Factoring polynomials and solving higher degree equations

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Recall. With respect to division polynomials behave a lot like natural numbers. It is not always possible to divide two polynomials and get a polynomial as a result. The result may sometimes be a polynomial but in general we will get a *rational expression*. The best we can hope in general is to get a *quotient* and a *reminder*. Last time we show that given two polynomials $a(x)$ and $b(x)$ we can perform *long division* find a quotient polynomial $r(x)$ and a remainder polynomial $r(x)$ so that:

$$\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)} \tag{1}$$

where the degree of the remainder $r(x)$ is less than the degree of the denominator $b(x)$.

Sometimes it is more convenient to write Formula (1) as

$$a(x) = q(x)b(x) + r(x) \tag{2}$$

Example 1. Find the quotient and the remainder of the division:

$$\frac{x^3 - 5x^2 - 3x + 4}{x - 2}$$

Answer. We will use the long division algorithm:

$$
\begin{array}{rr}
 & x^2 - 3x - 9 \\
x - 2 & \overline{)\; x^3 - 5x^2 - 3x + 4} \\
x^3 - 2x^2 & -x^3 + 2x^2 \qquad\qquad \\
\hline
 & -3x^2 - 3x + 4 \\
-3x^2 + 6x & 3x^2 - 6x \quad\;\; \\
\hline
 & -9x + 4 \\
-9x + 18 & 9x - 18 \\
\hline
 & -14
\end{array}
$$

Therefore the quotient is $x^2 - 3x - 9$ and the remainder is -14. We can write this as

$$x^3 - 5x^2 - 3x + 4 = (x^2 - 3x - 9)(x - 2) - 14$$

□

Let $p(x) = x^3 - 5x^2 - 3x + 4$. Then the previous example tells us that

$$p(x) = q(x)(x - 2) - 14$$

where $q(x) = x^2 - 3x - 9$ is the quotient of the division. If we now want to evaluate the polynomial $p(x)$ for $x = 2$ we get

$$\begin{aligned} p(2) &= q(2)(2-2) - 14 \\ &= q(2) \cdot 0 - 14 \\ &= -14 \end{aligned}$$

So the result of the evaluation $p(2)$ is the same as the remainder of the division of $p(x)$ by $(x - 2)$. Notice that in the previous calculation it doesn't really matter what $q(2)$ is. No matter what $q(2)$ is it is multiplied by 0 so it is "killed". Therefore, for any polynomial $p(x)$ we have that $p(2)$ is the same as the quotient of the division $p(x) \div (x - 2)$.

Example 2. What is the remainder of the division $(x^3 - 2x^3 + 7x^2 - 5x + 4) \div (x - 1)$.

Answer. Let $p(x) = x^3 - 2x^3 + 7x^2 - 5x + 4$. Assume that the quotient is $q(x)$ and r is the remainder[1]. We will have

$$p(x) = q(x)(x-1) + r$$

Plugging $x = 1$ in the above equation gives:

$$p(1) = r$$

So the remainder is the result of evaluating $p(x)$ at $x = 1$. Thus:

$$\begin{aligned} r &= p(1) \\ &= 1^3 - 2 \cdot 1^3 + 7 \cdot 1^2 - 5 \cdot 1 + 4 \\ &= 1 - 2 + 7 - 5 + 4 \\ &= 5 \end{aligned}$$

□

Example 3. Let $p(x) = x^3 + 8$. Find the remainder of the division:

$$\frac{p(x)}{x+2}$$

Answer. If $q(x)$ is the quotient and r the remainder then we will have:

$$p(x) = q(x)(x+2) + r$$

Plugging $x = -2$ in the equation above gives: $p(-2) = r$. So the remainder is:

$$\begin{aligned} r &= p(-2) \\ &= (-2)^3 + 8 \\ &= -8 + 8 \\ &= 0 \end{aligned}$$

□

[1] Why is the remainder a number and not a more general polynomial?

Now let's practice this ideas:

1. Find the remainder of the division
$$\frac{x^{23}-5}{x-1}$$

2. Find the remainder of the division: $(x^5+x^4+2x^3+2x^2-2x+7)\div(x+1)$

3. Find the remainder of the division: $$\frac{-2x^7+5x^5-21x^4+23x^3-14x^2+x-45}{x}$$

We have then the following:

Fact 1. *For any polynomial $p(x)$ and any real number a, the remainder of the division $p(x)\div(x-a)$ is always equal to $p(a)$.*

When the remainder of the division $p(x)\div b(x)$ is 0, we say that $b(x)$ *divides* $p(x)$ or that $b(x)$ *is a factor* of $p(x)$. We conclude this section by stating the following special case of Fact 1.

Fact 2. *For a real number a and a polynomial $p(x)$, $x-a$ is a factor of $p(x)$ exactly when $p(a)=0$.*

Factoring polynomials

In the previous chapter we show how we can get the simplified expanded form of a polynomial given as a product of two (or more) polynomials. In this chapter we will examine the reverse problem, we will start from the expanded form of the polynomial and we will try to write it as the product of two or more polynomials. We will start with some terminology:

A polynomial is called *reducible* if it has a *non constant* factor. In other words, a polynomial $p(x)$ is reducible if we can write it as a product of two polynomials
$$p(x)=a(x)b(x)$$

and neither $a(x)$ nor $b(x)$ are constants.

Example 4. The following polynomials are reducible:

A. $x^3 - 2x$ B. $x^2 - 4$ C. $x^3 + 8$ D. $x^2 - 2x - 15$ E. $6x^4 + x^3 - 22x^2 - 11x + 6$

Justification. We show how each of these polynomials can be written as a product of other polynomials.

A $x^3 - 2x = x(x^2 - 2)$

B $x^2 - 4 = (x - 2)(x + 2)$

C $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

D $x^2 - 2x - 15 = (x + 3)(x - 5)$

E $6x^4 + x^3 - 22x^2 - 11x + 6 = (x + 1)(2x + 3)(x - 2)(3x - 1)$

□

Later on we will see how we can get these results.

If a polynomial is not reducible it's called *irreducible*. So irreducible polynomials have only constant factors.

Recall from the previous section that if a polynomial $p(x)$ has a factor of the form $x - a$, where a is a real number then $p(a) = 0$. Therefore if no matter what value we substitute for x, $p(x)$ never evaluates to 0 we can conclude that $p(x)$ has no factors of the form $x - a$. We see then that values of x that make the polynomial evaluate to 0 are important so we give them a special name:

A *root* of a polynomial $p(x)$ is a number a such that $p(a) = 0$.

Example 5. The polynomial $p(x) = x^2 + 1$ is irreducible.

Justification. $p(x)$ is quadratic. Therefore if it has a non constant factor, this factor has to be linear[2]. Now every linear polynomial has a root[3]. In sum, if $p(x)$ was reducible then it would have a root. But, $p(x)$ does not have roots, because no matter what real number we substitute for x, x^2 will be greater or equal than 0, and adding one will make $x^2 + 1$ greater than 0. So, $p(x)$ will always evaluate to a positive number.

In sum, $p(x)$ has no roots and therefore it is irreducible. □

The previous example illustrates a very important test for deciding whether a *quadratic* polynomial is irreducible.

A *quadratic* polynomial is irreducible exactly when it has no roots .

Use this test to justify each of the following true statements:

[2] Why?

[3] Why?

1. The polynomial $x^2 + 7$ is irreducible.

2. The polynomial $-2x^2 - 5$ is irreducible.

3. The polynomial $(x-2)^2 + 21$ is irreducible.

4. The polynomial $x^2 - 9$ is reducible.

5. The polynomial $(x-2)^2 + (x+3)^2$ is irreducible.

Identifying common factors

The most straightforward method for factoring is identifying common factors among the terms of a polynomial. Then we can use the distributive property, in the *contracting*[4] direction. Let's start with some examples:

Example 6. Factor the polynomial: $ax + bx$.

Answer. Each of the terms of the polynomial that we have to factor has x as factor. So we can use the distributive property to get:

$$ax + bx = (a + b)x$$

□

Example 7. Factor: $2x^2 - 4x^3$.

Answer. In this case the coefficients have 2 as a common factor and the variable parts of the two terms have x^2 as a common factor. So $2x^2$ is a common factor of the two terms, and we have:

$$2x^2 - 4x^3 = 2x^2(1 - 2x)$$

□

In general to identify a possible common factor among the terms of a polynomial we can follow the following procedure:

1. Check if there is a common factor of *all* the coefficients. If there is such a common factor, it will be the coefficient of the common factor.

2. For each variable of the polynomial, check whether it occurs in *all* the terms. Each variable that occurs in all the terms will occur in the common factor, and its exponent in the common factor will be the smallest of the exponents in all occurrences.

Once the common factor has been identified we can find the *other factor* in the factorization of the polynomial we proceed as follows.

3. We divide every term of the original polynomial by the common factor. The quotient will be a term of the other factor.

Example 8. Factor the polynomial $p(x, y, z, w) = 3x^2y^3z - 6xy^2z^2 + 9x^3y^2w$.

Answer. The coefficient of the common factor will be 3.

x occurs in the first term with exponent 2, in the second term with exponent 1 and in the third term with exponent 3. So the common factor will have an x.

y occurs with exponent 3 in the first term, 2 in the second and 2 in the third. So the common factor will have a y^2.

z does not occur in the third term. So z doesn't occur in the common factor.

w does not occur in the first (or the second) term. So w doesn't occur in the common factor.

In sum the common factor is $3xy^2$.

Now we divide each term of $p(x, y, z, w)$ with the common factor to get the terms of the other factor:

$$3xy^2(xyz - 2z^2 + 3x^2w)$$

□

[4] As opposed to the expanding direction that we are using to get the expanded form of a product.

Let's practice:

1. Factor $6x^4y^3z - 12x^2yz^3 + 21x^3yz^2$.

2. Factor $4xy^2 - x^2y^3 + 8x^3y^4$.

3. Factor $2x^2y^3 - 3x^3z^4 + 5yz^2$.

4. Factor $6yw^3x^3 - 24w^3x^3y^3 - 12w^4x^2$.

5. Factor $-7x^4 - 14x^3 + 21x^2 + 7x$.

The technique of identifying common factors can be used any time that we have a polynomial written as a sum of products, even if the polynomial is not in simplified expanded form.

Example 9. Factor $2x(x+3) - 7x^5(x+3)$.

Answer. Both summands have $x+3$ as a factor. So we can write:

$$2(x+3) - 7x^5(x+3) = (x+3)(2-7x^5)$$

□

Example 10. Factor $(2x-3)xy^2 - 7(2x-3)x^2$.

Answer. Now we have $x(2x-3)$ as a common factor. So:

$$(2x-3)xy^2 - 7(2x-3)x^2 = x(2x-3)(y^2 - 7x^2)$$

□

Often the common factor is "in disguise", and we need to look carefully to be able to identify it. For example, the common factor may appear with opposite signs in different summands:

Example 11. Factor $x^2(3x+1) + 4y(-3x-1)$.

Answer. In this case $(3x+1)$ is a common factor, since $-3x-1 = -(3x+1)$. So:

$$x^2(3x+1) + 4y(-3x-1) = (3x+1)(x^2 - 4y)$$

□

Example 12. Factor $3x(x-2) + 5y(2-x)$.

Answer. $x-2$ is the common factor:

$$3x(x-2) - 5y(2-x) = (x-2)(3x+5y)$$

□

Another "disguise" to watch for, is the lack of parenthesis, sometimes the common factor may appear by itself:

Example 13. Factor $5x^2(3x-7) + 3x - 7$.

Answer. We can think of the last two terms together as forming one summand:

$$5x^2(3x-7) + (3x-7)$$

Once we insert the "missing" parentheses we can see that we have $3x-7$ as a common factor. So:

$$5x^2(3x-7) + 3x - 7 = (3x-7)(5x^2+1)$$

□

Factoring by grouping

Sometimes even though there is no common factor for *all* the terms of a polynomial, we can separate the terms into two or more groups, in such a way that each of the groups has a common factor and *after* we have factored these common factors the other factors are the same. Let's see some examples:

Example 14. Factor $ax + ay + bx + by$.

Answer. There is no common factor to all four terms. However the first two terms have common factor a, and the last two terms have common factor b. After "factoring" out these common factors we have:

$$a(x + y) + b(x + y)$$

The two summands now have a common factor $x + y$. So we can factor it out:

$$(a + b)(x + y)$$

□

Example 15. Factor $ax - ay + bx - by + cy - cx$

Answer. Again there is no factor common to all the terms of this polynomial. However the first two terms have a as a common factor, the middle two terms have b as a common factor, and the last two terms have c as a common factor. After "factoring out" these common factors we get:

$$a(x - y) + b(x - y) + c(y - x)$$

Each of the three summands now has $x - y$ as a common factor. So we finally get:

$$(x - y)(a + b - c)$$

□

Alternatively, we could have noticed that the first, third and sixth term have x as a common factor and the second, fourth and fifth have y as a common factor. Then we get:

$$x(a + b - c) - y(a + b - c)$$

which finally gives

$$(a + b - c)(x - y)$$

Notice that this is the same factoring, really. It has the same two factors but in different order. This is typical, usually there will be more than one ways to group the terms of a polynomial so that each group has a common factor and the after factoring the other terms are the same.

Example 16. Factor $10x^2y^3 + 8wxy - 15wxy^2 - 12w^2$

Answer. We have:

$$\begin{aligned} 10x^2y^3 + 8wxy - 15wxy^2 - 12w^2 &= 5xy^2(2xy - 3w) + 4w(2xy - 3w) \\ &= (2xy - 3w)(5xy^2 + 4w) \end{aligned}$$

□

Let's practice this technique:

1. Factor: $5za^2 + 3xa^2 - 5bz - 3bx$

2. Factor: $10yx^2 - 8x^2 + 15xy - 12x$

3. Factor: $4yx^2z^3 + 10zx^3 - 6y^2z^2 - 15xy$

4. Factor: $2a^2x^2 - 5ya^2 + 5by - 2bx^2 + 2c^3x^2 - 5yc^3$

Factoring trinomials by splitting the linear term

In this section we concentrate on *quadratic polynomials* in one variable. Such a polynomial *must* have a quadratic term, and it may (or may not) have a linear term and a constant term. It is customary to use the letters a, b and c for the coefficients of the quadratic, the linear and the constant term respectively. So the form of the polynomials we will deal with in this section is:

$$p(x) = ax^2 + bx + c, \quad a \neq 0$$

The quadratic term has to be present, hence its coefficient a cannot be 0. The linear term and the constant term however may be missing, that is b and/or c may be 0.

> For the remaining of this section, a, b, and c will stand for the coefficients of the quadratic, the linear, and the constant term respectively.

Example 17. Identify a, b, and c for each of the following polynomials:
A. $2x^2 - x + 6$ B. $-2x^2 - 5$ C. $x^2 + 5x$ D. $7x^2$

Answer.

A. $a = 2$, $b = -1$, $c = 6$

B. $a = -2$, $b = 0$, $c = -5$

C. $a = 1$, $b = 5$, $c = 0$

D. $a = 7$, $b = 0$, $c = 0$

□

The method we will use consists of "splitting" the linear term into the sum of two terms in such a way that grouping one of these new terms with the quadratic term and the other with the constant term "works". Before explaining how to choose this splitting let's see a couple of examples:

Example 18. Factor $x^2 + 5x + 6$

Answer. We will split the linear term in to two in such a way that we can perform factoring by grouping. So instead of $5x$ we will write $3x + 2x$ and write the polynomial as

$$x^2 + 3x + 2x + 6$$

Now we see that the first two terms have a common factor and that the last two terms have a common factor also. So we get,

$$x(x + 3) + 2(x + 3)$$

And after factoring out the common factor, we get:

$$(x + 3)(x + 2)$$

□

Example 19. Factor $6x^2 - 13x - 5$

Answer. We will split the linear term as $-13x = 2x - 15x$. We then have:

$$\begin{aligned} 6x^2 - 13x - 5 &= 6x^2 + 2x - 15x - 5 \\ &= 2x(3x+1) - 5(3x+1) \\ &= (3x+1)(2x-5) \end{aligned}$$

□

How did I know how to split the linear term?

Fact 3. *The splitting described below, works: We split the linear term as the sum of two terms $bx = b_1x + b_2x$, where the coefficients b_1 and b_2 are chosen in such a way that*

$$b_1 b_2 = ac$$

In other words, the correct splitting is given by two numbers b_1 and b_2 with the properties that

- $b_1 + b_2 = b$
- $b_1 b_2 = ac$

Because of the second condition, this method is often reffered to as *the ac method*.

In general the two coefficients b_1 and b_2 could be any two real numbers. However for this class, we will always be looking for two *integers*.

Let's practice this technique:

Example 20. Factor $x^2 + 2x - 15$.

Answer. We look for two integers whose product is -15 and whose sum is 2. To do this we will examine all possible ways that -15 can be written as a product of two integers until we find a pair whose sum is 2. We construct the following table, of all possible pairs:

b_1	b_2	$b_1 + b_2$
-1	15	14
1	-15	-4
-3	5	2
3	-5	-2

So we should split $2x$ as $-3x + 5x$. We have:

$$\begin{aligned} x^2 + 2x - 15 &= x^2 + -3x + 5x - 15 \\ &= x(x-3) + 5(x-3) \\ &= (x-3)(x+5) \end{aligned}$$

□

In general we will always find all possible ways that ac can be written as a product of two integers, and select that pair of integers whose sum is b.[5] Of course, in the previous example we could (and should) have stopped our search once we found the pair that has sum 2. Our search is often simplified by the following observation:

[5] The reason that we start with the product and not with the sum is that there are only finitely many ways to write an ac as a product of two integers while there are infinitely many ways to write b as a sum of two integers.

Observation 1. Depending on the sign of ac:

- if ac is positive then b_1 and b_2 must have the same sign, the sign of b.
- if ac is negative then b_1 and b_2 must have opposite signs, furthermore the largest one must have the same sign as b.

Example 21. Factor: $3x^2 + 17x - 6$

Answer. In this case $ac = -18$ and $b = 17$. Taking into account Observation 1, we will look for two numbers of opposite sign and the larger one should be positive. The first pair we try is 18 and -1. Thus the linear term should be split as $17x = 18x - x$. We have:

$$\begin{aligned} 3x^2 + 17x - 6 &= 3x^2 + 18x - x - 6 \\ &= 3x(x+6) - x - 6 \\ &= (x+6)(3x-1) \end{aligned}$$

□

Example 22. Factor: $6x^2 + 19x + 10$

Answer. We look for two positive integers whose product is 60 and whose sum is 19. We proceed by examining all possible numbers that go evenly into 60:

b_1	b_2	$b_1 + b_2$
1	60	61
2	30	32
3	20	23
4	15	19

So we have:

$$\begin{aligned} 6x^2 + 19x + 10 &= 6x^2 + 15x + 4x + 10 \\ &= 3x(2x+5) + 2(2x+5) \\ &= (2x+5)(3x+2) \end{aligned}$$

□

Example 23. Factor: $x^2 - 9x + 14$

Answer. We look for two negative numbers whose product is 14 and whose sum is -9.

b_1	b_2	$b_1 + b_2$
-1	-14	-15
-2	-7	-9

So we have:

$$
\begin{aligned}
x^2 - 9x + 14 &= x^2 - 7x - 2x + 14 \\
&= x(x-7) - 2(x-7) \\
&= (x-7)(x-2)
\end{aligned}
$$

□

Example 24. Factor: $4x^2 + 12x + 9$

Answer. We look for two positive integers whose product is 36 and whose sum is 12.

b_1	b_2	$b_1 + b_2$
1	36	37
2	18	20
3	12	15
4	9	13
6	6	12

So we have:

$$
\begin{aligned}
4x^2 + 12x + 9 &= 4x^2 + 6x + 6x + 9 \\
&= 2x(2x+3) + 3(2x+3) \\
&= (2x+3)(2x+3) \\
&= (2x+3)^2
\end{aligned}
$$

□

Example 25. Factor: $x^2 - 9$.

Answer. In this case there is no linear term which means that $b = 0$. Our method still works: we need two numbers that have product -9 and sum 0. Since the sum is 0 the two numbers must be opposite. The numbers are therefore -3 and 3. So we have to split the linear term as $0x = 3x - 3x$:

$$
\begin{aligned}
x^2 - 9 &= x^2 + 3x - 3x - 9 \\
&= x(x+3) - 3(x+3) \\
&= (x+3)(x-3)
\end{aligned}
$$

□

Notice that with this method we can identify factors that have integers coefficients, but it is hard to find more general factorizations. For example:

Example 26. Consider the polynomial:

$$x^2 + 4x + 1$$

We can not find two integers whose product is 1 and whose sum is 4. However, the polynomial is not irreducible, because

$$x^2 + 4x + 1 = (x + 2 + \sqrt{3})(x + 2 - \sqrt{3})$$

Notice that if we take $b_1 = 2 + \sqrt{3}$ and $b_2 = 2 - \sqrt{3}$ then $b_1 b_2 = 1$ and $b_1 + b_2 = 4$ so the splitting method works *in principle*. In practice however it is rather hard to come up with these b_1 and b_2. So we are still in need of a method that would allow us to detect such factors.

Now let's practice this technique:

1. Factor each of the following polynomials using the "splitting method".
   (a) $-8x^2 + 22x - 5$

   (b) $x^2 - 8x + 12$

   (c) $x^2 - 8x - 33$

   (d) $11x^2 + 32x - 3$

   (e) $24x^2 - 2x - 15$