Factoring II

Nikos Apostolakis

November 19, 2008

Factoring using identities

Sometimes we can recognize that a polynomial has the form that appears on one side of a well known identity. For example, we've seen that

$$(a-b)(a+b) = a^2 - b^2$$

We can write this identity as

$$a^2 - b^2 = (a+b)(a-b) \tag{1}$$

In this form this identity tell us that the difference of the squares of two expressions can be factored to a product of the sum of the two expressions and the difference of the expressions.

Example 1. Factor $x^2 - 16$

Solution. The polynomial we wish to factor is the difference of two squares: the square of x minus the square of 4. According to Identity (1) we have

$$x^2 - 16 = (x+4)(x-4)$$

□

Example 2. Factor $9y^2 - 4x^2$

Answer. Now we have the difference of the square of $3y$ and the square of $2x$. So we have:

$$9y^2 - 4x^2 = (3y+2x)(3y-2x)$$

□

Example 3. Factor $(x+2)^2 - (5x-3)^2$

Answer. We have:

$$\begin{aligned}(x+3)^2 - (5x-3)^2 &= [(x+2)+(5x-3)]\,[(x+2)-(5x-3)] \\ &= [x+2+5x-3]\,[x+2-5x+3] \\ &= (6x-1)\,(-4x+5)\end{aligned}$$

□

Example 4. Factor $x^4 - 81$

Answer. This polynomial is the difference of the square of x^2 and the square of 9. So we have:

$$x^4 - 81 = (x^2 + 9)(x^2 - 9)$$

Now, the first factor $x^2 + 9$ is irreducible[1], but the second factor $x^2 - 9$ is itself a difference of squares. So we can further factor to finally get:

$$x^4 - 81 = (x^2 + 9)(x - 3)(x + 3)$$

□

A difference is really a sum, where one of the summands is the opposite of an expression. So for example, instead of $a^2 - b^2$ we could write $-b^2 + a^2$. We should be able to recognize the difference of two squares even when the term with the negative sign is written first:

Example 5. Factor $-36 + x^2$

Answer. The given polynomial is the difference of the square of x and the the square of 6. So:

$$-36 + x^2 = (x - 6)(x + 6)$$

□

Sometimes, we can use Identity (1) only after we have factored out a common factor:

Example 6. Factor $3x^2 - 12$

Answer. 3 is a common factor for each of the terms. After we factor it out the other factor is a difference of squares:

$$\begin{aligned} 3x^2 - 12 &= 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

□

Let's practice:

1. Factor $x^2 - 1$

2. Factor $25x^2 - 49y^2$

[1]Why?

3. Factor $x^4 - y^4$

4. Factor $100x^2 - (x + 3)^2$

5. Factor $-7xy^2 + 28x$

6. Factor $9x^2 + 64$

Sum or difference of cubes

Two more identities are often used in factoring: The sum of two cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \tag{2}$$

and the difference of two cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \tag{3}$$

Example 7. Factor $x^3 - 8$

Answer. The given polynomial is the difference of two cubes, the cube of x and the cube of 2. So we have:

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$

□

Example 8. Factor $27x^3 + 1$

Answer. The given polynomial is the sum of two cubes, the cube of $3x$ and the cube of 1. So we have:

$$27x^3 + 1 = (3x + 1)(9x^2 + 3x + 1)$$

□

Example 9. Factor $x^6 - y^6$

Answer. This is the difference of two squares: the square of x^3 minus the square of y^3. After applying the "difference of squares" identity we get two factors: a difference of cubes and a sum of cubes.

$$\begin{aligned} x^6 - y^6 &= (x^3 + y^3)(x^3 - y^3) \\ &= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2) \end{aligned}$$

□

Let's practice:

1. Factor $125x^3 - 27$

2. Factor $8x^6 + 125y^3$

3. Factor $32x^3 - 4$

4. Factor $x^9 - y^9$

Monic quadratic trinomials

A polynomial of one variable is called *monic* if its higher degree term has coefficient 1. For quadratic trinomials, this means $a = 1$. We've already seen how to factor quadratic trinomials using the "splitting method". However, in the special case of a monic quadratic trinomial, there is a somewhat simpler method. Let's look at what happens when we multiply, two monic linear binomials:

Example 10. Let's expand the product: $(x + 6)(x + 7)$

$$\begin{aligned}(x + 6)(x + 7) &= x^2 + 7x + 6x + 42 \\ &= x^2 + 13x + 42\end{aligned}$$

We see that the linear term of the simplified form of the product is obtained by combining the terms $7x$ and $6x$, and therefore its coefficient is 13. We also see that the constant term of the product is obtained by multiplying the constant terms of the two factors and it is therefore 42.

Of course, there is nothing special about the numbers 6 and 7. Something similar will happen every time we expand the product of two monic linear binomials $(x + b_1)(x + b_2)$. Indeed:

$$\begin{aligned}(x + b_1)(x + b_2) &= x^2 + b_1x + b_2x + b_1b_2 \\ &= x^2 + (b_1 + b_2)x + b_1b_2\end{aligned}$$

Or in words:

- The *coefficient of the linear term* of the simplified expanded form of the product of two monic linear binomials, is the *sum* of the constant terms of the two binomials.
- The *constant term* of the simplified expanded form of the product of two monic linear binomials, is the *product* of the constant terms of the two binomials.

It follows then that to factor a monic quadratic polynomial $p(x) = x^2 + bx + c$ we should look for two numbers b_1 and b_2 such that

- $b_1 + b_2 = b$
- $b_1b_2 = c$

Once we find these two numbers the factorization is:

$$p(x) = (x + b_1)(x + b_2)$$

Notice that we have developed a method for finding *integers* b_1 and b_2 that satisfy this conditions in the section about the *splitting method.*

Example 11. Factor $x^2 + 6x - 55$

Answer. We look for two integers whose product is -55 and whose sum is 6. The numbers are -11 and 5. So the factorization is

$$x^2 + 6x - 55 = (x - 11)(x + 5)$$

□

Review of Factoring

As a summary, when presented with a polynomial to factor we proceed as follows:

1. First we should check whether there is a common factor for all the terms of the polynomial. In case there is we should "take out" the greatest common factor of all the terms of the trinomial.
2. Check whether the polynomial is the LHS of an identity, namely whether it is a difference of squares, a sum of cubes, or a difference of cubes.
3. Check whether factor by grouping is possible.
4. Check whether the polynomial is quadratic.
5. If any of the above steps is successful, repeat this procedure for each of the factors.

Let's see some examples:

Example 12. Factor $12x^2 - 75$

Answer. We have a common factor of 3. After "taking out" the common factor we get a difference of squares:

$$\begin{aligned} 12x^2 - 75 &= 3(4x^2 - 25) \\ &= 3(2x+5)(2x-5) \end{aligned}$$

□

Example 13. Factor $(5x-6)^2 - (3x+2)^2$

Answer. We have a difference of squares:

$$\begin{aligned} (5x-6)^2 - (3x+2)^2 &= [(5x-6)+(3x+2)]\,[(5x-6)-(3x+2)] \\ &= [5x-6+3x+2]\,[5x-6-3x-2] \\ &= (8x-4)(2x-8) \\ &= 8(2x-1)(x-4) \end{aligned}$$

where in the last line we took out a common factor of 4 from the first factor and a common factor of 2 from the second factor. □

Example 14. Factor $x^2 + 16$

Answer. This polynomial is irreducible, since it is a quadratic polynomial without roots.[2] □

Example 15. Factor $a^2x^2 + 3a^2x + 2a^2 - b^2x^2 - 3b^2x - 2b^2$.

[2] Why this polynomial doesn't have any roots?

Answer. We will try factoring by grouping: the first three terms have a common factor of a^2 and the last three terms have a common factor of $-b^2$. So we have:

$$
\begin{aligned}
a^2x^2 + 3a^2x + 2a^2 - b^2x^2 - 3b^2x - 2b^2 &= a^2(x^2 + 3x + 2) - b^2(x^2 + 3x + 2) \\
&= (a^2 - b^2)(x^2 + 3x + 2)
\end{aligned}
$$

Now we examine each of the two factors to see whether they can be farther factored. The first one is a difference of two squares so we have:

$$a^2 - b^2 = (a + b)(a - b)$$

The second factor is a monic quadratic polynomial. So we can factor it as:

$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

In sum we have:

$$a^2x^2 + 3a^2x + 2a^2 - b^2x^2 - 3b^2x - 2b^2 = (a + b)(a - b)(x + 1)(x + 2)$$

□

Sometimes, we may know a factor, for example by guessing, or because it is given to us. In this case we can perform long division to get the other factor and try to factor it. Here is an example:

Example 16. Given that $x + 5$ is a factor of $p(x) = 2x^3 + 11x^2 - x - 30$, factor $p(x)$ completely.

Answer. We first perform long division to find the other factor:

$$
\begin{array}{rr|r}
 & & 2x^2 + x - 6 \\ \cline{3-3}
 & x+5 & 2x^3 + 11x^2 - x - 30 \\
2x^3 + 10x^2 & & -2x^3 - 10x^2 \qquad\quad \\ \cline{3-3}
 & & x^2 - x - 30 \\
x^2 + 5x & & -x^2 - 5x \qquad \\ \cline{3-3}
 & & -6x - 30 \\
-6x - 30 & & 6x + 30 \\ \cline{3-3}
 & & 0
\end{array}
$$

So we have:

$$p(x) = (x + 5)(2x^2 + x - 6)$$

Now we need to factor the trinomial $2x^2 + x - 6$. We use the splitting method (the numbers are 4 and -3):

$$
\begin{aligned}
2x^2 + x - 6 &= 2x^2 + 4x - 3x - 6 \\
&= 2x(x + 2) - 3(x + 2) \\
&= (x + 2)(2x - 3)
\end{aligned}
$$

So, in sum we have:

$$p(x) = (x + 5)(x + 2)(2x - 3)$$

□