} \approx \frac{2.3}{12.649} \approx 0.18$

Within 0.5 day of the mean, means $\mu \pm 0.5$

so x is between 21.5 and 22.5.

The z-score for 21.5 is $z = \frac{21.5-22}{0.18} = \frac{-0.5}{0.18} = -2.78$

The z-score for 21.5 is $z = \frac{22.5-22}{0.18} = \frac{0.5}{0.18} = 2.78$

Thus $P(21.5 < x < 22.5) = P(-2.78 < z < 2.78)$

$= P(z < 2.78) - P(z < -2.78)$

$= 0.9972 - 0.0027$

$= \boxed{0.9945}$

⑤ We are sampling from a binomial distribution with $p = 0.20$, and the sample size is $n = 275$. To use the normal approximation we need $np > 5$ and $nq > 5$.

$n \cdot p = 275 \cdot 0.2 = 55 > 5$. $q = 1 - 0.2 = 0.8$

$nq = 275 \cdot 0.8 = 220 > 5$

So the normal approximation applies. So the sample proportion $\hat{p}$ can be approximated by a normal variable x with $\mu = p = .2$ and $\sigma = \sqrt{\frac{pq}{n}} =$

$= \sqrt{\frac{0.2 \cdot 0.8}{275}} = \sqrt{\frac{.16}{275}} \approx \sqrt{0.00058} \approx 0.024$

We want $P(.15 \le \hat{p} \le .25)$

z-score for .15 : $z = \frac{.15-.20}{0.024} = \frac{-.05}{.024} \approx -1.25$

z-score for .25 : $z = \frac{.25-.20}{0.024} = \frac{.05}{.024} \approx 1.25$

So $P(.15 \le \hat{p} \le .25) = P(-1.25 \le z \le 1.25)$

$= P(z \le 1.25) - P(z \le -1.25)$

$= .8944 - .1056$

$= \boxed{0.7888}$

⑥ Let x stand for the life of a light bulb. Then x follows a normal distribution with $\mu = 1{,}231$ hours and $\sigma = 106$ hours. We want to find the value x^* of x so that $P(x \ge x^*) = .9$ or equivalently $P(x \le x^*) = .1$

From the body of the z-table we see that the corresponding z^* value is $z^* = -1.28$.

So $x^* = z^*\sigma + \mu = -1.28 \cdot 106 + 1231 = -135.68 + 1231 = 1095.32$

So the advertised value should be $\boxed{1095}$ hours.

(7) Since σ is known and the sample is large we will use the standard normal distribution: For confidence level 90% we have $z_{0.90} = 1.645$

Since $\bar{x} = 8.84$ and $\sigma = 2.70$ we have that the maximal margin of error is

$$E = z_{0.90} \frac{\sigma}{\sqrt{n}} = 1.645 \frac{2.70}{\sqrt{400}}$$

$$= \frac{4.4415}{20}$$

$$\approx 0.222$$

The confidence interval is then

$$\bar{x} - E < \mu < \bar{x} + E$$

or

$$8.84 - 0.222 < \mu < 8.84 + 0.222$$

or

$$\boxed{8.62 < \mu < 9.06}$$

(8) We have the formula

$$n = \left(\frac{z_c \sigma}{E}\right)^2$$

where $c = 0.95$, $\sigma = 12$, and $E = 2$.

$z_{0.95} = 1.96$ so

$$n = \left(\frac{1.96 \cdot 12}{2}\right)^2$$

$$= \left(\frac{23.52}{2}\right)^2$$

$$= (11.76)^2$$

$$= 138.2976$$

Rounding up we have

$$\boxed{n = 139}$$

(9) We will use Student's t-distribution with 24 degrees of freedom (since $n = 25$). We have $\bar{x} = 350.69$ and $s = 70.30$

For $c = 0.90$ with 24 d.f. we have $t_{0.90} = 1.711$

So
$$E = t_{0.90} \frac{s}{\sqrt{n}}$$

$$= 1.711 \frac{70.30}{\sqrt{25}}$$

$$= \frac{120.2833}{5}$$

$$\approx 24.06$$

So the confidence interval is $\bar{x} - E < \mu < x + E$, or

$$350.69 - 24.06 < \mu < 350.69 + 24.06$$

or

$$\boxed{326.63 < \mu < 374.75}$$

(10) For a) and b) we use the table:

x	x^2
8.2	67.24
9.1	82.81
7.7	59.29
8.6	73.96
6.9	47.61
11.2	125.44
10.1	102.01
9.9	98.01
8.9	79.21
9.2	84.64
7.5	56.25
10.5	110.25
Σ 107.8	986.72

a) $$\bar{x} = \frac{\Sigma x}{n} = \frac{107.8}{12} \approx \boxed{9.0}$$

b) $$s = \sqrt{\frac{\Sigma x^2 - \frac{1}{n}(\Sigma x)^2}{n-1}}$$

$$= \sqrt{\frac{986.72 - \frac{(107.8)^2}{12}}{11}}$$

$$= \sqrt{\frac{986.72 - \frac{11620.84}{12}}{11}}$$

$$\approx \sqrt{\frac{986.72 - 968.40}{11}}$$

$$\approx \sqrt{\frac{18.32}{11}}$$

$$\approx \sqrt{1.67} \approx \boxed{1.29}$$

c) Since $n = 12$ we have 11 degrees of freedom and for $c = 0.95$ we have

$t_{0.95} = 2.201$

So the maximal margin of error is

$$E = t_{0.95} \frac{s}{\sqrt{n}} = 2.201 \frac{1.29}{\sqrt{12}}$$

$$\approx \frac{2.839}{3.4641} \approx 0.82$$

Confidence interval: $\bar{x} - E < \mu < \bar{x} + E$

$$9.0 - 0.82 < \mu < 9.0 + 0.82$$

$$\boxed{8.18 < \mu < 9.82}$$

(11) a) $$\hat{p} = \frac{42}{900} \approx 0.047$$

b) $\hat{q} = 1 - \hat{p} \approx 0.953$ $\quad n \cdot \hat{p} = 42$, $n\hat{q} = 858$

Both $n\hat{p}$ and $n\hat{q}$ are more than 5.

So the sample is large enough.

c) For $c = 0.80$ we have $z_{0.80} = 1.28$. So

$$E = z_{0.80}\sqrt{\frac{\hat{p}\hat{q}}{n}} = 1.28\sqrt{\frac{0.047 \cdot 0.953}{42}}$$

$$\doteq 1.28\sqrt{\frac{0.044791}{42}} \approx 1.28\sqrt{0.001}$$

$$\approx 0.033$$

The confidence interval is: $\hat{p} - E < p < \hat{p} + E$

$$0.047 - 0.033 < p < 0.047 + 0.033$$

$$\boxed{0.014 < p < 0.08}$$