Application of Iterated Integrals to Number Theory and Algebraic Geometry

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Preface

Introduction

Iterated integrals are used in topology to study the fundamental group of a manifold. They are also used to express Euler's multiple zeta values as (iterated) integrals. Manin has used such technique to construct non-commutative modular symbol.

The heart of the book studies applications of a higher dimensional analogue which we call iterated integrals over membrane. The applications are:

1. Non-commutative reciprocity laws on curves and surfaces.
2. Number theoretic analogue of multiple zeta values, which we call multiple Dedekind zeta values
3. Non-commutative Hilbert modular symbols.

We also give a detailed overview of iterated integrals.

Part I

The First Part

Chapter 1

Preliminary material on iterated path integrals

1.1 Iterated path integrals on complex curves

For proofs of theorems of this section, see Chen[Ch] or Goncharov[G].

Definition 1 *Let $\omega_1, \omega_2, \cdots, \omega_n$ be holomorphic 1-forms on a simply connected open subset U of the complex plane $\mathbb{C}$. Let $\gamma : [0,1] \to U$ be a path. Then we call the integral*

$$\int_\gamma \omega_1 \circ \cdots \circ \omega_n := \int \cdots \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \cdots \wedge \gamma^* \omega_n(t_n)$$

the iterated integral of the differential forms $\omega_1, \omega_2, \cdots, \omega_n$ over the path γ.

Theorem 2 *Let $\omega_1, \cdots, \omega_n$ be holomorphic 1-forms on a simply connected open subset U of the complex plane $\mathbb{C}$. Let $H : [0,1] \times [0,1] \to U$ be a homotopy, fixing the end points, of paths $\gamma_s : [0,1] \to U$ such that $\gamma_s(t) = H(s,t)$. Then*

$$\int_{\gamma_s} \omega_1 \circ \cdots \circ \omega_n$$

is independent of s.

Theorem 3 *[Shuffle relation] Let $\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{n+n}$ be differential 1-forms, where some of them could repeat. Let also γ be a path that does not pass through any of the poles of the given differential forms. Denote by $Sh(m,n)$ the shuffles, which are permutations τ of the set $\{1, ..., m, m+1, ..., m+n\}$ such that $\tau(1) < \tau(2) < \cdots < \tau(m)$ and $\tau(m+1) < \tau(m+2) < \cdots < \tau(m+n)$. Then*

$$\int_\gamma \omega_1 \circ \cdots \circ \omega_n \int_\gamma \omega_{n+1} \circ \cdots \circ \omega_{m+n} = \sum_{\tau \in Sh(m,n)} \int_\gamma \omega_{\tau(1)} \circ \omega_{\tau(2)} \cdots \circ \omega_{\tau(m+n)}.$$

Lemma 4 (Reversing the path)

$$\int_\gamma \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n = (-1)^n \int_{\gamma^{-1}} \omega_n \circ \omega_{n-1} \circ \cdots \circ \omega_1$$

Theorem 5 *[Composition of paths] Let $\omega_1, \omega_2, \cdots, \omega_n$ be differential forms, where some of them could repeat. Let γ_1 be a path that ends at Q and γ_2 be a path that starts at Q. Then*

$$\int_{\gamma_1\gamma_2} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_i \int_{\gamma_2} \omega_i \circ \omega_{i+1} \circ \cdots \circ \omega_n$$

This Section contains a definition and properties of iterated integrals, which will be used for the definition of bi-local symbols and for another proof of the Weil reciprocity law in Subsection 1.2.

Let C be a smooth complex curve. Let f_1 and f_2 be two non-zero rational functions on C. Let

$$\gamma : [0,1] \to C$$

be a path, which is a continuous, piecewise differentiable function on the unit interval.

Definition 6 *We define the following iterated integral*

$$\int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{0<t_1<t_2<1} \gamma^* \left(\frac{df_1}{f_1}\right)(t_1) \wedge \gamma^* \left(\frac{df_2}{f_2}\right)(t_2).$$

The two Lemmas below are due to K.-T. Chen [**?**].

Lemma 7 *An iterated integral over a path γ on a smooth curve C is homotopy invariant with respect to a homotopy, fixing the end points of the path γ.*

Lemma 8 *If $\gamma = \gamma_1\gamma_2$ is a composition of two paths, where the end of the first path γ_1 is the beginning of the second path γ_2, then*

$$\int_{\gamma_1\gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_{\gamma_1} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\gamma_1} \frac{df_1}{f_1} \int_{\gamma_2} \frac{df_2}{f_2} + \int_{\gamma_2} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.$$

Let σ be a simple loop around a point P on C with a base point Q. Let $\sigma = \gamma\sigma_0\gamma^{-1}$, where σ_0 is a small loop around P, with a base the point R and let γ be a path joining the points Q with R.

The following Lemma is essential for the proof of the Weil reciprocity (see also [**?**]).

Lemma 9 *With the above notation, the following holds*

$$\int_\sigma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_\gamma \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2}.$$

Proof. First, we use Lemma 8 for the composition $\gamma\sigma_0\gamma^{-1}$. We obtain

$$\begin{aligned} \textstyle\int_\sigma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} &= \textstyle\int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_\gamma \frac{df_1}{f_1} \int_{\sigma_0} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \\ &\quad \textstyle+ \int_\gamma \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\sigma_0} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \end{aligned} \tag{1.1}$$

Then, we use the homotopy invariance of iterated integrals, Lemma 7, for the path $\gamma\gamma^{-1}$. Thus,

$$0 = \int_{\gamma\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}.$$

Finally, we use Lemma 8 for the composition of paths $\gamma\gamma^{-1}$. That gives

$$0 = \int_{\gamma\gamma^{-1}} \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = \int_\gamma \frac{df_1}{f_1} \circ \frac{df_2}{f_2} + \int_\gamma \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} + \int_{\gamma^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2}. \tag{1.2}$$

The Lemma 9 follows from Equations (1.1) and (1.2). ■

1.2 Chen's construction for differential forms on loop spaces

Chapter 2

Reciprocity laws on curves

2.1 Weil reciprocity via iterated path integrals

Here, we present a proof of the Weil reciprocity law, based on iterated integrals and bi-local symbols. This method will be generalized in the later Subsections in order to prove reciprocity laws on complex surfaces. Similar ideas about the Weil reciprocity law are contained in [**?**], however, without bi-local symbols.

Let x be a rational function on a curve C, representing an uniformizer at P. Let

$$a_i = ord_P(f_i).$$

and let

$$g_i = x^{-a_i} f_i.$$

Then

$$\frac{df_i}{f_i} = a_i \frac{dx}{x} + \frac{dg_i}{g_i}.$$

Let σ_0^ϵ be a small loop around the point P, whose points are at most at distance ϵ from the point P. One can take the metric inherited from the Fubini-Study metric on $\mathbb{P}^k$. Put $\sigma_0^\epsilon = \sigma_0$ in Lemma 9, then

$$\int_\gamma \frac{df_1}{f_1} \int_{\sigma_0^\epsilon} \frac{df_2}{f_2} = 2\pi i a_2 \int_\gamma \frac{df_1}{f_1} = 2\pi i a_2 \left(a_1 \int_\gamma \frac{dx}{x} + \int_\gamma \frac{dg_1}{g_1} \right).$$

Similarly,

$$\int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \int_{\gamma^{-1}} \frac{df_2}{f_2} = 2\pi i a_1 \left(-a_2 \int_\gamma \frac{dx}{x} - \int_\gamma \frac{dg_2}{g_2} \right).$$

From [**?**], we have that

$$\lim_{\epsilon \to 0} \int_{\sigma_0^\epsilon} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \frac{(2\pi i)^2}{2} a_1 a_2. \tag{2.1}$$

Using Lemma 9, we obtain

$$\int_\sigma \frac{df_1}{f_2} \circ \frac{df_2}{f_2} = 2\pi i \left(a_2 \log(g_1) - a_1 \log(g_2) + \pi i a_1 a_2\right)|_Q^P.$$

After exponentiation, we obtain

Lemma 10 *With the above notation the following holds*

$$\exp\left(\frac{1}{2\pi i}\int_\sigma \frac{df_1}{f_2} \circ \frac{df_2}{f_2}\right) = (-1)^{a_1 a_2}\frac{g_1^{a_2}}{g_2^{a_1}}(P)\left(\frac{g_1^{a_2}}{g_2^{a_1}}(Q)\right)^{-1} = (-1)^{a_1 a_2}\frac{f_1^{a_2}}{f_2^{a_1}}(P)\left(\frac{f_1^{a_2}}{f_2^{a_1}}(Q)\right)^{-1}$$

Definition 11 *(Bi-local symbol on a curve) With the above notation, we define a bi-local symbol*

$$\{f_1, f_2\}_P^Q = (-1)^{a_1 a_2}\frac{f_1^{a_2}}{f_2^{a_1}}(P)\left(\frac{f_1^{a_2}}{f_2^{a_1}}(Q)\right)^{-1}. \tag{2.2}$$

Let the curve C be of genus g and let $P_1, \ldots, P_n$ be the points of the union of the support of the divisors of f_1 and f_2. Let $\sigma_1, \ldots, \sigma_n$ be simple loops around the points $P_1, \ldots, P_n$, respectively. Let also $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the $2g$ loops on the curve C such that

$$\pi_1(C, Q) =< \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n > / \sim,$$

where $\delta \sim 1$, for

$$\delta = \prod_{i=1}^n \sigma_i \prod_{j=1}^g [\alpha_j, \beta_j].$$

From Theorem 3.1 in [**?**], we have

Lemma 12

$$\int_{\alpha\beta\alpha^{-1}\beta^{-1}} \frac{df_1}{f_1} \circ \frac{df_2}{f_2} = \int_\alpha \frac{df_1}{f_1}\int_\beta \frac{df_2}{f_2} - \int_\alpha \frac{df_2}{f_2}\int_\beta \frac{df_1}{f_1}.$$

Using the above result, we obtain that

$$0 = \int_\delta \frac{df_1}{f_1} \circ \frac{df_2}{f_2} \in (2\pi i)^2\mathbb{Z} + \sum_{i=1}^n \int_{\sigma_i} \frac{df_1}{f_1} \cdot \frac{df_2}{f_2},$$

where the sum is over simple loops σ_i around each of the points P_i. Then we obtain:

Theorem 13 *(Reciprocity law for the bi-local symbol* (2.2)*) With the above notation, the following holds*

$$\prod_P \{f_1, f_2\}_P^Q = 1.$$

If we want to make the above reciprocity law into a reciprocity law for a local symbol we have to remove the dependency on the base point Q. This can be achieved in the following way: In the reciprocity law for the bi-local symbol, the dependency on Q is

$$\prod_P f_1(Q)^{a_2} f_2(Q)^{-a_1} = f_1(Q)^{(2\pi i)^{-1}\sum_P Res_P \frac{df_2}{f_2}} =$$

$$f_2(Q)^{-(2\pi i)^{-1}\sum_P Res_P \frac{df_1}{f_1}} = f_1(Q)^0 f_2(Q)^0 = 1.$$

Thus, we recover Weil reciprocity:

Theorem 14 *(Weil reciprocity) The local symbol*

$$\{f_1, f_2\}_P = (-1)^{a_1 a_2} \frac{f_1^{a_2}}{f_2^{a_1}}(P).$$

satisfies the following reciprocity law

$$\prod_P \{f_1, f_2\}_P = 1,$$

where the product is over all points P in C.

2.2 Contou-Carrère symbol and reciprocity law

2.3 Non-commutative reciprocity laws on complex curves for differential forms with logarithmic poles

2.4 Non-commutative reciprocity laws on complex curves for any differential forms

Chapter 3

Preliminary material on iterated integrals over membranes

3.1 Two foliations on a surface

The goal of this Subsection is to construct two foliations on a complex projective algebraic surface X in $\mathbb{P}^k$. This is an algebraic-geometric material, needed for the definition of iterated integrals on membranes, presented in Subsection 1.4.

Let f_1, f_2, f_3 and f_4 be four non-zero rational functions on the surface X. Let

$$C \cup C_1 \cup \cdots \cup C_n = \bigcup_{i=1}^{4} |div(f_i)|,$$

where we fix one of the irreducible components C. Let

$$\{P_1, \ldots, P_N\} = C \cap (C_1 \cup \cdots \cup C_n).$$

We can assume that the curves $C, C_1, \ldots, C_n$ are smooth and that the intersections are transversal (normal crossings) and no three of them intersect at a point, by allowing blow-ups on the surface X.

The two foliations have to satisfy the following

Conditions:

1. There exists a foliation F'_v such that

   (a) $F'_v = (f - v)_0$ are the level sets of a rational function

   $$f : X \to \mathbb{P}^1,$$

   for small values of v, (that is, for $|v| < \epsilon$ for a chosen ϵ);

(b) F'_v is smooth for all but finitely many values of v;

(c) F'_v has only nodal singularities;

(d) $ord_C(f) = 1$;

(e) $R_i \notin C_j$, for $i = 1, \ldots, M$ and $j = 1, \ldots, n$, where

$$\{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m)$$

and

$$F'_0 = (f)_0 = C \cup D_1 \cup \cdots \cup D_m.$$

2. There exists a foliation G_w such that

(a) $G_w = (g - w)_0$ are the level sets of a rational function

$$g : X \to \mathbb{P}^1;$$

(b) G_w is smooth for all but finitely many values of w;

(c) G_w has only nodal singularities;

(d) $g|_C$ is non constant.

3. Coherence between the two foliations F' and G:

(a) All but finitely many leaves of the foliation G are transversal to the curve C.

(b) $G_{g(P_i)}$ intersects the curve C transversally, for $i = 1, \ldots, N$. (For definition of the points P_i see the beginning of this Subsection.)

(c) $G_{g(R_i)}$ intersects the curve C transversally, for $i = 1, \ldots, M$. (For definition of the points R_i see condition 1(e).)

The existence of $f \in \mathbb{C}(X)^\times$ satisfying properties 1(a-d) is a direct consequence from the following result, which follows immediately from (a special case of) a result of Thomas ([**?**], Theorem 4.2).

Theorem 15 *Consider a smooth curve C in a smooth projective surface X, with hyperplane section H_X. There exists a large constant $N \in \mathbb{N}$ and a pencil in $|NH_X|$, given as the level sets $(f - x)_0$ of some rational function f such that $(f - x)_0$ is smooth for all but finitely many values of x, at which it has only nodal singularities, and $C \subset (f)_0$.*

Moreover, a general choice of $g \in \mathbb{C}(X)^\times$ will satisfy 2(a-d) and 3(a-c). (For instance, the quotient of two generic linear forms on $\mathbb{P}^k$ restricted to C will not have branch points in $\{P_i\} \cup \{R_j\}$.)

It remains to examine property 1(e). The proof of Theorem 4.2 in [op. cit.] contains the basic

Observation: *The base locus of the linear system $H^0(I_C(N))$ is the smooth curve C for$N >> 0$. So by Bertini's theorem the general element of the linear* *system is smooth away from C.*

Consider $C \subset X$. By the Observation, there exists $\mathcal{F} \in H^0(X, \mathcal{O}(N))$ such that $ord_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$, where D is a second smooth curve on X, meeting C transversally (if at all).

Claim: *We may choose $\mathcal{F}$ so that condition 1(e) holds, that is, $R_i \notin C_j$ for each i, j, where $\{R_1, \ldots, R_M\} = C \cap D$. Equivalently, $C \cap D \cap C_j = \emptyset$.*

Proof. Define $H^0(I_C(N))^{reg}$ to be the subset of $H^0(X, I_C(N))$ whose elements $\mathcal{F}$ satisfy $ord_C(\mathcal{F}) = 1$ and $(\mathcal{F}) = C + D$ as above. Assume that for every $N >> 0$ and $\mathcal{F} \in H^0(I_C(N))^{reg}$ we have $D \cap C \cap C_j \neq \emptyset$ for some particular j. If we obtain a contradiction (for some N) then the claim is proved, since this is a closed condition for each j.

According to our assumption, $(\mathcal{F})$ always has an ordinary double point at the intersection $\Delta := C \cap C_j \neq \emptyset$. In the exact sequence

$$0 \to H^0(X, I_C^2(N)) \to H^0(X, I_C(N)) \to H^0(C, \mathcal{N}^*_{C/X}(N)) \to H^1(X, I_C^2(N)),$$

the last term vanishes by ([**?**], Vanishing Theorem B) for N sufficiently large. Hence, every section over C of the twisted conormal sheaf $\mathcal{N}^*_{C/X}(N)$ has a zero along $\Delta = C \cap C_j$.

Next consider the exact sequence

$$0 \to H^0(C, I_\Delta \otimes \mathcal{N}^*_{C/X}(N)) \to H^0(C, \mathcal{N}^*_{C/X}(N)) \to \mathbb{C}^{|\Delta|} \to H^1(C, I_\Delta \otimes \mathcal{N}^*_{C/X}(N)).$$

The last term vanishes again by [loc. cit.]. Denote the third arrow by ev_Δ. Then we can take a section of $\mathcal{N}^*_{C/X}(N)$ not vanishing on Δ simply by taking an element in the preimage of $ev_\Delta(1, \ldots, 1)$. This produces the desired contradiction. ■

Consider a metric on the surface X, which respects the complex structure. For example, we can take the metric inherited from the Fubini-Study metric on $\mathbb{P}^k$ via the embedding $X \to \mathbb{P}^k$. Let $U_1^\epsilon, \ldots, U_M^\epsilon$ be disks of radii ϵ on C, centered respectively at $R_1, \ldots, R_M$. Let

$$C_0 = C - \bigcup_{j=1}^{M} U_j^\epsilon - \{P_1, \ldots, P_N\}.$$

Definition 16 *With the above notation, let F_v be the connected component of*

$$F'_v - \left(\bigcup_{i=1}^{M} G_{g(U_i^\epsilon)} \right) \cap F'_v,$$

containing C_0, for $|v| << \epsilon$, where

$$G_{g(U_i^\epsilon)} = \bigcup_{w \in U_i^\epsilon} G_{g(w)}$$

Lemma 17 *With the above notation, for small values of $|v|$, we have that each leaf F_v is a continuous deformation of $F_0 = C_0$, preserving homotopy type.*

Proof. From Property 3(c), it follows that C and D_i meet at R_j (if at all) at a non-zero angle. At the intersection R_i, locally we can represent the curves by $xy = 0$. The deformation leads to $v - xy = 0$, which is a leaf of F', locally near R_i. Consider a disk U of radius ϵ_i at $(x, y) = (0, 0)$ in the xy-plane. Then for $|v| << \epsilon_i$ we have that U separates F'_v into 2 components, one close to the x-axis and the other close to the y-axis. We do the same for each of the points $R_1, \ldots, R_M$ and we take the minimum of the bounds ϵ_i. Then F_v will consist of points close to the curve C_0.

3.2 Iterated integrals on a membrane. Definitions and properties

In this Subsection, we define types of iterated integrals over membranes, needed in most of this manuscript.

Let τ be a simple loop around C_0 in $X - C_0 - \left(\bigcup_{i=1}^{M} G_{g(U_i^\epsilon)}\right)$. Let σ be a loop on the curve C_0. We define a *membrane* m_σ associated to a loop σ in C^0 by

$$m_\sigma : [0, 1]^2 \to X$$

and

$$m_\sigma(s, t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}.$$

Note that for fixed values of s and t, we have that

$$F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$

consists of finitely many points, where F and G are foliations satisfying the Conditions in Subsection 3.1 and Lemma 17.

Claim: The image of m_σ is a torus.

Indeed, consider a tubular neighborhood around a loop σ on the curve C_0. One can take the following tubular neighborhood:

$$\bigcup_{|v|<\epsilon} F_v \cap G_{g(\sigma)}$$

of $F_v \cap G_{g(\sigma)}$. Its boundary is $F_{f(\tau)} \cap G_{g(\sigma)}$, where τ is a simple loop around C_0 on $X - \bigcup_{i=1}^{n} C_i - \bigcup_{j=1}^{m} D_j$ and $|f(\tau(t))| = \epsilon$.

We shall define the simplest type of iterated integrals over membranes. Also, we are going to construct local symbols in terms of iterated integrals I_1, I_2, I_3, I_4 on membranes, defined below.

We define the following differential forms

$$A(s, t) = m^* \left(\frac{\mathrm{d}f_1}{f_1} \wedge \frac{\mathrm{d}f_2}{f_2} \right) (s, t)$$

$$b(s,t) = m^*\left(\frac{\mathrm{d}f_3}{f_3}\right)(s,t)$$

and

$$B(s,t) = m^*\left(\frac{\mathrm{d}f_3}{f_3} \wedge \frac{\mathrm{d}f_4}{f_4}\right)(s,t).$$

The first diagram

$$\begin{array}{cc} t & \boxed{A} \\ & s \end{array}$$

denotes

$$I_1 = \int_0^1 \int_0^1 A(s,t).$$

The second diagram

$$\begin{array}{c|c|} \cline{2-2} t_2 & b \\ \cline{2-2} t_1 & A \\ \cline{2-2} \multicolumn{1}{c}{} & \multicolumn{1}{c}{s} \end{array}$$

denotes

$$I_2 = \int_0^1 \int\int_{0<t_1<t_2<1} A(s,t_1) \wedge b(s,t_2).$$

Note that the iteration happens with respect to t_1 and t_2. In other word,

$$\int\int_{0<t_1<t_2<1} A(s,t_1) \wedge b(s,t_2)$$

is a differential 1-form on the loop space of X(see [?]). These differential 1-forms on the loop space of X are closed, since $dA = db = 0$ and $A(s,t) \wedge b(s,t) = 0$.

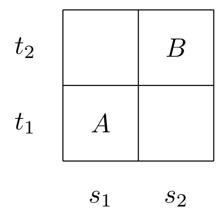
The third diagram

$$\begin{array}{c|c|c|} \cline{2-3} t & A & b \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{s_1} & \multicolumn{1}{c}{s_2} \end{array}$$

denotes

$$I_3 = \int\int_{0<s_1<s_2<1} \int_0^1 A(s_1,t) \wedge b(s_2,t).$$

And the fourth diagram



denotes

$$I_4 = \int\int_{0<s_1<s_2<1}\int\int_{0<t_1<t_2<1} A(s_1,t_1)\wedge B(s_2,t_2).$$

The integral I_4 is a homotopy invariant function with variable the torus of integration m. The proof of the homotopy invariance for iterated integrals over membranes, such at I_4, can be found in [**?**] and in more general form in [**?**].

Local symbols will be defined via the above four types of iterated integrals. The integrals that we define below, used for defining bi-local symbols, are a technical tool for proving reciprocity laws for the local symbols. Bi-local symbols also satisfy reciprocity laws.

Consider the dependence of $\log(f_i(m(s,t)))$ on the variables s and t via the parametrization of the membrane m.

Definition 18 *Let*

$$l_i(s,t) = \log(f_i(m(s,t)))$$

We have

$$\mathrm{d}l_i(s,t) = \frac{\partial l_i(s,t)}{\partial s}\mathrm{d}s + \frac{\partial l_i(s,t)}{\partial t}\mathrm{d}t.$$

$$b(s,t) = \mathrm{d}l_3(s,t)$$

$$A(s,t) = \frac{\partial l_1(s,t)}{\partial s}\frac{\partial l_2(s,t)}{\partial t}\mathrm{d}s\wedge\mathrm{d}t - \frac{\partial l_1(s,t)}{\partial t}\frac{\partial l_2(s,t)}{\partial s}\mathrm{d}s\wedge\mathrm{d}t \qquad (3.1)$$

$$B(s,t) = \frac{\partial l_3(s,t)}{\partial s}\frac{\partial l_4(s,t)}{\partial t}\mathrm{d}s\wedge\mathrm{d}t - \frac{\partial l_3(s,t)}{\partial t}\frac{\partial l_4(s,t)}{\partial s}\mathrm{d}s\wedge\mathrm{d}t \qquad (3.2)$$

The above equations express the differential forms A, B and b is terms of monomials in terms of first derivatives of l_1, l_2, l_3, l_4. We are going to define bi-local symbols associated to monomials in first derivatives of l_1, l_2, l_3, l_4, which occur in

$$A(s,t),\; A(s,t_1)\wedge b(s,t_2),\; A(s_1,t)\wedge b(s_2,t),\text{ and } A(s_1,t_2)\wedge B(s_2,t_2)$$

Definition 19 *(Iterated integrals on membranes) Let $f_1,\ldots,f_{k+l}$ be rational functions on X, where the pairs (k,l) will be superscripts of the integrals. Let m be a membrane as above. We define:*

(a) $I^{(1,1)}(m; f_1, f_2) =$

$$= \int_0^1\int_0^1 \left(\frac{\partial l_1(s,t)}{\partial s}\mathrm{d}s\right)\wedge\left(\frac{\partial l_2(s,t)}{\partial t}\mathrm{d}t\right)$$

(b) $I^{(1,2)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\leq s\leq 1;\ 0\leq t_1\leq t_2\leq 1} \left(\frac{\partial l_1(s,t_1)}{\partial s}\frac{\partial l_2(s,t_1)}{\partial t_1}\mathrm{d}s\wedge\mathrm{d}t_1\right)\wedge\left(\frac{\partial l_3(s,t_2)}{\partial t_2}\mathrm{d}t_2\right)$$

(c) $I^{(2,1)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\leq s_1\leq s_2\leq 1;\ 0\leq t\leq 1} \left(\frac{\partial l_1(s_1,t)}{\partial s_1}\frac{\partial l_2(s_1,t)}{\partial t}\mathrm{d}s_1\wedge\mathrm{d}t\right)\wedge\left(\frac{\partial l_3(s_2,t)}{\partial s_2}\mathrm{d}s_2\right)$$

(d) $I^{(2,2)}(m; f_1, f_2, f_3, f_4) =$

$$= \int\int\int\int_{0\leq s_1\leq s_2\leq 1;\ 0\leq t_1\leq t_2\leq 1} \left(\frac{\partial l_1(s_1,t_1)}{\partial s_1}\frac{\partial l_2(s_1,t_1)}{\partial t_1}\mathrm{d}s_1\wedge\mathrm{d}t_1\right)\wedge$$
$$\wedge\left(\frac{\partial l_3(s_2,t_2)}{\partial s_2}\frac{\partial l_4(s_2,t_2)}{\partial t_2}\mathrm{d}s_2\wedge\mathrm{d}t_2\right)$$

Proposition 20 *(a)* $I_1 = I^{(1,1)}(m; f_1, f_2) - I^{(1,1)}(m; f_2, f_1)$;
(b) $I_2 = I^{(1,2)}(m; f_1, f_2, f_3) - I^{(1,2)}(m; f_2, f_1, f_3)$;
(c) $I_3 = I^{(2,1)}(m; f_1, f_2, f_3) - I^{(2,1)}(m; f_2, f_1, f_3)$;
(d) $I_4 = I^{(2,2)}(m; f_1, f_2, f_3, f_4) - I^{(2,2)}(m; f_2, f_1, f_3, f_4) - I^{(2,2)}(m; f_1, f_2, f_4, f_3) + I^{(2,2)}(m; f_2, f_1, f_4, f_3)$;

Consider a metric on the projective surface X inherited from the Fubini-Study metric on $\mathbb{P}^k$. Let τ be a simple loop around the curve C of distance at most ϵ from C. We are going to take the limit as $\epsilon \to 0$. Informally, the radius of the loop τ goes to zero. Then we have the following lemma.

Lemma 21 *With the above notation the following holds:*
(a)

$$\lim_{\epsilon\to 0} I^{(1,1)}(m_\sigma, f_1, f_2) = (2\pi i) Res\frac{df_2}{f_2}\int_\sigma \frac{df_1}{f_1}$$

(b)

$$\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3) = \frac{(2\pi i)^2}{2} Res\frac{df_2}{f_2} Res\frac{df_3}{f_3}\int_\sigma \frac{df_1}{f_1}$$

(c)

$$\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3) = -(2\pi i) Res\frac{df_2}{f_2}\int_\sigma \frac{df_1}{f_1}\circ\frac{df_3}{f_3}$$

(d)

$$\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4) = -\frac{(2\pi i)^2}{2} Res\frac{df_2}{f_2} Res\frac{df_4}{f_4}\int_\sigma \frac{df_1}{f_1}\circ\frac{df_3}{f_3}$$

Proof. First, we consider the integrals in parts (a) and (c), where there is integration with respect to the variable t in the definition of the membrane m. Let $m(s,\cdot)$ denote the loop obtained by fixing the first variable s and varying the second variable t.Then, there is no iteration along the loop $m(s,\cdot)$ around the curve C, for fixed value of s. Using Properties 1(d) and e(b), the integration over the loop $m(s,\cdot)$ gives us a single residue. This process is independent of the base point of the loop $m(s,\cdot)$. That proves parts (a) and (c).

For parts (b) and (d), we have a double iteration along the loop $m(s,\cdot)$ around the curve C, where the value of s is fixed and the second argument varies. After taking the limit as ϵ goes to 0, the integral along $m(s,\cdot)$, with respect to t_1 and t_2, becomes a product of two residues (see Equation (2.1)), which are independent of a base point. That proves parts (b) and (d).

Let τ be a simple loop around C in $X-D$, based at R. Let σ be a loop on the curve $C^0 = C_0 - (D_1 \cup \cdots \cup D_m) \cap C_0$. We define a *membrane* m_σ associated to a loop σ in C^0 by

$$m_\sigma : [0,1]^2 \to X,$$
$$m_\sigma(s,t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$
$$m_\sigma(0,0) = R.$$

Note that for fixed values of s and t, we have that

$$F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$

consists of finitely many points, where F and G are foliations satisfying the Conditions in Subsection 2.1 and Lemma 2.2.

Consider the dependence of $\log(f_i(m(s,t))$ on the variables s and t via the parametrization of the membrane m. We have

$$d\log(f_i(m(s,t)) = \frac{\partial \log(f_i(m(s,t)))}{\partial s} ds + \frac{\partial \log(f_i(m(s,t)))}{\partial t} dt.$$

In order to use a more compact notation, we will use

$$\log(f_i),_s(s,t) = \frac{\partial \log(f_i(m(s,t)))}{\partial s}$$

and similarly

$$\log(f_i),_t(s,t) = \frac{\partial \log(f_i(m(s,t)))}{\partial t}.$$

Definition 22 *(Interior Iterated integrals on membranes) Let $f_1,\ldots,f_{k+l}$ be rational functions on X, where the integers (k,l) will be superscripts. Let m be a membrane as above. We define:*

(a) $I^{(1,1)}(m; f_1, f_2) =$

$$\int_0^1 \int_0^1 \log(f_1),_s(s,t)ds \wedge \log(f_1),_t(s,t)dt;$$

(b) $I^{(1,2)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\leq s_1\leq s_2\leq 1; 0\leq t\leq 1} \log(f_1),_{s_1}(s_1,t)ds_1 \wedge \log(f_2),_t(s_1,t)dt \wedge$$
$$\wedge \log(f_3),_{s_2}(s_2,t)ds_2;$$

(c) $I^{(2,1)}(m; f_1, f_2, f_3) =$

$$= \int\int\int_{0\leq s\leq 1; 0\leq t_1\leq t_2\leq 1} \log(f_1),_s(s,t_1)ds \wedge \log(f_2),_{t_1}(s,t_1)dt_1 \wedge$$
$$\wedge \log(f_3),_{t_2}(s,t_2)dt_2;$$

Define any smooth metric on X. Let τ be a simple loop around the curve C of distance at most ϵ from C. We are going to take the limit as $\epsilon \to 0$. Informally, the radius of the loop τ goes to zero. Then we have the following lemma.

Using Chen [**?**] we obtain the following Lemma.

Lemma 23 *Let α and β be two loops on the surface X with a common base. Put*

$$\theta_1 = \frac{df_1}{f_1}$$

and

$$\theta_2 = \frac{df_2}{f_2} \wedge \frac{df_3}{f_3}.$$

Put $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Then

$$\int_{[\alpha,\beta]} \theta_1 \cdot \theta_2 = \int_\alpha \theta_1 \int_\beta \theta_2 - \int_\beta \theta_1 \int_\alpha \theta_2.$$

Proof. It follows directly from Lemma 5 by applying it to each ingredient of the commutator.

As a direct consequence, we obtain the following:

Corollary 24

$$\int_{m_{[\alpha,\beta]}} \frac{df_1}{f_1} \cdot \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) \in (2\pi i)^3\mathbb{Z}.$$

Following Chen, we obtain the 1-form

$$\int \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right)$$

on the loop space is closed since

(1) $\frac{df_1}{f_1}$ and $\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}$ are closed and

(2) $\frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = 0$.

Since, we have a closed form it follows that the integral is homotopy invariant. Thus, we can take a relation in the fundamental group of a curve embedded in the surface. More precisely, we take

$$\delta = [\alpha_1, \beta_1] \dots [\alpha_g, \beta_g]\sigma_1 \dots \sigma_n$$

for a curve C_0 of genus g with n punctures.

For each of the above loops we associate a torus.

Let τ be a simple loop around C_0 in $X - C_0 - \left(\bigcup_{i=1}^{M} G_{g(U_i^\epsilon)}\right)$. Let σ be a loop on the curve C_0. We define a *membrane* m_σ associated to a loop σ in C^0 by

$$m_\sigma : [0,1]^2 \to X$$

and

$$m_\sigma(s,t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}.$$

Note that for fixed values of s and t, we have that

$$F_{f(\tau(t))} \cap G_{g(\sigma(s))}$$

consists of finitely many points, where F and G are foliations satisfying the Conditions in Subsection 3.1.

Claim: The image of m_σ is a torus.

Indeed, consider a tubular neighborhood around a loop σ on the curve C_0. One can take the following tubular neighborhood:

$$\bigcup_{|v|<\epsilon} F_v \cap G_{g(\sigma)}$$

of $F_v \cap G_{g(\sigma)}$. Its boundary is $F_{f(\tau)} \cap G_{g(\sigma)}$, where τ is a simple loop around C_0 on $X - \bigcup_{i=1}^{n} C_i - \bigcup_{j=1}^{m} D_j$ and $|f(\tau(t))| = \epsilon$.

In the last section we will associate a Contou-Carrere symbol to a simple loop σ_i, namely, $I^{1,2}_{m_{\sigma_i}}(f_1, f_2, f_3)$. By the above corollary we have that $I^{1,2}_{m_{[\alpha_i,\beta_i]}}(f_1, f_2, f_3)$ is an integer multiple of $(2pii)^3$

Then $I^{1,2}_{m_\delta} = 0$, since δ is homotopic to the trivial path. Also, by the Lemma 5, we have $0 = I^{1,2}_{m_\delta}(f_1, f_2, f_3) = \sum_{i=1}^{n} I^{1,2}_{m_{\sigma_i}}(f_1, f_2, f_3) + (2\pi i)^3\mathbb{Z}$,

Chapter 4

Reciprocity laws on surfaces

4.1 Reciprocity laws for bi-local symbols

In this Subsection, we define bi-local symbols and prove their reciprocity laws. Using them, in the following two Sections, we establish the first type of reciprocity laws for the Parshin symbol and for a new 4-function new symbol. By a first type of reciprocity law, we mean that the product of the local symbols is taken over all points P of a fixed curve C on the surface X.

Consider the fundamental group of C_0. We recall that C_0 is essentially the curve C without several intersection points and without several open neighborhoods. More precisely,

$$C_0 = C - \left(\bigcup_{j=1}^{m} G_{U_j^\epsilon}\right) \cap C - \left(\bigcup_{i=1}^{n} C_i\right) \cap C.$$

where U_j^ϵ is a small neighborhood of R_j on the complex curve C. We recall the notation for the intersection points

$$\{P_1, \ldots, P_N\} = C \cap (C_1 \cup \cdots \cup C_n),$$

$$\{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m),$$

Let

$$\pi_1(C_0, Q) =< \sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g > / \sim$$

be a presentation of the fundamental group, where

$$\delta \sim 1,$$

for

$$\delta = \prod_{i=1}^{n} \sigma_i \prod_{j=1}^{g} [\alpha_j, \beta_j].$$

We are going to drop the indices i and j. Thus, we are going to write P instead of P_i or R_j and σ instead of σ_i. Consider the definition of a membrane m_σ, associated to a loop σ, given in the beginning of Subsection 3.2. Let $m_\sigma(s,\cdot)$ be the loop obtained by fixing the variable s and letting the second argument vary. Similarly, $m_\sigma(\cdot,t)$ denotes the loop obtained by fixing the variable t and letting the first argument vary.

Definition 25 *Let $a_k = ord_C(f_k)$ and $b_k = ord_P((x^{-a_k}f_k)|_C)$, where x is a rational function, representing an uniformizer such that $ord_C(x) = 1$ and P is not an intersection of any two of the components of the divisor of x.*

It is straightforward to represent the order of vanishing as residues, given by the following:

Lemma 26 *We have*

$$a_k = \frac{1}{2\pi i}\int_{m_\sigma(s,\cdot)}\frac{df_k}{f_k} \qquad \textit{and} \qquad b_k = \frac{1}{2\pi i}\int_{m_\sigma(\cdot,t)}\frac{df_k}{f_k}.$$

Using properties 1(d) and 3(b), we should think of $m_\sigma(\cdot,t)$ and $m_\sigma(s,\cdot)$ as translates of σ and of τ, respectively. Then the above integrals are residues, which detect the order of vanishing. For example a_k is the order of vanishing of f_k along a generic point of C. Then the following theorem holds, whose proof is immediate from Lemmas 21 and 26.

Theorem 27 *(a)*

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,1)}(m_\sigma, f_1, f_2) = a_2 b_1,$$

(b)

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3) = (\pi i)a_2 a_3 b_1,$$

(c)

$$\exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3)\right) = \left(\{f_2, f_3\}_P^Q\right)^{-a_1},$$

(d)

$$\exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4)\right) = \left(\{f_1, f_3\}_P^Q\right)^{-a_2 a_4}.$$

Let us denote by α the loop α_j and by β the loop β_j. Then the following lemma holds

Lemma 28 *(a)*

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,1)}(m_{[\alpha,\beta]} f_1, f_2) = 0,$$

(b)

$$(2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_{[\alpha,\beta]}, f_1, f_2, f_3) = 0,$$

(c)

$$\exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_{[\alpha,\beta]}, f_1, f_2, f_3)\right) = 1,$$

(d)

$$\exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_{[\alpha,\beta]}, f_1, f_2, f_3, f_4)\right) = 1.$$

Proof. It follows from Lemmas 21 and 23. A more modern proof follows from the well-definedness of the integral Beilinson regulator on K_2 on the level of homology (see [**?**].)

Definition 29 *(Bi-local symbols on a surface) For a simple loop σ around a point P in C_0, based at Q, let*

$$Log^{(i,j)}[f_1, \ldots, f_{i+j}]^{(1),Q}_{C,P} = \lim_{e\to 0} I^{i,j}(m_\sigma, f_1, \ldots, f_{i+j}),$$

$${}^{1,2}[f_1, f_2, f_3]^{(1),Q}_{C,P} = \exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(1,2)}(m_\sigma, f_1, f_2, f_3)\right),$$

$${}^{2,1}[f_1, f_2, f_3]^{(1),Q}_{C,P} = \exp\left((2\pi i)^{-2}\lim_{\epsilon\to 0} I^{(2,1)}(m_\sigma, f_1, f_2, f_3)\right),$$

$${}^{2,2}[f_1, f_2, f_3, f_4]^{(1),Q}_{C,P} = \exp\left(\frac{2}{(2\pi i)^3}\lim_{\epsilon\to 0} I^{(2,2)}(m_\sigma, f_1, f_2, f_3, f_4)\right).$$

The following reciprocity laws hold for the above bi-local symbols.

Theorem 30 *(a)* $\sum_P Log^{1,1}[f_1, f_2]^{(1),Q}_{C,P} = 0.$
(b) $\prod_P {}^{1,2}[f_1, f_2, f_3]^{(1),Q}_{C,P} = 1.$
(c) $\prod_P {}^{2,1}[f_1, f_2, f_3]^{(1),Q}_{C,P} = 1.$
(d) $\prod_P {}^{2,2}[f_1, f_2, f_3, f_4]^{(1),Q}_{C,P} = 1.$

Proof. Parts (b), (c) and (d) follow directly from Theorem 27 and from Weil reciprocity. Part (a) follows again from Theorem 27 and the theorem that the sum of the residues of a differential form on a curve is zero. ∎

4.2 Parshin symbol and its first reciprocity law.

In this Subsection, we construct a refinement of the Parshin symbol in terms of six bi-local symbols. Using this presentation of the Parshin symbol, Definition 31 and Theorem 32, we prove the first reciprocity of the Parshin symbol (Theorem 34).

Definition 31 *We define the following bi-local symbol*

$$Pr^{Q}_{C,P} = \left({}^{1,2}[f_1,f_2,f_3]^{(1),Q}_{C,P}\right)\left({}^{1,2}[f_2,f_3,f_1]^{(1),Q}_{C,P}\right)\left({}^{1,2}[f_3,f_1,f_2]^{(1),Q}_{C,P}\right)\times$$
$$\times\left({}^{2,1}[f_1,f_2,f_3]^{(1),Q}_{C,P}\right)\left({}^{2,1}[f_2,f_3,f_1]^{(1),Q}_{C,P}\right)\left({}^{2,1}[f_3,f_1,f_2]^{(1),Q}_{C,P}\right)$$

at the points $P = P_i \in C\cap(C_1\cup\cdots\cup C_n)$ and a fixed point Q in $C - C\cap(C_1\cup\cdots\cup C_n)$.

Using Theorem 27 parts (b) and (c), we obtain:

Theorem 32 *(Refinement of the Parshin symbol) We have the following explicit formula*

$$Pr^{Q}_{C,P} = (-1)^K\frac{\left(f_1^{D_1}f_2^{D_2}f_3^{D_3}\right)(P)}{\left(f_1^{D_1}f_2^{D_2}f_3^{D_3}\right)(Q)},$$

where

$$D_1 = \begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix},\quad D_2 = \begin{vmatrix} a_3 & a_1\\ b_3 & b_1\end{vmatrix},\quad D_3 = \begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

and

$$K = a_1a_2b_3 + a_2a_3b_1 + a_3a_1b_2 + b_1b_2a_3 + b_2b_3a_1 + b_3b_1a_2.$$

Note that $Pr^{Q}_{C,P}$ is essentially the Parshin symbol, which can be defined in the following way

Definition 33 *(The Parshin symbol)*

$$\{f_1,f_2,f_3\}_{C,P} = (-1)^K\left(f_1^{D_1}f_2^{D_2}f_3^{D_3}\right)(P).$$

The only difference between the two symbols is the constant factor in $Pr^{Q}_{C,P}$, depending only on the base point Q (the denominator of $Pr^{Q}_{C,P}$). Rescaling by that constant leads to the Parshin symbol.

Theorem 34 *(First reciprocity law for the Parshin symbol) For the Parshin symbol, the following reciprocity law holds*

$$\prod_P \{f_1,f_2,f_3\}_{C,P} = 1,$$

where the product is taken over points P in $C\cap(C_1\cup\cdots\cup C_n)$. (When P is another point of C then the symbol is trivial.) Here we assume that the union of the support of the divisors $\bigcup_{i=1}^3|div(f_i)|$ in X have normal crossing and no three components have a common point.

Proof. We are going to use the reciprocity laws for bi-local symbols stated in Theorem 30 parts (b) and (c). Then the reciprocity law for the bi-local symbol Pr_P^Q follows. There is relation between the Parshin symbol and Pr_P^Q, namely,

$$\{f_1, f_2, f_3\}_{C,P} = Pr_{C,P}^Q \left(f_1^{D_1} f_2^{D_2} f_3^{D_3} \right)(Q).$$

Now, we remove the dependence on the base point Q. In order to do that, note that

$$\prod_P f_1(Q)^{D_1} = g_1(Q)^{\sum_P D_1}.$$

Here $g_1 = x^{-a_1} f_1$, where x is a rational function on the surface X, representing an uniformazer at the curve C, such that the components of the divisor of x do not intersect at the points P or Q. Moreover,

$$D_1 = (2\pi i)^{-2} \left(Log^{1,1}[f_2, f_3]_P^{(1),Q} - Log^{1,1}[f_3, f_2]_P^{(1),Q} \right)$$

by Theorem 27 part (a) and Proposition 20 part (a). Using Theorem 30 part (a), for the above equality, we obtain

$$\sum_P D_1 = 0.$$

Therefore,

$$\prod_P g_1(Q)^{D_1} = 1.$$

Similarly,

$$\prod_P g_2(Q)^{D_2} = 1 \text{ and } \prod_P g_3(Q)^{D_3} = 1,$$

where $g_k = x^{-a_k} f_k$.

4.3 New 4-function local symbol and its first reciprocity law

In this Subsection, we define a new 4-function local symbol on a surface. We also express the new 4-function local symbol as a product of bi-local symbols (Definition 35 and Proposition 36), which serves as a refinement similar to the refinement of the Parshin symbol in Subsection 2.2. Using the reciprocity laws for bi-local symbols established in Subsection 2.1, we obtain the first type of reciprocity law for the new 4-function local symbol (Theorem 38).

Definition 35 *We define the following bi-local symbol, which will lead to the 4-function local symbol on a surface.*

$$PR_{C,P}^Q = \left({}^{2,2}[f_1, f_2, f_3, f_4]_P^{(1),Q} \right) \left({}^{2,2}[f_1, f_2, f_4, f_3]_P^{(1),Q} \right)^{-1} \times$$
$$\times \left({}^{2,2}[f_2, f_1, f_3, f_4]_P^{(1),Q} \right)^{-1} \left({}^{2,2}[f_2, f_1, f_4, f_3]_P^{(1),Q} \right).$$

Using Theorem 27, part (d), we obtain:

Proposition 36 *Explicitly, the bi-local symbol $PR^Q_{C,P}$ is given by*

$$PR^Q_{C,P} = (-1)^L \frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(P) \cdot \left(\frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(Q)\right)^{-1}, \tag{4.1}$$

where

$$L = (a_1b_2 - a_2b_1)(a_3b_4 - a_4b_3).$$

Definition 37 *(4-function local symbol) With the above notation, we define a 4-function local symbol*

$$\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = (-1)^L \frac{\left(\frac{f_1^{a_2}}{f_2^{a_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4}}{f_4^{a_3}}\right)^{a_1b_2-b_1a_2}}(P).$$

It is an easy exercise to check that the symbol $\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P}$ is independent of the choices of local uniformizers. See also the Appendix for K-theoretical approach for the 4-function local symbol. Note that the relation between the bi-local symbol $PR^C_{C,P}$ and the local symbol $\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P}$ is only a constant factor depending on the base point Q. There is a similar relation between the bi-local symbol $Pr^Q_{C,P}$ and the Parshin symbol $\{f_1, f_2, f_3\}_{C,P}$.

Theorem 38 *(Reciprocity law for the 4-function local symbol) The following reciprocity law for the 4-function local symbol on a surface holds*

$$\prod_P \{f_1, f_2, f_3, f_4\}^{(1)}_{C,P} = 1,$$

where the product is taken over points P on a fixed curve C. Here we assume that the union of the support of the divisors $\bigcup_{i=1}^4 |div(f_i)|$ in X have normal crossing and no three components have a common point.

Proof. Using Theorem 30 part (d), we obtain that the bi-local symbol $PR^Q_{C,P}$ satisfies a reciprocity law, namely,

$$\prod_P PR^Q_{C,P} = 1, \tag{4.2}$$

where the product is over all points P in $C \cap (C_1 \cup \cdots \cup C_n)$. In order to complete the proof of Theorem 38, we proceed similarly to the proof of the first Parshin reciprocity law. Namely,

$$\prod_P g_1(Q)^{a_2(a_3b_4-a_4b_3)} = g_1(Q)^{a_2 \sum_P a_3b_4-a_4b_3} = g(Q)^{b_2 \cdot 0} = 1, \tag{4.3}$$

where $g_1 = x^{-a_1} f_1$ and x is a rational function representing an uniforminzer at the curve C, such that the components of the divisor of x do not intersect at the points P or Q. The last equality of (4.3) holds, because

$$a_3b_4 - a_4b_3 = (2\pi i)^{-2}\left(Log^{1,1}[f_3, f_4]^{(1),Q}_{C,P} - Log^{1,1}[f_4, f_3]^{(1),Q}_{C,P}\right) = 0$$

and

$$\sum_P (2\pi i)^{-2}\left(Log^{1,1}[f_3, f_4]^{(1),Q}_{C,P} - Log^{1,1}[f_4, f_3]^{(1),Q}_{C,P}\right) = 0,$$

by Theorem 27 (a) and Theorem 30 (a), respectively. ■

There is one more interesting relation for the 4-function symbol, whose is a direct consequence of the explicit formula of the symbol.

Theorem 39 *Let*

$$R_{ijkl} = \{f_i, f_j, f_k, f_l\}_{C,P}.$$

Then R_{ijkl} has the same symmetry as the symmetry of a Riemann curvature tensor with respect to permutations of the indices, namely

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = -R_{klij}.$$

4.4 Bi-local symbols revisited

In this Subsection, we define bi-local symbols, designed for proofs of the second type of reciprocity laws for local symbols. These bi-local symbols also satisfy reciprocity laws. Using them, in the following two sections, we establish the second type of reciprocity laws for the Parshin symbol and for a new 4-function new symbol. By a second type of reciprocity law, we mean that the product of the local symbols is taken over all curves C on the surface X, passing through a fixed point P.

Let $C_1, \ldots, C_n$ be curves in X intersecting at a point P. Assume that $C_1, \ldots, C_n$ are among the divisors of the rational functions $f_1, \ldots, f_4$. Let $\tilde{X}$ be the blow-up of X at the point P. Assume that after the blow-up the curves above $C_1, \ldots, C_n$ meet transversally the exceptional curve E and no two of them intersect at a point on the exceptional curve E.

Let D be a curve on $\tilde{X}$ such that D intersects E in one point. Setting

$$\tilde{P}_k = E \cap \tilde{C}_k,$$

where $\tilde{C}_k$ is the curve above C_k after the blow-up, and

$$Q = E \cap D,$$

Definition 40 *We define the following bi-local symbols*

$$^{i,j}[f_1, \ldots, f_{i+j}]^{(2),D}_{C_k,P} :=^{i,j} [f_1, \ldots, f_{i+j}]^{(1),Q}_{E,\tilde{P}_k}.$$

Theorem 41 *The following reciprocity laws for bi-local symbols hold:*

(a)
$$\prod_{C_k} {}^{1,2}[f_1, f_2, f_3]^{(2),D}_{C_k,P} = 1,$$

(b)
$$\prod_{C_k} {}^{2,1}[f_1, f_2, f_3]^{(2),D}_{C_k,P} = 1,$$

(c)
$$\prod_{C_k} {}^{2,2}[f_1, f_2, f_3, f_4]^{(2),D}_{C_k,P} = 1,$$

where the product is over the curves C, among the divisors of at least one of the rational functions $f_1, \dots, f_4$, which pass through the point P.

The proof is reformulation of Theorem 30, where the triple (C_k, P, D) in the above Theorem correspond to the triple (P, Q, C) with $P = C_k \cap E$ and $Q = D \cap E$ in Theorem 30, where the curve C in Theorem 30 corresponds to the curve E.

4.5 Parshin symbol and its second reciprocity law.

In this Subsection, we present an alternative refinement of the Parshin symbol in terms of bi-local symbols (Definition 42). This implies the second reciprocity law for the Parshin symbol, since each of the bi-local symbols satisfy the second type of reciprocity laws (see Subsection 3.1).

Definition 42 *We define the following bi-local symbol, useful for the proof of the second reciprocity law of the Parshin symbol*

$$\begin{aligned} Pr^D_{C,E} = &\left({}^{1,2}[f_1, f_2, f_3]^{(2),D}_{C,P}\right)\left({}^{1,2}[f_2, f_3, f_1]^{(2),D}_{C,P}\right)\left({}^{1,2}[f_3, f_1, f_2]^{(2),D}_{C,P}\right) \times \\ &\times \left({}^{2,1}[f_1, f_2, f_3]^{(2),D}_{C,P}\right)\left({}^{2,1}[f_2, f_3, f_1]^{(2),D}_{C,P}\right)\left({}^{2,1}[f_3, f_1, f_2]^{(2),D}_{C,P}\right), \end{aligned}$$

Let $\tilde{P} = \tilde{C} \cap E$, $Q = D \cap E$. Then

$$Pr^D_{C,E} = Pr^Q_{E,\tilde{P}}.$$

Similarly to the proof of Theorem 34, we can remove the dependence of the bi-local symbol $Pr^Q_{E,\tilde{P}}$ on the base point Q.

Definition 43 *The second Parshin symbol $\{f_1, f_2, f_3\}^{(2)}_{C,E}$ is the symbol, explicitly given by*

$$\{f_1, f_2, f_3\}^{(2)}_{C,E} = (-1)^K \left(f_1^{D_1} f_2^{D_2} f_3^{D_3}\right)(\tilde{P}),$$

where

$$D_1 = \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix}, \; D_2 = \begin{vmatrix} c_3 & c_1 \\ d_3 & d_1 \end{vmatrix}, \; D_3 = \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix}$$

and

$$K = c_1c_2d_3 + c_2c_3d_1 + c_3c_1d_2 + d_1d_2c_3 + d_2d_3c_1 + d_3d_1c_2,$$

with $c_k = ord_E(f_k)$ *and* $d_i = ord_{\tilde{P}}((y^{-c_k} f_k)|_E)$. *Here* y *is a rational function representing an uniformizer at* E *such that the components of the divisor of* y *do not intersect at the point* $\tilde{P}$.

Proposition 44 *The second Parshin symbol is equal to the inverse of the Parshin symbol. More precisely,*

$$\{f_1, f_2, f_3\}^{(2)}_{C,E} = (\{f_1, f_2, f_3\}_{C,P})^{-1}$$

Let

$$a_i = ord_C(f_i)$$

and

$$b_i = ord_P((x^{-a_i} f_i)|_C),$$

where x is a rational function representing a uniformizer at C, whose support does not contain other components passing through the point P.

Lemma 45 *With the above notation, the following holds*

$$ord_E(f_i) = c_i = a_i + b_i.$$

Proof. We still assume that after the blow-up the union of the support of the rational functions f_1, f_2, f_3 have normal crossings and no three curves intersect at a point. Before the blow-up, let $C_1, \ldots, C_n$ be all the components of the union of the support of the three rational functions that meet at the point P. And let E be the exceptional curve above the point P. Then for $C = C_1$, we have

$$b_i = \sum_{j=2}^{n} ord_{C_j}(f_i)$$

and

$$ord_E(f_i) = \sum_{j=1}^{n} ord_{C_j}(f_i).$$

That proves the Lemma.

Proof. (of Proposition 44) Consider the pairs (C, P) on the surface X and $(E, \tilde{C})$ on on the blow-up $\tilde{X}$. Then by the above Lemma, we have

$$\begin{bmatrix} c_i \\ d_i \end{bmatrix} = \begin{bmatrix} ord_E(f_i) \\ \\ ord_{\tilde{C}}(f_i) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ \\ b_i \end{bmatrix} \tag{4.4}$$

The Parshin symbol is invariant under change of variables given by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Also the Parshin symbol is send to its reciprocal when we change the variables by a matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. That proves the Proposition.

Theorem 46 *(Second reciprocity law for the Parshin symbol) We have*

$$\prod_C \{f_1, f_2, f_3\}_{C,P} = 1,$$

where the product is over the curves C from the support of the divisors of the rational functions $\bigcup_{i=1}^3 |div(f_i)|$, which pass through the point P. (For all other choices of curves C, the Parshin symbol will be equal to 1.) Here we assume that the union of the support of the divisors $\bigcup_{i=1}^3 |div(f_i)|$ in $\tilde{X}$ have normal crossings and no two components have a common point with the exceptional curve E in $\tilde{X}$ above the point P. We denote by $\tilde{X}$ the blow-up of X at the point P.

Proof. We can use Proposition 44 and the first reciprocity law for the Parshin symbol given in Theorem 34. Then Theorem 46 follows.

4.6 The second 4-function local symbol and its second reciprocity law

In this Subsection, We define a second type of 4-function local symbol (Definition 49), which satisfies the second type reciprocity laws. By a second reciprocity law, we mean that the product of the local symbols is taken over all curves C on the surface X, which pass through a fixed point P. The 4-function local symbol has a refinement (see Definition 47, which provides a proof of the second reciprocity law (Theorem 50).

Definition 47 *We define a bi-local symbol, useful for the second reciprocity law for a new 4-function local symbol. Let*

$$PR^D_{C,P} = \left({}^{2,2}[f_1, f_2, f_3, f_4]^{(2),D}_{C,E}\right)\left({}^{2,2}[f_1, f_2, f_4, f_3]^{(2),D}_{C,E}\right)^{-1} \times$$
$$\times \left({}^{2,2}[f_2, f_1, f_3, f_4]^{(2),D}_{C,E}\right)^{-1}\left({}^{2,2}[f_2, f_1, f_4, f_3]^{(2),D}_{C,E}\right).$$

Let

$$L = (c_1d_2 - c_2d_1)(c_3d_4 - c_4d_3),$$

where

$$c_i = ord_E(f_i),$$
$$d_i = ord_{\tilde{P}}((x^{-a_i}f_i)|_E),$$

for a rational function x, representing a uniformizer at E, whose support does not contain other components passing through the point $\tilde{P} = E \cap \tilde{C}$.

Lemma 48

$$PR^D_{C,E} = (-1)^L \left(\frac{\left(\frac{f_1^{c_2}}{f_2^{c_1}}\right)^{c_3d_4-c_4d_3}}{\left(\frac{f_3^{c_4}}{f_4^{c_3}}\right)^{c_1d_2-c_2d_1}}(\tilde{P}) \right)^{-1} \frac{\left(\frac{f_1^{c_2}}{f_2^{c_1}}\right)^{c_3d_4-c_4d_3}}{\left(\frac{f_3^{c_4}}{f_4^{c_3}}\right)^{c_1d_2-c_2d_1}}(Q),$$

where $Q = D \cap E$.

It follows directly from Equation 4.1 and Lemma 45.

Definition 49 *The second 4-function local symbol has the following explicit representation:*

$$\{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = (-1)^L \left(\frac{\left(\frac{f_1^{a_2+b_2}}{f_2^{a_1+b_1}}\right)^{a_3b_4-b_3a_4}}{\left(\frac{f_3^{a_4+b_4}}{f_4^{a_3+b_3}}\right)^{a_1b_2-b_1a_2}}(P) \right)^{-1}.$$

Theorem 50 *(Reciprocity law for the second 4-function local symbol) We have the following reciprocity law*

$$\prod_C \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = 1,$$

where the product is over the curves C from the support of the divisors of the rational functions $\bigcup_{i=1}^4 |div(f_i)|$, which pass through the point P. Here we assume that the union of the support of the divisors $\bigcup_{i=1}^4 |div(f_i)|$ in $\tilde{X}$ have normal crossings and no two components have a common point with the exceptional curve E in $\tilde{X}$ above the point P. We denote by $\tilde{X}$ the blow-up of X at the point P.

Proof. Using Theorem 41, we obtain a reciprocity law for the bi-local symbol $PR^{(2),D}_{C,E}$. Multiplying each symbol by the same constant, depending only on Q, we can remove the dependence on Q. Explicitly, the separation between the dependence on D and the second 4 function local symbol are given in Lemma 48. Then we can use Lemma 45 in order to express the coefficients c_i and d_i in terms of a_i and b_i, which implies the reciprocity law stated in the Theorem 50.

4.7 A K-theoretic proof of the reciprocity laws for the 4-function local symbols

In this Section, we give alternative proofs of the two reciprocity laws of the 4-function local symbol, based in Milnor K-theory. We will use the K-theoretic interpretation of the 4-function local symbol, presented in the Appendix by M. Kerr.

Definition 51 *The quotient of the K-theoretic symbol ${}^K[f_1, f_2, f_3, f_4]^{(1)}_{C,P}$, from the Appendix, and the 4-function local symbol $\{f_1, f_2, f_3, f_4\}^{(1)}_{C,P}$ from Definition 37 is given by*

$$(f_1, f_2, f_3, f_4)^{(1)}_{C,P} = (-1)^{a_1a_2a_3b_4+a_2a_3a_4b_1+a_3a_4a_1b_2+a_4a_1a_2b_3}.$$

Here $a_k = ord_C(f_k)$ and $b_k = ord_P((x^{-a_k}f_k)|_C)$, where x is a rational function representing an uniformizer at C such that P is not an intersection point of the irreducible components of the support of the divisor (x).

For each point P on a fixed curve C the values a_k remain the same. Therefore, we have the following interpretation in terms of integrals. Let

$$\omega_k = (-a_k\frac{dx}{x} + \frac{df_k}{f_k})|C$$

be a differential form on the curve C. Then

$$b_k(P) = \frac{1}{2\pi i}Res_P(\omega_k)$$

Proposition 52 *We can express the sign $(f_1, f_2, f_3, f_4)_{C,P}$ in terms of residues*

$$\begin{aligned}(f_1, f_2, f_3, f_4)^{(1)}_{C,P} = \quad & \exp\left(\tfrac{1}{2}\left(a_1a_2a_3Res_P(\omega_4) + a_2a_3a_4Res_P(\omega_1) + \right.\right.\\ & \left.\left.+ a_3a_4a_1Res_P(\omega_2) + a_4a_1a_2Res_P(\omega_3)\right)\right)\end{aligned}$$

Theorem 53 *The sign $(f_1, f_2, f_3, f_4)^{(1)}_{C,P}$ is also a symbol, satisfying the following reciprocity law:*

$$\prod_P (f_1, f_2, f_3, f_4)^{(1)}_{C,P} = 1,$$

where the product is over all points P of the curve C.

Proof. It follows from the fact that the sum of the residues on a curve is equal to zero and from the previous Proposition.

Theorem 54 *The K-theoretic symbol satisfies the following reciprocity law*

$$\prod_C {}^K[f_1, f_2, f_3, f_4]^{(1)}_{C,P} = 1.$$

where the product is over all points P of the curve C.

The proof follows directly from the K-theoretic definition given in the Appendix.

Proof. (an alternative proof of Theorem 38) Using the reciprocity law for the K-theoretic symbol ${}^K[f_1, f_2, f_3, f_4]^{(1)}_{C,P}$ such as in the Appendix and the above Theorem, we obtain another proof of the reciprocity law for the 4-function local symbol.

Now, we proceed toward an alternative proof of the second type of reciprocity laws for the new 4-function local symbol.

Let E be the exceptional curve for the blowup of X at the point P. Let $\tilde{C}$ be the irreducible component sitting above the curve C in the blow-up. We define $\tilde{P} = \tilde{C} \cap E$. A direct observation leads to

$$\{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = \left(\{f_1, f_2, f_3, f_4\}^{(1)}_{E,\tilde{P}}\right)^{-1}$$

for the 4-function local symbols. Similarly we define

$$(f_1, f_2, f_3, f_4)^{(2)}_{C,P} = \left((f_1, f_2, f_3, f_4)^{(1)}_{E,\tilde{P}}\right)^{-1} \tag{4.5}$$

for the sign and

$$^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = \left(\ ^K[f_1, f_2, f_3, f_4]^{(1)}_{E,\tilde{P}}\right)^{-1} \tag{4.6}$$

for the K-theoretic symbol.

Theorem 55 *For the sign and the K-theoretic symbol we have a second type of reciprocity laws.*

$$\prod_C (f_1, f_2, f_3, f_4)^{(2)}_{C,P} = 1$$

and

$$\prod_C\ ^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = 1,$$

where the product is taken over all curves C, passing through the point P. Here we assume that the union of the support of the divisors $\bigcup_{i=1}^4 |div(f_i)|$ in $\tilde{X}$ have normal crossings and no two components have a common point with the exceptional curve E in $\tilde{X}$ above the point P. We denote by $\tilde{X}$ the blow-up of X at the point P.

Proof. For the K-theoretic symbol we have

$$\prod_C\ ^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = \left(\prod_{\tilde{P}}\ ^K[f_1, f_2, f_3, f_4]^{(1)}_{E,\tilde{P}}\right)^{-1} = 1.$$

The first equality follows from the definition of $^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P}$ and the second equality from Theorem 54.

Proof. (an alternative proof of Theorem 50) We have the following equalities

$$\prod_C \{f_1, f_2, f_3, f_4\}^{(2)}_{C,P} = \prod_C\ ^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} \prod_C\ ^K[f_1, f_2, f_3, f_4]^{(2)}_{C,P} = 1.$$

The first equality follows from Definition 51 and Equations (4.5) and (4.6). The second equality follows from Theorem 55.

4.8 On the Contou-Carrere symbol

In this subsection we present computation of the Contou-Carrere symbol for all possible factors from the formal infinite product. Let

$$f_1 = x^{\nu_1(f_1)} y^{\nu_2(f_1)} \prod_{i_1 > -N} \prod_{j_1 > -N_{i_1}} (1 - a_{i_1,j_1} x^{i_1} y^{j_1}),$$

$$f_2 = x^{\nu_1(f_2)} y^{\nu_2(f_2)} \prod_{i_2 > -N} \prod_{j_2 > -N_{i_2}} (1 - a_{i_2,j_2} x^{i_2} y^{j_2}),$$

$$f_3 = x^{\nu_1(f_3)} y^{\nu_2(f_3)} \prod_{i_3 > -N} \prod_{j_3 > -N_{i_3}} (1 - a_{i_3,j_3} x^{i_3} y^{j_3}).$$

We consider an integral over a torus of $\log(f_1)\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}$ in the following cases: The function f_1 is either $x^{i_1}y^{j_1}$ or $1 - x^{i_1}y^{j_1}$. The function f_2 is either $x^{i_2}y^{j_2}$ or $1-x^{i_2}y^{j_2}$. The function f_3 is either $x^{i_3}y^{j_3}$ or $1-x^{i_3}y^{j_3}$. For each of the functions there are two possibilities. For the triple (f_1, f_2, f_3) there are 2^3 possibilities, which we list in the following $8 = 2^3$ cases. We define the two dimensional Contou-Carrere symbol as a cyclic symmetrization of

$$\exp\left(\int\int_T \log(f_1)\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right),$$

where T is a torus of the type m_σ from Section 1. At the next 8 cases, we examine the logarithm of the Countou-Carrere symbol, when each of the functions f_1, f_2, f_3 consists of a single factor of the above infinite products.

At the end of the paper, we compute a more complicated case, which will be useful when we consider complex analytic products instead of products coming from Witt parameters.

Case 1: Let $f_1 = 1 - ax^{i_1}y^{j_1}$, $f_2 = 1 - bx^{i_2}y^{j_2}$ and $f_3 = 1 - cx^{i_3}y^{j_3}$. Then

$$
\begin{aligned}
&\int_T \log(f_1)\frac{df_2}{f_2}\wedge\frac{df_3}{f_3} = \\
&\int_0^1\int_0^1 \left(\int_{(0,0)}^{(\theta_1,\theta_2)} \sum_{n_1=1}^{\infty} -i_1 a^{n_1}\epsilon^{n_1 i_1}\epsilon^{n_1 j_1}(\exp(2\pi\sqrt{-1}n_1 i_1\theta_1')d\theta_1' + \exp(2\pi\sqrt{-1}n_1 i_1\theta_1')d\theta_2'\right)\times \\
&\times \sum_{n_2,n_3=1}^{\infty} (i_2 j_3 - i_3 j_2) b^{n_2} c^{n_3}\epsilon_1^{n_2 i_2 + n_3 i_3}\epsilon_2^{n_2 j_2 + n_3 j_3}\times \\
&\times \exp(2\pi\sqrt{-1}(n_2 i_2 + n_3 i_3)\theta_1)\exp(2\pi\sqrt{-1}(n_2 j_2 + n_3 j_3)\theta_2)d\theta_1\wedge d\theta_2 = \\
=&\int_0^1\int_0^1 -i_1 i_2 j_3\times \\
&\times \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{i_1 n_1} a^{n_1} b^{n_2} c^{n_3}\epsilon_1^{n_1 i_1 + n_2 i_2 + n_3 i_3}\epsilon_2^{n_1 j_1 + n_2 j_2 + n_3 j_3}\times \\
&\times (\exp(2\pi\sqrt{-1}(n_1 i_1 + n_2 i_2 + n_3 i_3)\theta_1)\exp(2\pi\sqrt{-1}(n_1 j_1 + n_2 j_2 + n_3 j_3)\theta_2) - \\
&- \exp(2\pi\sqrt{-1}(n_2 i_2 + n_3 i_3)\theta_1)\exp(2\pi\sqrt{-1}(n_2 j_2 + n_3 j_3)\theta_2))d\theta_1\wedge d\theta_2 = \\
=&(i_2 j_3 - i_3 j_2)\times \\
&\times \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{n_1} a^{n_1} b^{n_2} c^{n_3}\epsilon_1^{n_1 i_1 + n_2 i_2 + n_3 i_3}\epsilon_2^{n_1 j_1 + n_2 j_2 + n_3 j_3}\times \\
&\times \int_0^1\int_0^1 \exp(2\pi\sqrt{-1}(n_2 i_2 + n_3 i_3)\theta_1)\exp(2\pi\sqrt{-1}(n_2 j_2 + n_3 j_3)\theta_2)d\theta_1\wedge d\theta_2 = \\
&- (i_2 j_3 - i_3 j_2)\times \\
&\times \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{n_1} a^{n_1} b^{n_2} c^{n_3}\epsilon_1^{n_1 i_1 + n_2 i_2 + n_3 i_3}\epsilon_2^{n_1 j_1 + n_2 j_2 + n_3 j_3}\times \\
&\times \int_0^1\int_0^1 \exp(2\pi\sqrt{-1}(n_1 i_1 + n_2 i_2 + n_3 i_3)\theta_1)\times \\
&\times \exp\left(2\pi\sqrt{-1}(n_1 j_1 + n_2 j_2 + n_3 j_3)\theta_2\right)d\theta_1\wedge d\theta_2 = \\
=&-(i_2 j_3 - i_3 j_2) \sum_{\substack{n_1,n_2,n_3=1 \\ \mathbf{n}\cdot\mathbf{i}=0 \\ \mathbf{n}\cdot\mathbf{j}=0}}^{\infty} \frac{a^{n_1} b^{n_2} c^{n_3}}{n_1}
\end{aligned}
$$

Put $m_k = \begin{vmatrix} i_{k+1} & i_{k+2} \\ j_{k+1} & j_{k+2} \end{vmatrix}$, where the indices vary modulo 3. Let $d = gcd(m_1, m_2, m_3)$. Then

$$n_1 = k\frac{|m_1|}{d},\ \ n_2 = k\frac{|m_2|}{d},\ \ n_3 = k\frac{|m_3|}{d}.$$

$$i_2 j_3 \sum_{\substack{n_1, n_2, n_3 = 1 \\ \mathbf{n}\cdot\mathbf{i}=0 \\ \mathbf{n}\cdot\mathbf{j}=0}}^{\infty} \frac{a^{n_1} b^{n_2} c^{n_3}}{n_1} =$$

$$= \sum_{k=1}^{\infty} -i_2 j_3 \frac{a^{k|m_1|/d} b^{k|m_2|/d} c^{k|m_3|/d}}{k|m_1|/d} =$$

$$= sign(m_1)\cdot d \cdot \sum_{k=1}^{\infty} \frac{a^{k|m_1|/d} b^{k|m_2|/d} c^{kn_3}}{k} =$$

$$= sign(m_1)\cdot d \cdot \log\left(1 - a^{|m_1|/d} b^{|m_2|/d} c^{|m_3|/d}\right)$$

Case 2: Let $f_1 = a x^{i_1} y^{j_1}$, $f_2 = 1 - b x^{i_2} y^{j_2}$, $f_3 = 1 - c x^{i_3} y^{j_3}$.

$$\int\int_{T_0} \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) =$$

$$= \int_0^1 \int_0^1 i_1 \log(x) \times$$

$$\times \sum_{n_2, n_3 = 1}^{\infty} (i_2 j_3 - i_3 j_2) b^{n_2} c^{n_3} x^{n_2 i_2 + n_3 i_3} y^{n_2 j_2 + n_3 j_3} \frac{dx}{x} \wedge \frac{dy}{y}$$

If $n_2 j_2 + n_3 j_3 \neq 0$ then the integral vanishes. Let $n_2 j_2 + n_3 j_3 = 0$. Then

$$n_2 = k|j_3|/gcd(j_2, j_3)$$

and

$$n_3 = k|j_2|/gcd(j_2, j_3).$$

Moreover, we have a geometric series under the integral, namely,

$$g(x) = \sum_{k=1}^{\infty} b^{n_2} c^{n_3} x^{n_2 i_2 + n_3 i_3} =$$

$$= \sum_{k=1}^{\infty} \left(b^{|j_3|/gcd(j_2,j_3)} c^{|j_2|/gcd(j_2,j_3)} x^{(|j_3| i_2 + |j_2| i_3)/gcd(j_2,j_3)}\right)^k =$$

$$= b^{|j_3|/gcd(j_2,j_3)} c^{|j_2|/gcd(j_2,j_3)} x^{sign(j_3) m_1/gcd(j_2,j_3)} \times$$

$$\times \left(1 - b^{|j_3|/gcd(j_2,j_3)} c^{|j_2|/gcd(j_2,j_3)} x^{sign(j_3) m_1/gcd(j_2,j_3)}\right)^{-1}$$

Then

$$g(x)\frac{dx}{x} = -d \log\left(1 - b^{|j_3|/gcd(j_2,j_3)} c^{|j_2|/gcd(j_2,j_3)} x^{sign(j_3) m_1/gcd(j_2,j_3)}\right)$$

Let

$$h(x) = 1 - b^{|j_3|/gcd(j_2,j_3)} c^{|j_2|/gcd(j_2,j_3)} x^{sign(j_3)m_1/gcd(j_2,j_3)}.$$

Then we have that Case 2 is the logarithm of the 1 dimensional Contou-Carrere symbol of x^{i_1} and $h(x)$ times $i_1(i_2j_3 - i_3j_2)$.

Alternatively, if we sum term by term we can use the following Lemma.

Lemma 56 *If $k \in \mathbb{Z}$ and $k \neq 0$ then* $\int_0^1 \theta e^{2\pi\sqrt{-1}k\theta} d\theta = \frac{1}{2\pi\sqrt{-1}k}$.

Proof.

$$\begin{aligned}\int_0^1 \theta e^{2\pi\sqrt{-1}k\theta} d\theta &= \frac{1}{2\pi\sqrt{-1}k}\int_0^1 \theta d e^{2\pi\sqrt{-1}k\theta} = \\ &= \frac{1}{2\pi\sqrt{-1}k}(\theta e^{2\pi\sqrt{-1}k\theta}|_0^1 - \int_0^1 e^{2\pi\sqrt{-1}k\theta} d\theta) = \\ &= \frac{1}{2\pi\sqrt{-1}k}\end{aligned}$$

Then

$$\begin{aligned}&I^{1,2}(f_1, f_2, f_3) = \\ &\int_0^1\int_0^1 2\pi\sqrt{-1}i_1\theta_1 \times \\ &\times \sum_{n_2,n_3=1}^{\infty} (i_2j_3 - i_3j_2) b^{n_2} c^{n_3} \epsilon_1^{n_2i_2+n_3i_3} \times \\ &\times \exp\left(2\pi\sqrt{-1}(k|j_3|i_2/d + k|j_2|i_3/d)\theta_1\right) d\theta_1 \wedge d\theta_2 = \\ &= \sum_{k_1=1}^{\infty} 2\pi\sqrt{-1}i_1(i_2j_3 - i_3j_2)\frac{\left(b^{|j_3|/d}c^{|j_2|/d}\epsilon_1^{(|j_3|i_2/d+|j_2|i_3/d)}\right)^{k_1}}{2\pi\sqrt{-1}k_1|j_3|i_2/d + k_1|j_2|i_3/d} + \\ &= sign(j_3)\cdot i_1 \cdot d \cdot \sum_{k_1=1}^{\infty}\frac{\left(b^{|j_3|/d}c^{|j_2|/d}\epsilon_1^{(|j_3|i_2/d+|j_2|i_3/d)}\right)^{k_1}}{k_1} = \\ &= -\, sign(j_3)\cdot i_1 \cdot d \cdot \log\left(1 - b^{|j_3|/d}c^{|j_2|/d}\epsilon_1^{sign(j_3)m_1/d}\right)\end{aligned}$$

Then

$$
\begin{aligned}
&I^{2,1}(f_1,f_2,f_3)=\\
&\int_0^1\int_0^1 2\pi\sqrt{-1}j_1\theta_2\times\\
&\times\sum_{n_2,n_3=1}^{\infty}(i_2j_3-i_3j_2)b^{n_2}c^{n_3}\epsilon_1^{n_2i_2+n_3i_3}\times\\
&\times\exp\left(2\pi\sqrt{-1}(n_2j_2+n_3j_3)\theta_2\right)d\theta_1\wedge d\theta_2=\\
=&\sum_{k_2=1}^{\infty}2\pi\sqrt{-1}j_2(i_2j_3-i_3j_2)\frac{\left(b^{|i_3|/d}c^{|i_2|/d}\epsilon_1^{(|i_3|j_2/d+|i_2|j_3/d)}\right)^{k_2}}{2\pi\sqrt{-1}k_2|i_3|j_2/d+k_2|i_2|j_3/d}+\\
=&sign(i_3)\cdot i_1\cdot d\cdot\sum_{k_2=1}^{\infty}\frac{\left(b^{|i_3|/d}c^{|i_2|/d}\epsilon_2^{(|i_3|j_2/d+|i_2|j_3/d)}\right)^{k_2}}{k_2}=\\
=&-sign(i_3)\cdot j_1\cdot d\cdot\log\left(1-b^{|i_3|/d}c^{|i_2|/d}\epsilon_2^{sign(i_3)m_1/d}\right)
\end{aligned}
$$

Case 3: Let $f_1=1-ax^{i_1}y^{j_1}$, $f_2=bx^{i_2}y^{j_2}$, $f_3=1-cx^{i_3}y^{j_3}$.

$$
\begin{aligned}
&\int\int_{T_0}\frac{df_1}{f_1}\circ\left(\frac{df_2}{f_2}\wedge\frac{df_3}{f_3}\right)=\\
=&\int_0^1\int_0^1\left(\int_0^{\theta_1}\sum_{n_1=1}^{\infty}-i_1a^{n_1}\epsilon_1^{n_1i_1}\epsilon_2^{n_1j_1}\exp\left(2\pi\sqrt{-1}n_1(i_1\theta_1'+j_1\theta_2)\left(d\theta_1'\right)\right)\right)\times\\
&\times(2\pi\sqrt{-1})(i_2d\theta_1+j_2d\theta_2)\wedge\\
&\wedge\sum_{n_3=1}^{\infty}c^{n_3}\epsilon_1^{n_3i_3}\epsilon_2^{n_3j_3}\exp\left(2\pi\sqrt{-1}(n_3i_3)\theta_1\right)\exp\left(2\pi\sqrt{-1}(n_3j_3)\theta_2\right)(i_3d\theta_1+j_3d\theta_2)=\\
=&\sum_{n_1,n_3=1}^{\infty}\frac{i_1(i_2j_3-i_3j_2)}{n_1i_1}a^{n_1}c^{n_3}\epsilon_1^{n_1i_1+n_3i_3}\epsilon_2^{n_1j_1+n_3j_3}\times\\
&\times\int_0^1\int_0^1\exp\left(2\pi\sqrt{-1}(n_1i_1+n_3i_3)\theta_1\right)\exp\left(2\pi\sqrt{-1}(n_1j_1+n_3j_3)\theta_2\right)d\theta_1\wedge d\theta_2=
\end{aligned}
$$

The last double integral vanishes if $n_1i_1+n_3i_3\neq 0$ or if $n_1j_1+n_3j_3\neq 0$. If both $n_1i_1+n_3i_3=0$ and $n_1j_1+n_3j_3=0$ then the two vectors (i_1,i_3) and (j_1,j_3) are linearly dependent. Let $n_1=k|i_3|/(i_1,i_3)$ and $n_3=k|i_1|/(i_1,i_3)$.

Note that $i_2j_3 - i_3j_2 = 0$. Then the contribution from Case 3: becomes

$$\begin{aligned}
I_3 =& \sum_{k=1}^{\infty} \frac{i_2j_3}{n_1} a^{n_1} c^{n_3} = \\
=& \sum_{k_1=1}^{\infty} \frac{i_2j_3(j_1,j_3)}{|j_3|k_1} \left(a^{|j_3|/(j_1,j_3)} c^{|j_1|/(j_1,j_3)}\right)^{k_1} - \\
& - \sum_{k_2=1}^{\infty} \frac{i_3j_2(i_1,i_3)}{|i_3|k_2} \left(a^{|i_3|/(i_1,i_3)} c^{|i_1|/(i_1,i_3)}\right)^{k_2} = \\
=& - sign(j_3) j_2 (j_1,j_3) \log\left(1 - a^{|j_3|/(j_1,j_3)} c^{|j_1|/(j_1,j_3)}\right) + \\
& + sign(i_3) j_2 (i_1,i_3) \log\left(1 - a^{|i_3|/(i_1,i_3)} c^{|i_1|/(i_1,i_3)}\right)
\end{aligned}$$

Case 4: Let $f_1 = 1 - ax^{i_1}y^{j_1}, f_2 = 1 - bx^{i_2}y^{j_2}, f_3 = cx^{i_3}y^{j_3}$.

The contribution from Case 4 is similar to Case 3.

$$\begin{aligned}
I_4 =& \sum_{k=1}^{\infty} \frac{i_2j_3 - i_3j_2}{n_1} a^{n_1} b^{n_2} = \\
=& \sum_{k_1=1}^{\infty} \frac{i_2j_3(i_1,i_2)}{|i_2|k_1} \left(a^{|i_2|/(i_1,i_2)} b^{|i_1|/(i_1,i_2)}\right)^{k_1} - \\
& - \sum_{k_2=1}^{\infty} \frac{i_3j_2(j_1,j_2)}{|j_2|k_2} \left(a^{|j_2|/(j_1,j_2)} b^{|j_1|/(j_1,j_2)}\right)^{k_2} = \\
=& - sign(i_2) j_3 (i_1,i_2) \log\left(1 - a^{|i_2|/(i_1,i_2)} b^{|i_1|/(i_1,i_2)}\right) + \\
& + sign(j_2) i_3 (j_1,j_2) \log\left(1 - a^{|j_2|/(j_1,j_2)} b^{|j_1|/(j_1,j_2)}\right)
\end{aligned}$$

Case 5: Let $f_1 = x^{i_1}y^{j_1}, f_2 = x^{i_2}y^{j_2}, f_3 = C(1 - cx^{i_3}y^{j_3})$.

$$\begin{aligned}
&\int\int_{T_0} \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
=&\int_0^1 \int_0^1 \left(\int_{(0,0)}^{(\theta_1,\theta_2)} (i_1 d\theta_1' + j_1 d\theta_2' \right) \times \\
&\times (2\pi\sqrt{-1})(i_2 d\theta_1 + j_2 d\theta_2) \wedge \\
&\wedge \sum_{n_3=1}^{\infty} -c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp\left(2\pi\sqrt{-1}(n_3 i_3)\theta_1\right) \exp\left(2\pi\sqrt{-1}(n_3 j_3)\theta_2\right) (i_3 d\theta_1 + j_3 d\theta_2) = \\
=&\int_0^1 \int_0^1 m_1 i_1 \theta_1 \times \\
&\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp\left(2\pi\sqrt{-1}(n_3 i_3)\theta_1\right) \exp\left(2\pi\sqrt{-1}(n_3 j_3)\theta_2\right) d\theta_1 \wedge d\theta_2 + \\
&+ \int_0^1 \int_0^1 j_1 m_1 \theta_2 \times \\
&\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp\left(2\pi\sqrt{-1}(n_3 i_3)\theta_1\right) \exp\left(2\pi\sqrt{-1}(n_3 j_3)\theta_2\right) d\theta_1 \wedge d\theta_2
\end{aligned}$$

The last integral is different from zero only if $i_3 = 0$. In that case, we have

$$
\begin{aligned}
&\int\int_{T_0} \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) = \\
=&\int_0^1 \int_0^1 i_1\theta_1(-i_3 j_2)\times \\
&\times \sum_{n_3=1}^{\infty} c^{n_3}\epsilon_1^{n_3 i_3}\epsilon_2^{n_3 j_3} \exp\left(2\pi\sqrt{-1}(n_3 i_3)\theta_1\right)\exp\left(2\pi\sqrt{-1}(n_3 j_3)\theta_2\right) d\theta_1\wedge d\theta_2 + \\
&+\int_0^1 \int_0^1 j_1\theta_2(i_2 j_3)\times \\
&\times \sum_{n_3=1}^{\infty} c^{n_3}\epsilon_1^{n_3 i_3}\epsilon_2^{n_3 j_3} \exp\left(2\pi\sqrt{-1}(n_3 i_3)\theta_1\right)\exp\left(2\pi\sqrt{-1}(n_3 j_3)\theta_2\right) d\theta_1\wedge d\theta_2 = \\
=&\sum_{n_3=1}^{\infty} \frac{-i_1 i_3 j_2}{n_3 i_3} c^{n_3}\epsilon_1^{n_3 i_3} + \\
&+\sum_{n_3=1}^{\infty} \frac{i_1 i_2 j_3}{n_3 i_3} c^{n_3}\epsilon_2^{n_3 j_3} = \\
=&-\frac{i_1 i_3 j_2}{i_3}\sum_{n_3=1}^{\infty}\frac{(c\epsilon_1^{i_3})^{n_3}}{n_3} + \frac{i_1 i_2 j_3}{j_3}\sum_{n_3=1}^{\infty}\frac{(c\epsilon_1^{i_3})^{n_3}}{n_3} = \\
=&i_1 j_2\log(1-c\epsilon_1^{i_3}) - i_2 j_1 \log(1-c\epsilon_2^{i_3})
\end{aligned}
$$

Thus,

$$I_5 = i_1 j_2 \log(1-c(-P_1)^{i_3}) - i_2 j_1 \log(1-c(-P_2)^{j_3})$$

6: Let $f_1 = x^{i_1}y^{j_1}, f_2 = B(1-bx^{i_2}y^{j_2}), f_3 = x^{i_3}y^{j_3}$. Similarly to case 5, we obtain

$$I_6 = i_3 j_1 \log(1-b(-P_1)^{i_2}) - i_1 j_3 \log(1-b(-P_2)^{j_2})$$

7: Let $f_1 = A(1-ax^{i_1}y^{j_1}), f_2 = x^{i_2}y^{j_2}, f_3 = x^{i_3}y^{j_3}$.

$$
\begin{aligned}
&\int\int_{T_0} \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) = \\
=&\int_0^1 \int_0^1 \left(\int_0^{\theta_1} \sum_{n_1=1}^{\infty} -i_1 a^{n_1}\epsilon_1^{n_1 i_1}\epsilon_2^{n_1 j_1} \exp(2\pi\sqrt{-1}n_1 i_1\theta_1')\exp(2\pi\sqrt{-1}n_1 j_1\theta_2)d\theta_1'\right)\times \\
&\times i_2 j_3 1 d\theta_1\wedge d\theta_2 = \\
=&\int_0^1 \int_0^1 \sum_{n_1=1}^{\infty} -\frac{i_1 i_2 j_3 a^{n_1}\epsilon_1^{n_1 i_1}\epsilon_2^{n_1 j_1}}{n_1 i_1}\times \\
&\times\left(\exp\left(2\pi\sqrt{-1}(n_1 i_1)\theta_1\right) - 1\right)\exp(2\pi\sqrt{-1}n_1 j_1\theta_2)d\theta_1\wedge d\theta_2
\end{aligned}
$$

The integral is zero if $j_1 \neq 0$. If $j_2 = 0$ then we obtain

$$\int\int_{T_0} \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) =$$

$$= -\int_0^1 \sum_{n_1=1}^{\infty} -\frac{i_1 i_2 j_3 a^{n_1} \epsilon_1^{n_1 i_1}}{n_1 i_1} \exp\left(2\pi\sqrt{-1}(n_1 i_1)\theta_1\right) d\theta_1 -$$

$$\sum_{n_1=1}^{\infty} -\frac{i_1 i_2 j_3 a^{n_1} \epsilon_1^{n_1 i_1}}{n_1 i_1} = 0 - i_2 j_3 \log(1 - a\epsilon_1^{i_1})$$

Note that the last integral vanishes if $i_1 \neq 0$. However, if $i_1 = 0$ and $j_1 = 0$ then $f_1 = 1$ and the integral vanishes again. Thus,

$$I_7 = -m_1 \log(1 - aR_1^{i_1})$$

8: Let $f_1 = x^{i_1} y^{j_1}, f_2 = x^{i_2} y^{j_2}, f_3 = x^{i_3} y^{j_3}$.

For this case it is better to use the notation $I^{1,2}(f_1, f_2, f_3)$ and $I^{2,1}(f_1, f_2, f_3)$ from Subsection 1.3. Using that the logarithm of the Parshin symbol (see [**?**]) can be written as

$$I^{1,2}(f_1, f_2, f_3) + I^{1,2}(f_3, f_1, f_2) + I^{1,2}(f_2, f_3, f_1) -$$
$$- I^{2,1}(f_1, f_2, f_3) - I^{2,1}(f_3, f_1, f_2) - I^{2,1}(f_1, f_2, f_3, f_3)$$

The iteration $\frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right)$ gives 4 from the above 6 terms, namely

$$\frac{(2\pi i)^3}{2}(i_1 i_2 j_3 + i_3 i_1 j_3 - i_3 j_1 j_2 - i_2 j_3 j_1)$$

The ones that are not present are monomials $i_2 i_3 j_1$ and $j_2 j_3 i_1$.

Definition 57 *(Contou-Carrere symbol for surfaces) Let*

$$f_1 = a_1 x^{\nu_1(f_1)} y^{\nu_2(f_1)} \prod_{i_1 > -N} \prod_{j_1 > -N_{i_1}} (1 - a_{i_1, j_1} x^{i_1} y^{j_1}),$$

$$f_2 = a_2 x^{\nu_1(f_2)} y^{\nu_2(f_2)} \prod_{i_2 > -N} \prod_{j_2 > -N_{i_2}} (1 - a_{i_2, j_2} x^{i_2} y^{j_2}),$$

$$f_3 = a_3 x^{\nu_1(f_3)} y^{\nu_2(f_3)} \prod_{i_3 > -N} \prod_{j_3 > -N_{i_3}} (1 - a_{i_3, j_3} x^{i_3} y^{j_3}).$$

Let

$$T(f_1, f_2, f_3) = \prod_{i_1, i_2, i_3, j_1, j_2, j_3} \left(1 - a_{i_1, j_1}^{|m_1|/d} a_{i_2, j_2}^{|m_2|/d} a_{i_3, j_3}^{|m_3|/d}\right)^{sign(m_1)d},$$

where $m_k = i_{k+1}j_{k+2} - i_{k+2}j_{k+1}$ *for* k *modulo* 3 *and* d *is the greatest common divisor of* m_1, m_2, m_3.

$$Q_1(f_1, f_2, f_3, P_1) = \prod_{\nu_1(f_1), i_2, i_3, j_2, j_3} \left(1 - a_{i_2,j_2}^{|j_3|/d} a_{i_3,j_3}^{|j_2|/(j_2,j_3)} P_1^{sign(j_3)m_1/(j_2,j_3)}\right)^{-sign(j_3)\nu_1(f_1)(j_2,j_3)}$$

$$Q_2(f_1, f_2, f_3, P_2) = \prod_{\nu_2(f_1), i_2, i_3, j_2, j_3} \left(1 - a_{i_2,j_2}^{|i_3|/(i_2,i_3)} a_{i_3,j_3}^{|i_2|/(i_2,i_3)} P_2^{sign(i_3)m_1/d}\right)^{sign(i_3)\nu_2(f_1)(i_2,i_3)}$$

$$Q_3(f_1, f_2, f_3) = \prod_{\nu_1(f_2), i_1, i_3, j_1, j_3} \left(1 - a_{i_1,j_1}^{|i_3|/(i_1,i_3)} a_{i_3,j_3}^{|i_1|/(i_1,i_3)}\right)^{sign(i_3)\nu_2(f_2)(i_1,i_3)}$$

$$Q_4(f_1, f_2, f_3) = \prod_{\nu_2(f_2), i_1, i_3, j_1, j_3} \left(1 - a_{i_1,j_1}^{|j_3|/(j_1,j_3)} a_{i_3,j_3}^{|j_1|/(j_1,j_3)}\right)^{-sign(j_3)\nu_1(f_2)(j_1,j_3)}$$

$$R_1(f_1, f_2, f_3, P_1) = \prod_{\nu_1(f_2), \nu_2(f_3), \nu_1(f_3), \nu_2(f_2), i_1} (1 - a_{i_1,0}P_1^{i_1})^{\nu_1(f_2)\nu_2(f_3) - \nu_1(f_3)\nu_2(f_2)}$$

$$R_2(f_1, f_2, f_3, P_2) = \prod_{\nu_2(f_2), \nu_2(f_3), \nu_1(f_3), \nu_2(f_2), j_1} (1 - a_{0,j_1}P_2^{j_1})^{\nu_1(f_2)\nu_2(f_3) - \nu_1(f_3)\nu_2(f_2)}$$

$S(f_1, f_2, f_3) = (-1)^A$, *where* $A = i_1i_2j_3 + i_2i_3j_1 + i_3i_1j_2 - i_1j_2j_3 - i_2j_3j_1 - i_3j_1j_2$

Then the Contou-Carrere symbol is a formal product

$$\begin{aligned}\{f_1, f_2, f_3\}_{C,0}^P =& T(f_1, f_2, f_3)S(f_1, f_2, f_3)\times \\ &\times \prod_{cyclic} Q_1(f_1, f_2, f_3, P_1)Q_2(f_1, f_2, f_3, P_2)Q_3(f_1, f_2, f_3)Q_4(f_1, f_2, f_3)\times \\ &\times \prod_{cyclic} R_1(f_1, f_2, f_3, P_1)R_2(f_1, f_2, f_3, P_2),\end{aligned}$$

where $\prod_{cyclic}$ *is a product over a cyclic permutation of the order of the functions* f_1, f_2, f_3.

4.8.1 Semi-local formulas

Let $f_1 = 1 - a_1x^{i_1}y^{j_1}$, $f_2 = 1 - a_2x^{i_2}y^{j_2}$, $f_1 = 1 - a_3x^{i_3}y^{j_3} - a_4x^{i_4}y^{j_4}$,

$$\begin{aligned}&\int_T \frac{df_1}{f_1} \circ \left(\frac{df_2}{f_2} \wedge \frac{df_3}{f_3}\right) = \\ &= \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} \int_T \sum_{n_1,n_2,n_3=1}^{\infty} \sum_{n_4=0}^{\infty} -\frac{2}{n_1} \frac{(n_3+n_4)!}{n_3!n_4!} \prod_{k=1}^{4} a_k^{n_k} x^{i_k n_k} y^{j_k n_k} \frac{dx}{x} \wedge \frac{dy}{y} = \\ &- \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix} \int_T \sum_{n_1,n_2,n_4=1}^{\infty} \sum_{n_3=0}^{\infty} \frac{2}{n_1} \frac{(n_3+n_4)!}{n_3!n_4!} \prod_{k=1}^{4} a_k^{n_k} x^{i_k n_k} y^{j_k n_k} \frac{dx}{x} \wedge \frac{dy}{y}\end{aligned}$$

$$\left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| \int_T \sum_{n_1,n_2,n_3=1}^{\infty} \sum_{n_4=0}^{\infty} -\frac{2}{n_1} \frac{(n_3+n_4)!}{n_3!n_4!} \prod_{k=1}^{4} a_k^{n_k} x^{i_k n_k} y^{j_k n_k} \frac{dx}{x} \wedge \frac{dy}{y} =$$

Let $\left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| \neq 0$. Then by row reduction we obtain the following

$$\left[\begin{array}{cccc} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{array} \right] \to \left[\begin{array}{cccc} \left| \begin{array}{cc} i_1 & i_3 \\ j_1 & j_3 \end{array} \right| & \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| & 0 & \left| \begin{array}{cc} i_4 & i_3 \\ j_4 & j_3 \end{array} \right| \\ \left| \begin{array}{cc} i_2 & i_1 \\ j_2 & j_1 \end{array} \right| & 0 & \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| & \left| \begin{array}{cc} i_2 & i_4 \\ j_2 & j_4 \end{array} \right| \end{array} \right]$$

Then the sums have to vanish, $\sum k = 1^4 i_k n_k = \sum_{k=1}^4 j_k n_k = 0$. Using the above row reduction, we obtain

$$\begin{aligned} n_1 &= \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| k_1/d \\ n_2 &= \left(- \left| \begin{array}{cc} i_4 & i_3 \\ j_4 & j_3 \end{array} \right| k_4 - \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| k_1 \right) /d \\ n_3 &= \left(- \left| \begin{array}{cc} i_2 & i_4 \\ j_2 & j_4 \end{array} \right| k_4 - \left| \begin{array}{cc} i_2 & i_1 \\ j_2 & j_1 \end{array} \right| k_1 \right) /d \\ n_4 &= \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| k_4/d, \end{aligned}$$

where d is the greatest common divisor of $\left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right|$, $\left| \begin{array}{cc} i_4 & i_3 \\ j_4 & j_3 \end{array} \right|$, $\left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right|$, $\left| \begin{array}{cc} i_2 & i_4 \\ j_2 & j_4 \end{array} \right|$ and $\left| \begin{array}{cc} i_2 & i_1 \\ j_2 & j_1 \end{array} \right|$.

Let

$$M = \left| \begin{array}{cc} i_2 & i_3 \\ j_2 & j_3 \end{array} \right| /d - \left| \begin{array}{cc} i_2 & i_4 \\ j_2 & j_4 \end{array} \right| /d$$

and

$$N = \left| \begin{array}{cc} i_2 & i_1 \\ j_2 & j_1 \end{array} \right| /d.$$

Let also ξ_M and ξ_N be a primitive M-th and N-th root of unity.

$$\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} \int_T \sum_{n_1,n_2,n_3=1}^{\infty} \sum_{n_4=0}^{\infty} -\frac{2}{n_1} \frac{(n_3+n_4)!}{n_3!n_4!} \prod_{k=1}^{4} a_k^{n_k} x^{i_k n_k} y^{j_k n_k} \frac{dx}{x} \wedge \frac{dy}{y} =$$

$$= -d \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{k_1 \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix}} a_1^{n_1} a_2^{n_2} a_3^{n_3} a_4^{n_4} \frac{(n_3+n_4)!}{n_3!n_4!} =$$

$$= -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{k_1} a_1^{n_1} a_2^{n_2} (a_3+a_4)^{n_3+n_4} =$$

$$= -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{k_1} a_1^{n_1} a_2^{n_2} (a_3+a_4)^{n_3+n_4} =$$

$$= -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{k_1} a_1^{\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_1/d} a_2^{\left(-\begin{vmatrix} i_4 & i_3 \\ j_4 & j_3 \end{vmatrix} k_4 - \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_1\right)/d} \times$$

$$\times \sum_{m_1,m_2=1}^{M} (\xi_M^{m_1} a_3 + \xi_M^{m_2} a_4))^{\left(\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_4 - \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix} k_4\right)/d} \times$$

$$\times \sum_{n=1}^{N} (\xi_N^{n_1} a_3 + \xi_N^{n_2} a_4))^{-\begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} k_1/d} =$$

$$= \sum_{m_1,m_2=1}^{M} \left(1 - a_2^{\left(-\begin{vmatrix} i_4 & i_3 \\ j_4 & j_3 \end{vmatrix}\right)/d} (\xi_M^{m_1} a_3 + \xi_M^{m_2} a_4)^{\left(\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} - \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix}\right)/d}\right)^{-1} \times$$

$$d \times \sum_{n_1,n_2=1}^{N} \log \left(1 - a_1^{\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix}/d} a_2^{-\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix}/d} (\xi_N^{n_1} a_3 + \xi_N^{n_2} a_4)^{-\begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix}/d}\right)$$

4.9 Non-commutative reciprocity laws on surfaces

4.9.1 Non-commutative reciprocity laws on surfaces for differential forms of with logarithmic poles

4.9.2 Non-commutative reciprocity laws on surfaces for any differential forms

Part II

The Second Part

Chapter 5

Multiple zeta values and multiple Dedekind zeta values

Multiple Dedekind zeta functions generalize Dedekind zeta functions in the same way the multiple zeta functions generalize the Riemann zeta function. Let us recall known definitions of the above functions. The Riemann zeta function is defined as

$$\zeta(s)=\sum_{n>0}\frac{1}{n^s},$$

where n is an integer. Multiple zeta functions are defined as

$$\zeta(s_1,\dots,s_m)=\sum_{0<n_1<\dots<n_m}\frac{1}{n_1^{s_1}\dots n_m^{s_m}},$$

where $n_1,\dots,n_m$ are integers. Special values of the Riemann zeta function $\zeta(k)$ and of the multiple zeta functions $\zeta(k_1,\dots,k_m)$ were defined by Euler [?]. The Riemann zeta function is closely related to the ring of integers.

Dedekind zeta function $\zeta_K(s)$ is an analogue of the Riemann zeta function, which is closely related to the algebraic integers $\mathcal{O}_K$ in a number field K. It is defined as

$$\zeta_K(s)=\sum_{\mathfrak{a}\neq(0)}\frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where the sum is over all ideals $\mathfrak{a}$ different from the zero ideal (0) and $N(\mathfrak{a})=\#|\mathcal{O}_K/\mathfrak{a}|$ is the norm of the ideal $\mathfrak{a}$.

A definition of multiple Dedekind zeta functions should combine ideas from multiple zeta functions and from Dedekind zeta functions.

There is a definition of multiple Dedekind zeta functions due to Masri [?]. Let us recall his definition. Let $K_1,\dots,K_m$ be number fields and let $\mathcal{O}_{K_i}$, for

$i = 1, \ldots, m$, be the corresponding rings of integers. Let $\mathfrak{a}_i$, for $i = 1, \ldots, m$, be ideals in $\mathcal{O}_{K_i}$, respectively. Then he defines

$$\zeta(K_1, \ldots, K_d; s_1, \ldots s_m) = \sum_{0 < N(\mathfrak{a}_1) < \cdots < N(\mathfrak{a}_m)} \frac{1}{N(\mathfrak{a}_1)^{s_1} \ldots N(\mathfrak{a}_m)^{s_m}}.$$

We propose a different definition. The advantage of our definition is that it leads to more properties: analytic, topological and algebraic-geometric. Let us give an explicit formula for a multiple Dedekind zeta function, in a case when it is easier to formulate. Let K be a number field with ring of integers $\mathcal{O}_K$. Let U_K be the group of units in $\mathcal{O}_K$. Let C be a cone inside of a fundamental domain of $\mathcal{O}_K$ modulo U_K. (More precisely, C has to be a positive unimodular simple cone as defined in Section 5.2.2. A fundamental domain for $\mathcal{O}_K$ modulo U_K can be written as a finite union of unimodular simple cones.) For such a cone C, we define a multiple Dedekind zeta function

$$\zeta_{K;C}(s_1, \ldots, s_1; \ldots; s_m, \ldots, s_m) = \tag{5.1}$$

$$\sum_{\alpha_1, \ldots, \alpha_m \in C} \frac{1}{N(\alpha_1)^{s_1} N(\alpha_1 + \alpha_2)^{s_2} \cdots N(\alpha_1 + \cdots + \alpha_m)^{s_m}}. \tag{5.2}$$

The key new ingredient in the definition of multiple Dedekind zeta functions is the definition of iterated integrals on a membrane. This is a higher dimensional analogue of iterated path integrals. In the iterated integrals on a membrane the iteration happens in n-directions. Such iterated integrals were defined in [?] generalizing Manin's non-commutative modular symbol [?] to higher dimensions in some cases, essentially for Hilbert modular surfaces.

Structure of the paper:

In Subsection 5.1.1, we recall definitions of multiple zeta values and of polylogarithms by giving many explicit formulas. In Subsection 5.1.2, we generalize the previous formulas to multiple Dedekind zeta values over the Gaussian integers via many examples.

In Section 5.2.1, we give two Definitions of iterated integrals on a membrane. The first definition is more intuitive. It can be used to generalize the first few formulas for MDZV over the Gaussian integers from Subsection 5.1.2. The second Definition is the one needed for the definition of multiple Dedekind zeta values. It is needed in order to express special values of the multiple Eisenstein series via MDZV, when the modular parameter has a value in an imaginary quadratic field.

In Section 5.2.2, we use some basic algebraic number theory (see [?]), in order to construct the functions that we integrate. We use an idea of Shintani (see [?], [?]) for defining a cone. We associate a product of geometric series to every unimodular simple cone. This is the type of functions that we integrate. Lemma 75 shows that a fundamental domain for the non-zero integer $\mathcal{O}_K - \{0\}$ modulo the units U_K can be written as a finite union of unimodular simple cones.

In Section 5.3, we define Dedekind polylogarithms associated to a positive unimodular simple cone. Theorem 78 expresses Dedekind zeta values in terms of Dedekind polylogarithms. The heart of the section is Definition 80 of multiple Dedekind zeta values (MDZV) as an iterated integral over a membrane and Definition 82 of multiple Dedekind zeta functions (MDZF) in terms of an integral representation. Theorems 81 and 83 express MDZV and MDZF as an infinite sum. At the end of the Section 5.3, we give many examples. Examples 1 and 2 are the simplest multiple Dedekind zeta values. Example 3 expresses partial Eisenstein-Kronecker series associated to an imaginary quadratic ring as multiple Dedekind zeta values (see [**?**], section 8.1). Example 4 considers multiple Eisenstein-Kronecker series (for an alternative definition see [**?**], Section 8.2). Examples 5 give the simplest multiple Dedekind zeta functions. Example 6 is a double Dedekind zeta function.

In Section 5.4, we prove an analytic continuation of multiple Dedekind zeta functions, which allows us to consider special values of multiple Eisenstein series, examined by Gangl, Kaneko and Zagier (see [**?**]), as values of multiple Dedekind zeta functions, (see Examples 7, 8, 9 in Subsection 5.4). Examples 10 and 11 are particular cases of analytic continuation and of a multiple residue at $(1, \dots, 1)$. The proof of analytic continuation is based on a generalization of Example 11 and a Theorem of Gelfand-Shilov (Theorem 84). At the end of Section 5.4.3, based on Examples 10 and 11, we state two conjectures about MDZV.

5.1 Examples

We are going to present several examples of Riemann zeta values and multiple zeta values in order to introduce key examples of multiple Dedekind zeta value as iterated integrals. Instead of considering the ring of integers in a general number field, which we will do in the later sections, we will examine only the ring of Gaussian integers. Also, here we will ignore questions about convergence. Such questions will be addressed in Subsection 5.2.2.

5.1.1 Classical cases: Multiple zeta values and (multiple) polylogarithms

Let us recall the m-th polylogarithm and its relation to Riemann zeta values.

If the first polylogarithm is defined as

$$Li_1(x_1) = \int_0^{x_1} \frac{dx_0}{1-x_0} = \int_0^{x_1} (1 + x_0 + x_0^2 + \dots) dx_0 = x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \dots$$

and the second polylogarithm is

$$Li_2(x_2) = \int_0^{x_2} Li_1(x_1) \frac{dx_1}{x_1} = x_2 + \frac{x_2^2}{2^2} + \frac{x_2^3}{3^2} + \dots$$

(Note that $\zeta(2) = Li_2(1)$), then the m-th polylogarithm is defined by iteration

$$Li_m(x_m) = \int_0^{x_m} Li_{m-1}(x_{m-1}) \frac{dx_{m-1}}{x_{m-1}}. \tag{5.3}$$

This is a presentation of the m-th polylogarithm as an iterated integral. By a direct computation it follows that

$$Li_m(x) = x + \frac{x^2}{2^m} + \frac{x^3}{3^m} + \dots$$

and the relation

$$\zeta(m) = Li_m(1)$$

is straightforward. Using Equation 5.3, we can express the m-th polylogarithm as

$$Li_m(x_m) = \int_{0<x_0<x_1<\cdots<x_m} \frac{dx_0}{1-x_0} \wedge \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{m-1}}{x_{m-1}}.$$

Let $x_i = e^{-t_i}$. Then the m-th polylogarithm can be written in the variables $t_0, \dots, t_m$ in the following way

$$Li_m(e^{-t_m}) = \int_{t_0>t_1>\cdots>t_m} \frac{dt_0 \wedge \cdots \wedge dt_{m-1}}{e^{t_0} - 1}. \tag{5.4}$$

This is achieved, first, by changing the variables in the differential forms

$$\frac{dx_0}{1-x_0} = \frac{d(-t_0)}{e^{t_0}-1}, \text{ and } \frac{dx_i}{x_i} = d(-t_i),$$

and second, by reversing the bounds of integration $0 < x_0 < x_1 < \cdots < x_m$ v.s. $t_0 > t_1 > \cdots > t_m$, which absorbs the sign. As an infinite sum, we have

$$Li_m(e^{-t}) = \sum_{n>0} \frac{e^{-nt}}{n^m}. \tag{5.5}$$

In Subsection 5.1.2, we present a key analogy of Equations (5.4) and (5.5) leading to Dedekind polylogarithms over the Gaussian integers. Equations (5.4) and (5.5) will be generalized to Dedekind polylogarithms in Subsection 5.3.1 and to multiple Dedekind zeta values in Subsection 5.3.2.

Below we present similar formulas for multiple polylogarithms with exponential variables. We will construct their generalizations in Subsection 5.1.2.

Let us recall the definition of double logarithm

$$Li_{1,1}(1, x_2) = \int_0^{x_2} Li_1(x_1) \frac{dx_1}{1-x_1} = \int_0^{x_2} \left(\sum_{n_1=1}^{\infty} \frac{x_1^{n_1}}{n_1} \right) \left(\sum_{n_2=1}^{\infty} x_1^{n_2} \right) \frac{dx_1}{x_1} =$$

$$= \sum_{n_1,n_2=1}^{\infty} \frac{x_2^{n_1+n_2}}{n_1(n_1+n_2)}.$$

Let $x_i = e^{-t_i}$. Then the $Li_{1,1}(1, e^{-t_2})$ can be written as an iterated integral in terms of the variables t_0, t_1, t_2 in the following way

$$Li_{1,1}(1, e^{-t_2}) = \int_{t_0>t_1>t_2>0} \frac{dt_0 \wedge dt_1}{(e^{t_0}-1)(e^{t_1}-1)}.$$

As an infinite sum, we have

$$Li_{1,1}(1, e^{-t}) = \sum_{n_1,n_2=1}^{\infty} \frac{e^{-(n_1+n_2)t}}{n_1(n_1+n_2)}. \tag{5.6}$$

An example of a multiple zeta value is

$$\zeta(1,2) = \sum_{n_1,n_2=1}^{\infty} \frac{1}{n_1(n_1+n_2)^2} = \int_0^1 Li_{1,1}(x_2)\frac{dx_2}{x_2}.$$

Thus, an integral representation of $\zeta(1,2)$ is

$$\zeta(1,2) = \int_{t_0>t_1>t_2>0} \frac{dt_0}{(e^{t_0}-1)} \wedge \frac{dt_1}{(e^{t_1}-1)} \wedge dt_2. \tag{5.7}$$

Similarly,

$$\zeta(2,2) = \int_{t_0>t_1>t_2>t_3>0} \frac{dt_0}{(e^{t_0}-1)} \wedge dt_1 \wedge \frac{dt_2}{(e^{t_2}-1)} \wedge dt_3. \tag{5.8}$$

5.1.2 Dedekind polylogarithms over the Gaussian integers

In this Subsection, we are going to construct analogues of polylogarithms (and of some multiple polylogarithms), which we call Dedekind (multiple) polylogarithms over the Gaussian integers. We will denote by f_m the m-th Dedekind polylogarithm, which will be an analogue the m-th polylogarithm $Li_m(e^{-t})$ with an exponential variable. Each of the analogues will have an integral representation, resembling an iterated integral and an infinite sum representation, resembling the classical Dedekind zeta values over the Gaussian integers. We also draw diagrams that represent integrals in order to give a geometric view of the iterated integrals on membranes in dimension 2. We will give examples of multiple Dedekind zeta values (MDZV) over the Gaussian integers, using the Dedekind (multiple) polylogarithms.

We are going to generalize Equations (5.5) and (5.6) for (multiple) polylogarithms to their analogue over the Gaussian integers. We will recall some properties and definitions related to Gaussian integers. For more information one may consider [**?**].

By Gaussian integers we mean all numbers of the form $a + ib$, where a and b are integers and $i = \sqrt{-1}$. The ring of Gaussian integers is denoted by $\mathbb{Z}[i]$. We call the following set C a *cone*

$$C = \mathbb{N}\{1+i, 1-i\} = \{\alpha \in \mathbb{Z}[i] \mid \alpha = a(1+i) + b(1-i);\ a, b \in \mathbb{N}\},$$

where $\mathbb{N}$ denotes the positive integers. Note that 0 does not belong to the cone C, since the coefficients a and b are positive integers. We are going to use two sequences of inequalities

$$t_1 > u_1 > v_1 > w_1 \text{ and } t_2 > u_2 > v_2 > w_2,$$

when we deal with a small number of iterations. The reason for introducing them is to make the examples easier to follow. However, for generalizations to higher order of iteration we will use the following notation for the two sequences

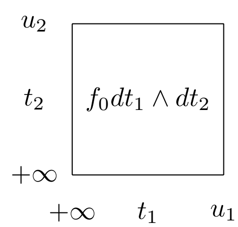
$$t_{1,0} > t_{1,1} > t_{1,2} > t_{1,3} \text{ and } t_{2,0} > t_{2,1} > t_{2,2} > t_{2,3}.$$

We are going to define a function f_1, which will be an analogue of $Li_1(e^{-t})$. Let

$$f_0(C; t_1, t_2) = \sum_{\alpha \in C} \exp(-\alpha t_1 - \overline{\alpha} t_2). \tag{5.9}$$

$$f_1(C, u_1, u_2) = \int_{\infty}^{u_1} \int_{\infty}^{u_2} f_0(C; t_1, t_2) dt_1 \wedge dt_2.$$

We can draw the following diagram for the integral representing f_1.



The diagram represents that the integrant is $f_0(C; t_1, t_2)dt_1 \wedge dt_2$, depending on the variables t_1 and t_2, subject to the restrictions $+\infty > t_1 > u_1$ and $+\infty > t_2 > u_2$.

We need the following:

Lemma 58 *(a)*

$$\int_{\infty}^{u} e^{-kt} dt = \frac{e^{-ku}}{k};$$

(b) Let $N(\alpha) = \alpha\overline{\alpha}$. Then

$$\int_{\infty}^{u_1} \int_{\infty}^{u_2} \exp(-\alpha t_1 - \overline{\alpha} t_2) dt_1 \wedge dt_2 = \frac{\exp(-\alpha u_1 - \overline{\alpha} u_2)}{N(\alpha)}.$$

The proof is straight forward.

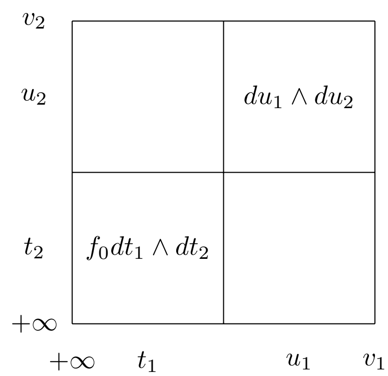
Using the above Lemma, we obtain

$$f_1(C; u_1, u_2) = \sum_{\alpha \in C} \frac{\exp(-\alpha u_1 - \overline{\alpha} u_2)}{N(\alpha)}$$

We define a Dedekind dilogarithm f_2 by

$$f_2(C; v_1, v_2) = \int_\infty^{v_1} \int_\infty^{v_2} f_1(C; u_1, u_2) du_1 \wedge du_2 =$$
$$= \int_{t_1 > u_1 > v_1;\ t_1 > u_1 > v_2} f_0(C; t_1, t_2) dt_1 \wedge dt_2 \wedge du_1 \wedge du_2 \qquad (5.10)$$

We can associate a diagram to the integral representation of the Dedekind dilogarithm f_2 (see Equation (5.10)).



The diagram represents that the variables under the integral are t_1, t_2, u_1, u_2, subject to the conditions $+\infty > t_1 > u_1 > v_1$ and $+\infty > t_2 > u_2 > v_2$. Also, the function f_0 in the diagram depends on the variables t_1 and t_2.

Similarly to Equation (5.3), we define inductively the m-th Dedekind polylogarithm over the Gaussian integers

$$f_m(C; t_{1,m}, t_{2,m}) = \int_\infty^{t_{1,m}} \int_\infty^{t_{2,m}} f_{m-1}(C; t_{1,m-1}, t_{2,m-1}) dt_{1,m-1} \wedge dt_{2,m-1}, \qquad (5.11)$$

where $t_{1,0} > t_{2,1} > \cdots > t_{1,m-1} > t_{1,m}$ and $t_{2,0} > t_{2,1} > \cdots > t_{2,m-1} > t_{2,m}$. The above integral is the key example of an iterated integral over a membrane, which is the topic of Subsection 5.2.1.

From Equation (5.11), we can derive an analogue of the infinite sum representation of a polylogarithm (see Equation (5.5)).

$$f_m(C; t_{1,m}, t_{2,m}) = \sum_{\alpha \in C} \frac{\exp(-\alpha t_{1,m} - \overline{\alpha} t_{2,m})}{N(\alpha)^m}. \qquad (5.12)$$

The above Equation gives an infinite sum representation of the m-th Dedekind polylogarithm over the Gaussian integers.

We derive one relation between the Dedekind m-polylogarithm f_m, a Dedekind zeta value over the Gaussian integers and a Riemann zeta value. For arithmetic over the Gaussian integers one can consider [**?**].

Lemma 59 *For the Dedekind polylogarithm f_m, associated to the above cone C, we have*

$$f_m(C;0,0) = 2^{-m}(\zeta_{\mathbb{Q}(i)}(m) - \zeta(2m)),$$

where $\zeta_{\mathbb{Q}(i)}(m)$ is a Dedekind zeta value and $\zeta(2m)$ is a Riemann zeta value.

Proof. We are going to prove the following equalities, which give the lemma.

$$f_m(C;0,0) = \sum_{\alpha \in C} \frac{1}{N(\alpha)^m} = 2^{-m}\left(\sum_{(\alpha)\neq(0)\subset\mathbb{Z}[i]} - \sum_{\alpha\in\mathbb{N}}\right)\frac{1}{N((\alpha))^m} =$$
$$= 2^{-m}(\zeta_{\mathbb{Q}(i)}(m) - \zeta(2m)), \tag{5.13}$$

The first equality follows from (5.12). The second and the third equalities relate our integral to classical zeta values. The second equality uses two facts: (1) for the Gaussian integers the norm of an element α, $N(\alpha)$, is equal to the norm of the principal ideal generated by α, denoted by $N((\alpha))$, namely $N(\alpha) = N((\alpha))$. Recall that for the Gaussian integers the norm of an element α, is $N(\alpha) = \alpha\overline{\alpha}$, and the norm of a principal ideal $N((\alpha))$ is equal to the number of elements in the quotient module

$$N((\alpha)) = \#|\mathbb{Z}[i]/(\alpha)|,$$

where

$$(\alpha) = \alpha\mathbb{Z}[i] = \{\mu \in \mathbb{Z}[i] \mid \mu = \alpha\beta \text{ for some } \beta \in \mathbb{Z}[i]\}$$

is view as a $\mathbb{Z}[i]$-submodule of $\mathbb{Z}[i]$. (2) the set of non-zero principal ideals can be parametrized by the non-zero integers modulo the units. Since the units are $\pm 1, \pm i$, we have that $(\alpha) \subset \mathbb{Z}[i]$, $(\alpha) \neq (0)$ can be parametrized by elements of the Gaussian integers with positive real part and non-negative imaginary part, which we will denote by C_0. Multiplying each element of C_0 by $1-i$, we obtain the union of the cone C and the set $\{a + ai \mid a \in \mathbb{N}\}$. Summing over C_0 gives the Dedekind zeta value. Summing over $(1-i)C$ gives $2{-}m\zeta_{\mathbb{Q}(i)}(m)$. Such a sum can be separated to a sum over C, which contributes f_m and a sum over the set $\{a + ai \mid a \in \mathbb{N}\}$, which gives $2^{-m}\zeta(2m)$.

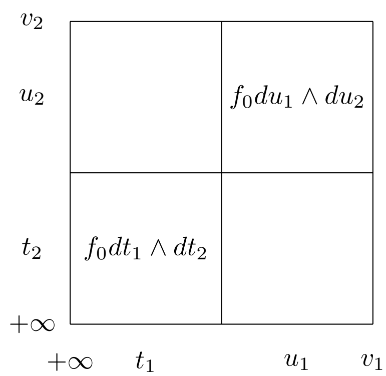
Now we can define an analogue of the double logarithm $Li_{1,1}(1, e^{-t})$ over the Gaussian integers, using the following integral representation

$$f_{1,1}(C; v_1, v_2) = \int_{\infty}^{v_1}\int_{\infty}^{v_2} f_1(C; u_1, u_2) f_0(C; u_1, u_2) du_1 \wedge du_2,$$

called a Dedekind double logarithm. Such an integral will be considered as an example of an iterated integral over a membrane in Subsection 2.1. As an analog for Equation (5.10), we can express $f_{1,1}$ only in terms of f_0 by

$$f_{1,1}(C; v_1, v_2) = \int_{t_1>u_1>v_1;\ t_2>u_2>v_2} (f_0(C; t_1, t_2) dt_1 \wedge dt_2) \wedge (f_0(u_1, u_2) du_1 \wedge du_2).$$

It allows us to associate a diagram to the Dedekind double logarithm $f_{1,1}$:



The variables t_1, t_2, u_1, u_2 in the diagram are variables in the integrant. They are subject to the conditions $t_1 > u_1 > v_1$ and $t_2 > u_2 > v_2$. Also, the lower left function f_0 in the diagram depends on the variables t_1 and t_2 and the upper right function f_0 depends on u_1 and u_2.

The similarity between $f_{1,1}(C; v_1, v_2)$ and $Li_{1,1}(1, e^{-t_2})$ can be noticed by the infinite sum representation in the following:

Lemma 60

$$f_{1,1}(C; v_1, v_2) = \sum_{\alpha,\beta \in C} \frac{\exp(-(\alpha+\beta)v_1 - (\overline{\alpha}+\overline{\beta})v_2)}{N(\alpha)N(\alpha+\beta)}.$$

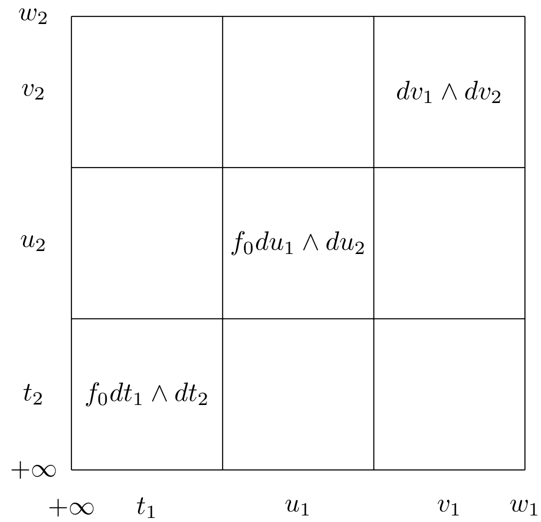
Proof.

$$\begin{aligned} f_{1,1}(C; v_1, v_2) &= \int_\infty^{v_1} \int_\infty^{v_2} f_1(C; u_1, u_2) f_0(C; u_1, u_2) du_1 \wedge du_2 = \\ &= \int_\infty^{v_1} \int_\infty^{v_2} \sum_{\alpha \in C} \frac{\exp(-\alpha u_1 - \overline{\alpha} u_2)}{N(\alpha)} \sum_{\beta \in C} \exp(-\beta u_1 - \overline{\beta} u_2) du_1 \wedge du_2 = \\ &= \int_\infty^{v_1} \int_\infty^{v_2} \sum_{\alpha,\beta \in C} \frac{\exp(-(\alpha+\beta)u_1 - (\overline{\alpha}+\overline{\beta})u_2)}{N(\alpha)} du_1 \wedge du_2 = \\ &= \sum_{\alpha,\beta \in C} \frac{\exp(-(\alpha+\beta)v_1 - (\overline{\alpha}+\overline{\beta})v_2)}{N(\alpha)N(\alpha+\beta)}. \end{aligned}$$

Similarly to the Dedekind double logarithm $f_{1,1}$, we define a multiple Dedekind polylogarithm

$$f_{1,2}(C, w_1, w_2) = \int_\infty^{w_1} \int_\infty^{w_2} f_{1,1}(C; v_1, v_2) dv_1 \wedge dv_2.$$

We can associate the following diagram to the multiple Dedekind polylogarithm $f_{1,2}$



The diagram represents the following: The variables of the integrant are t_1, t_2, u_1, u_2, v_1, v_2. The variables are subject to the conditions $t_1 > u_1 > v_1 > w_1$ and $t_2 > u_2 > v_2 > w_2$. The lower left function f_0 depends on the variables t_1 and t_2. And the middle function f_0 depends on u_1 and u_2. Thus, the diagram represents the following integral:

$$f_{1,2}(C; w_1, w_2) = \tag{5.14}$$
$$= \int_{D_{w_1,w_2}} (f_0(C; t_1, t_2) dt_1 \wedge dt_2) \wedge (f_0(C; u_1, u_2) du_1 \wedge du_2) \wedge (dv_1 \wedge dv_2),$$

where the domain of integration is

$$D_{w_1,w_2} = \{(t_1, t_2, u_1, u_2, v_1, v_2) \in \mathbb{R}^6 \mid t_1 > u_1 > v_1 > w_1 \text{ and } t_2 > u_2 > v_2 > w_2\}$$

A direct computation leads to

$$f_{1,2}(C; w_1, w_2) = \sum_{\alpha,\beta \in C} \frac{\exp(-(\alpha+\beta)w_1 - (\overline{\alpha} + \overline{\beta})w_2)}{N(\alpha)N(\alpha+\beta)^2}.$$

We define a multiple Dedekind zeta value as

$$\zeta_{\mathbb{Q}(i);C}(1,1;2,2) = f_{1,2}(C; 0, 0) = \sum_{\alpha,\beta \in C} \frac{1}{N(\alpha)N(\alpha+\beta)^2}.$$

Now let us give a relation between multiple Dedekind zeta values and iterated integrals. We use the following pair of inequalities in the following Sections $t_{1,0} > t_{1,1} > t_{1,2}$ and $t_{2,0} > t_{2,1} > t_{2,2}$, instead of $t_1 > u_1 > v_1 > w_1$ and $t_2 > u_2 > v_2 > w_2$, since using such notation it is easier to write higher order iterated integrals. In this notation, from Equation (5.11) , we obtain

$$f_2(C;t_{1,2},t_{2,2}) = \tag{5.15}$$
$$= \int_{t_{1,0}>t_{1,1}>t_{1,2};\ t_{2,0}>t_{2,1}>t_{2,2}} (f_0(C;t_{1,0},t_{2,0})dt_{1,0}\wedge dt_{2,0})\wedge(dt_{1,1}\wedge dt_{2,1}).$$

and

$$f_{1,1}(C;t_{1,2},t_{2,2}) = \tag{5.16}$$
$$= \int_{t_{1,0}>t_{1,1}>t_{1,2};\ t_{2,0}>t_{2,1}>t_{2,2}} (f_0(C;t_{1,0},t_{2,0})dt_{1,0}\wedge dt_{2,0})\wedge(f_0(C;t_{1,1},t_{2,1})dt_{1,1}\wedge dt_{2,1}).$$

In the next Section, we generalize the (iterated) integrals appearing in Equations (5.11) (5.14), (5.15), and (5.16), called iterated integrals over a membrane, (see Definition 61).

The next two Examples are needed in order to relate multiple Dedekind zeta values to values of Eisenstein series and values of multiple Eisenstein series (see [**?**], see also Examples 7, 8, 9 at the end of Section 5.3). The integrals below will present another type of iterated integral on a membrane, (see Definition 63) leading to a multiple Dedekind zeta values.

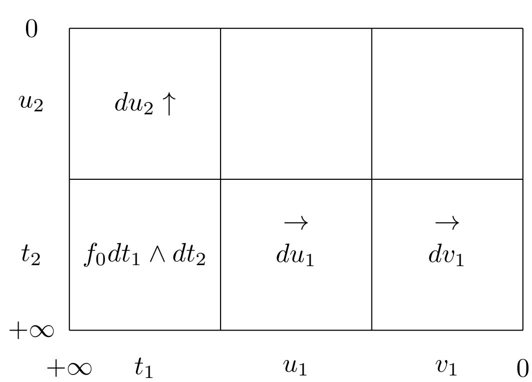
We define the following iterated integral to be a multiple Dedekind zeta value

$$\zeta_{\mathbb{Q}(i);C}(3;2) = \int_{t_1>u_1>v_1>0;\ t_2>u_2>0} f_0(C;t_1,t_2)dt_1\wedge du_1\wedge dv_1\wedge dt_2\wedge du_2. \tag{5.17}$$

The reason for such a definition is its infinite sum representation

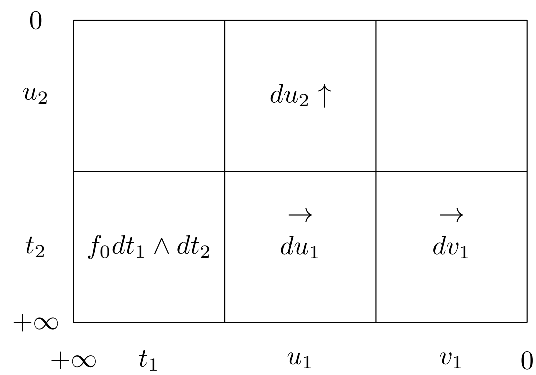
$$\zeta_{\mathbb{Q}(i);C}(3;2) = \sum_{\alpha\in C}\frac{1}{\alpha^3\overline{\alpha}^2}, \tag{5.18}$$

which can be achieved essentially in the same way as for the other multiple Dedekind zeta values. We can associate the following diagram to the integral representation of $\zeta_{\mathbb{Q}(i);C}(3;2)$ in Equation (5.17).

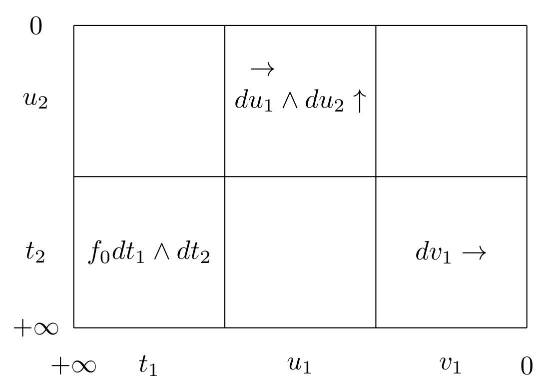


The arrows in the diagram signify the direction of decrease of the variables in differential 1-forms. It is important to consider the variables only in the horizontal direction and then only in vertical direction. In horizontal direction, we have $f_0(C; t_1, t_2)dt_1$ followed by du_1 and dv_1, The integration with respect to the variables t_1, u_1, v_1 leads to $1/\alpha^3$ in the summation of Equation (5.18). In vertical direction, we have a double iteration. First we have $f_0(C; t_1, t_2)dt_2$ followed by u_2. That leads to $1/\overline{\alpha}^2$ in the summation in Equation (5.18).

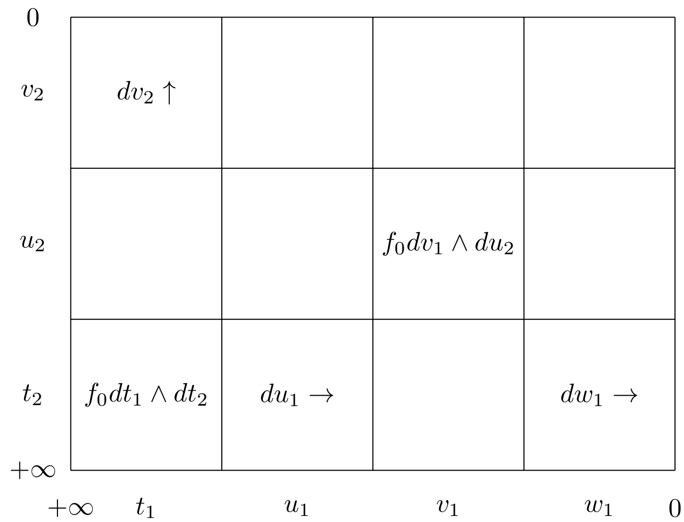
Diagrams associated to the integral representation of $\zeta_{\mathbb{Q}(i);C}(3; 2)$ (Equations (5.17)) are not unique. Alternatively, we could have used the diagrams



or



Consider the following diagram, associated to a more complicated MDZV, for the purpose of establishing notation.



The diagram encodes that $+\infty > t_1 > u_1 > v_1 > w_1 > 0$ and $+\infty > t_2 > u_2 > v_2 > 0$. Consider the horizontal direction of the diagram and Equation (5.8). We have $f_0 dt_1$, followed by du_1, $f_0 dv_1$ and dw_1. That gives an analogue of $\zeta(a,c) = \zeta(2,2)$ in horizontal direction for $(a,c) = (2,2)$. Consider the vertical direction of the diagram and Equation (5.7). We have $f_0 dt_2$, followed by $f_0 du_2$ and dv_2. That gives an analogue of $\zeta(b,d) = \zeta(1,2)$ in vertical direction, for $(b,d) = (1,2)$. We write

$$\zeta_{\mathbb{Q}(i);C,C}(a,b;c,d) = \zeta_{\mathbb{Q}(i);C,C}(2,1;2,2)$$

for the multiple Dedekind zeta function associated to the above diagram. We leave proof of the following statement is left to the reader

$$\zeta_{\mathbb{Q}(i);C,C}(2,1;2,2) = \sum_{\alpha,\beta \in C} \frac{1}{\alpha^2 \overline{\alpha}^1 (\alpha+\beta)^2 (\overline{\alpha}+\overline{\beta})^2}.$$

In Section 5.3, we use iterated integrals over a membrane to define multiple Dedekind zeta values associated to any number field.

5.2 Arithmetic and geometric tools

5.2.1 Iterated integrals on a membrane

Let D be a domain defined in terms of the real variables $t_{i,j}$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, by

$$D = \{(t_{1,1}, \dots, t_{n,m}) \in \mathbb{R}^{nm} \mid t_{i,1} > t_{i,2} > \cdots > t_{i,m} > 0 \text{ for } i = 1, \dots, n\}.$$

For each $j = 1, \dots, m$, let ω_j be a differential n-form on $\mathbb{C}^n$. Let

$$g : (0, +\infty)^n \to \mathbb{C}^n$$

be a smooth map, whose pull-back sends the coordinate-wise foliation on $\mathbb{C}^n$ to a coordinate-wise foliation on $(0, +\infty)^n$. We will call such a map *a membrane.* One should think of the n-forms $g^*\omega_j$ as an analogue of $f_0(C; t_1, t_2)dt_1 \wedge dt_2$ from Equation (5.9).

Definition 61 *An iterated integral on a membrane g, in terms of n-forms ω_j, $j = 1, \ldots, m$, is defined as*

$$\int_g \omega_1 \ldots \omega_m = \int_D \bigwedge_{j=1}^{m} g^*\omega_j(t_{1,j}, \ldots, t_{n,j}). \tag{5.19}$$

Definition 62 *A shuffle between two ordered sets*

$$S_1 = \{1, \ldots, p\}$$

and

$$S_2 = \{p+1, \ldots, p+q\}$$

is a permutation τ of the union $S_1 \cup S_2$, such that

1. *for $a, b \in S_1$, we have $\tau(a) < \tau(b)$ if $a < b$;*
2. *for $a, b \in S_2$, we have $\tau(a) < \tau(b)$ if $a < b$;*

We denote the set of shuffles between two ordered sets of orders p and q, respectively, by $Sh(p, q)$.

The definition of an iterated integral on a membrane is associated with the following objects:

1. $g : (0, +\infty)^n \to \mathbb{C}^n$, a membrane (that is a smooth map, whose pull-back sends the coordinate-wise foliation on $\mathbb{C}^n$ to a coordinate-wise foliation on $(0, +\infty)^n$).
2. $\omega_1, \ldots, \omega_m$ differential n-forms on $\mathbb{C}^n$;
3. m_i copies of differential 1-forms $\mathrm{d}z_i$ on $\mathbb{C}^n$, for $i = 1, \ldots, n$;
4. a shuffle $\tau_i \in Sh(m, m_i)$ for each $i = 1, \ldots, n$;
5. $\tau = (\tau_1, \ldots, \tau_n)$, the set of n shuffles $\tau_1, \ldots, \tau_n$.

Definition 63 *Given the above data, we define an iterated integral on a membrane g, involving n-forms and 1-forms, as*

$$\int_{(g,\tau)} \omega_1 \ldots \omega_m (\mathrm{d}z_1)^{m_1} \ldots (\mathrm{d}z_n)^{m_n} =$$

$$= \int_D \left(\prod_{j=1}^{m} g^*\omega_j(t_{1,\tau_1(j)}, \ldots, t_{n,\tau_n(j)}) \right) \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m+m_i} g^* dz_{i,j}, \tag{5.20}$$

where $t_{i,j} = g^* z_{i,j}$ and also $t_{i,j}$ belong to the domain

$$D = \{(t_{1,1}, \ldots, t_{n,m}) \in \mathbb{R}^{mn} \mid t_{i,1} > t_{i,2} > \cdots > t_{i,m+m_i} > 0\}.$$

Remark: Comparing the Definitions 61 and 63, one can notice that there is no sign occurring. The reason for that is the following:

1. In Definition 61 we use a domain D, whose coordinates are ordered by $t_{1,1}, \dots, t_{n,1},\ t_{1,2}, \dots, t_{n,2},\ \dots, t_{1,m}, \dots, t_{n,m}$. It is the same as the order of the differential 1-forms under the integral in Equation (5.19).

2. In Definition 63 we use a domain D, whose coordinates are ordered by $t_{1,1}, \dots, t_{1,m+m_1},\ t_{2,1}, \dots, t_{2,m+m_2},\ \dots, t_{n,1}, \dots, t_{n,m+m_n}$. It is the same as the order of the differential 1-forms under the integral in Equation (5.20).

Thus, if $m_1 = \dots = m_n = 0$, both definitions lead to the same value, since the permutation of the differential forms coincides with the permutation of the coordinates of the domain of integration. Thus, the change of orientation of the domain of integration coincides with the sign of permutation acting on the differential forms.

Theorem 64 *(homotopy invariance) The iterated integrals on membranes from Definition 63 are homotopy invariant, when the homotopy preserves the boundary of the membrane.*

Proof. Let g be a homotopy between the two membranes g_0 and g_1. Let

$$\Omega = \left(\prod_{j=1}^{m} \omega_j(z_{1,\tau_1(j)}, \dots, z_{n,\tau_n(j)}) \right) \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m+m_i} dz_{i,j}$$

Note that Ω is a closed form, since ω_i is a form of top dimension and since $dz_{i,j}$ is closed. By Stokes Theorem, we have

$$\begin{aligned} 0 &= \int_{s=0}^{s=1} \int_D g^* d\Omega = \\ &= \int_{(g_1,\tau)} \Omega - \int_{(g_0,\tau)} \Omega \pm && (5.21) \\ &\pm \int_{s=0}^{s=1} \sum_{i=1}^{n} \sum_{j=1}^{m+m_i} \int_{D|(z_{i,j}=z_{i,j+1})} g^* \Omega \pm && (5.22) \\ &\pm \int_{s=0}^{s=1} \sum_{i=1}^{n} \int_{D|(z_{i,m+m_i}=0)} g^* \Omega && (5.23) \end{aligned}$$

We want to show that the difference in the terms in (6.4) is zero. It is enough to show that each of the terms (6.6) and (6.7) are zero. If $z_{i,j} = z_{i,j+1}$, then the wedge of the corresponding differential forms will vanish. Thus the terms in (6.6) are zero. If $z_{i,m+m_i} = 0$ then $dt_{i,m+m_i} = 0$, defined via the pull-back g^*. Then the terms (6.7) are equal to zero.

5.2.2 Cones and geometric series

Let $n = [K : \mathbb{Q}]$ be the degree of the number field K over $\mathbb{Q}$. Let $\mathcal{O}_K$ be the ring of integers in K. And let U_k be the group of units in K. We are going to use an idea of Shintani [**?**] by examining Dedekind zeta functions in terms of a cone inside the ring of integers.

We define cone for any number ring. The meaning of cones is roughly the following: summation over the elements of finitely many cones would give multiple Dedekind zeta values or multiple Dedekind zeta functions.

Definition 65 *We define a cone C to be*

$$C = \mathbb{N}\{e_1, \ldots, e_k\} = \{\alpha \in \mathcal{O}_K \mid \alpha = a_1e_1 + \cdots a_ke_k \textit{ for } e_i \in \mathcal{O}_K \textit{ and } a_i \in \mathbb{N}\}$$

with generators $e_1, \ldots, e_k$.

For the next definition, we are going to use that a number field K can be viewed as an n-dimensional vector space over the rational numbers $\mathbb{Q}$.

Definition 66 *An unimodular cone is a cone with generators $e_1, \ldots, e_k$ such that $e_1, \ldots, e_k$ as elements of K are linearly independent over $\mathbb{Q}$, when we view the field K as a vector space over $\mathbb{Q}$.*

Note that if C is an unimodular cone then $0 \notin C$, since $e_1, \ldots, e_k$ are linearly independent over $\mathbb{Q}$ and the coefficients $a_1, \ldots, a_n$ are positive integers.

Definition 67 *We call C an unimodular simple cone if for any embedding σ_i of K into the complex numbers, $\sigma_i : K \to \mathbb{C}$ and a suitable branch of the functions* $\arg(z)$*, we have that the closure of the set* $\arg(\sigma(C))$ *is an interval $[\theta_0, \theta_1]$, such that its lengths is less than π, namely, $\theta_1 - \theta_0 \in [0, \pi)$.*

In particular, the cone

$$C = \{\alpha \in \mathbb{Z}[i] \mid \alpha = a(1 + i) + b(1 + i), a, b \in \mathbb{N}\},$$

considered in Subsection 5.1.2, is an unimodular simple cone, since $\arg(\sigma_1(\alpha)) \in (-\pi/4, \pi/4)$ and $\arg(\sigma_2(\alpha)) \in (-\pi/4, \pi/4)$, for each $\alpha \in C$. The maps σ_1 and σ_2 are complex conjugates of each other.

Definition 68 *(Dual cone) For an unimodular simple cone C with generators $e_1, \ldots, e_k$, we define a dual cone of C to be*

$$C^* = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid Re(z_i\sigma_i(e_j)) > 0 \textit{ for } i = 1, \ldots, n, \textit{ and } j = 1 \ldots, k\}$$

Clearly, if C is an unimodular simple cone then the dual cone C^* is a non-empty set. One can prove that by considering each coordinate of C^*, separately.

Definition 69 *For an unimodular simple cone C, we define a function*

$$f_0(C; z_1, \ldots, z_n) = \sum_{\alpha \in C} \exp(-\sum_{i=1}^{n} \sigma_i(\alpha) z_i), \tag{5.24}$$

where $\sigma_1, \ldots, \sigma_n$ are all embeddings of the number field K into the complex numbers $\mathbb{C}$ and the domain of the function f_0 is the dual cone C^*.

Lemma 70 *The function f_0 is uniformly convergent for $(z_1, \ldots, z_n)$ in any compact subset B of the dual cone C^* of an unimodular simple cone C.*

Proof. From the Definition 68, we have $\mathrm{Re}(\sigma_i(e_j)z_i) > 0$. Let

$$y_j = \prod_{i=1}^{n} \exp(-\sigma_i(e_j) z_i). \tag{5.25}$$

Then $|y_j| < 1$ on the domain B. Moreover, $|y_i|$ achieves a maximum on the compact subset B. Let $|y_j| \leq c_j < 1$ on the compact set B for some constant c_j. Then the rate of convergence of the geometric sequence in y_j is uniformly bounded by c_j on the compact set B. Therefore, we have a uniform convergence of the geometric series in y_j. The function $f_0(C; z_1, \ldots, z_n)$ is a product of k geometric series in the variables $y_1, \ldots, y_k$ each of which is uniformly bounded in absolute value by the constants $c_1, \ldots, c_k$ on the domain B, respectively. Then, we obtain that

$$f_0(C; z_1, \ldots, z_n) = \prod_{j=1}^{k} \frac{y_j}{1 - y_j}. \tag{5.26}$$

Corollary 71 *The function $f_0(C; z_1, \ldots, z_n)$ has analytic continuation to all values of $z_1, \ldots, z_n$, except at*

$$\sum_{i=1}^{n} \sigma_i(e_j) z_i \in 2\pi i \mathbb{Z},$$

for $j = 1, \ldots, k$.

Proof. Using the product formula 5.26 in terms of geometric series in y_j, we see that the right hand side of 5.26 makes sense for all $y_j \neq 1$. This gives analytic continuation from the domain C^* to the domain consisting of points $(y_1, \ldots, y_n)$ with $y_i \neq 1$.

Definition 72 *(positive cone) We call C a positive unimodular simple cone if C is an unimodular simple cone and the product of the positive real coordinates is in C^*, namely*

$$(\mathbb{R}_{>0})^n \subset C^*$$

as subsets of $\mathbb{C}^n$.

Lemma 73 *If C is an unimodular simple cone then for some $\alpha \in \mathcal{O}_K$ we have that*
$$\alpha C = \{\alpha\beta \mid \beta \in C\}$$
is a positive unimodular simple cone.

Proof. In order to find such an elements α, we need to recall properties of real or complex embeddings of a number field K.

The degree of a number field $n = [K : \mathbb{Q}]$ is the dimension of K as vector space over $\mathbb{Q}$. Then there are exactly n distinct embeddings $K \to \mathbb{C}$. Let the first r_1 embeddings, $\sigma_1, \ldots, \sigma_{r_1}$, be the ones whose image is inside the real numbers. They are called real embeddings. Let the next r_2 embeddings be complex embeddings, which are not pair-wise complex conjugates of each other. Let us denote them by $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$. Let the last r_2 embeddings be the complex conjugates of previously counted complex embeddings, namely,
$$\sigma_{r_1+r_2+i}(\beta) = \overline{\sigma_{r_1+i}(\beta)},$$
for $i = 1, \ldots, r_2$. We also have that $n = r_1 + 2r_2$.

Let $V_{\mathbb{R}}$ be a n dimensional real vector subspace of $\mathbb{C}^n$ defined in the following way:
$$\begin{aligned} V_{\mathbb{R}} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid\ & (z_1, \ldots, z_{r_1}) \in \mathbb{R}^{r_1}, \\ & (z_{r_1+1}, \ldots, z_{r_1+r_2}) \in \mathbb{C}^{r_2}, \text{ and} \\ & (z_{r_1+r_2+1}, \ldots, z_{r_1+2r_2}) = (\overline{z}_{r_1+1}, \ldots, \overline{z}_{r_1+r_2})\} \end{aligned}$$
Now, we proceed with the proof of the Lemma in six Steps.

Step 1. K is dense in $V_{\mathbb{R}}$.
Step 2. $V_{\mathbb{R}} \cap C^*$ in non-empty.
Step 3. $V_{\mathbb{R}} \cap C^*$ is an open subset of $V_{\mathbb{R}}$.
Step 4. $K \cap C^*$ is non-empty.
Step 5. $\mathcal{O}_K \cap C^*$ is non-empty.
Step 6. αC is a positive unimodular simple cone for any $\alpha \in \mathcal{O}_K \cap C^*$.

(Step 1) Recall also the product of the n embeddings of K to the space $V_{\mathbb{R}}$,
$$\prod_{i=1}^{n} \sigma_i : K \to V_{\mathbb{R}},$$
mapping $\beta \in K$ to $(\sigma_1(\beta), \ldots, \sigma_n(\beta)) \in V_{\mathbb{R}}$ has a dense image.

(Step 2) Indeed, let z_i be the i-th coordinate of C^*. The first r_1 coordinates $z_1, \ldots, z_{r_1}$ of C^* can be real numbers (positive or negative), since $\sigma_i(K) \subset \mathbb{R}$ for $i = 1, \ldots, r_1$. Thus, the first r_i coordinates can be both in $V_{\mathbb{R}}$ and in C^*. For the coordinates $z_{r_1+1}, \ldots, z_{r_1+r_2}$ of C^* there are no restrictions when we intersect C^* with $V_{\mathbb{R}}$. For the last r_2 coordinates of C^* we must have that $z_{r_1+r_2+i} = \overline{z}_{r_1+i}$ in order for the coordinates to be in the intersection $C^* \cap V_{\mathbb{R}}$. Since, $\sigma_{r_1+r_2+i}(\beta) = \overline{\sigma_{r_1+i}(\beta)}$, we have the conditions on the (r_1+i)-coordinate and on the $(r_1 + r_2 + i)$-coordinate of a point in C^* to be in $V_{\mathbb{R}}$ are
$$Re(z_{r_1+i}\sigma_{r_1+i}(\beta)) > 0,$$

for $\beta \in C$ and $z_{r_1+r_2+i} = \overline{z}_{r_1+i}$. The last condition implies that

$$Re(z_{r_1+r_2+i}\sigma_{r_1+r_2 i}(\beta)) = Re(\overline{z_{r_1+i}\sigma_{r_1+i}(\beta)}) > 0.$$

Thus, such a point $(z_1, \ldots, z_n)$ is in $C^* \cap V_{\mathbb{R}}$.

(Step 3) It is true, since C^* is an open subsets of $\mathbb{C}^n$

(Step 4) Since K is dense in $V_{\mathbb{R}}$ (Step 1) and $V_{\mathbb{R}} \cap C^*$ is open in $V_{\mathbb{R}}$ (Steps 2 and 3), we have that $K \cap C^*$ is non-empty.

(Step 5) If $\alpha \in K \cap C^*$ then for some positive integer L, we have that $L\alpha \in \mathcal{O}_K$, and also, $L\alpha \in C^*$, since C^* is invariant under rescaling by a positive (real) number L.

(Step 6) Let $(t_1, \ldots, t_n) \in \mathbb{R}^n_{>0}$ and let $\beta \in C$. Put $z_i = t_i\sigma_i(\alpha)$. Then $(z_1, \ldots, z_n) \in C^*$.

$$Re(t_i\sigma_i(\alpha\beta)) = Re(t_i\sigma_i(\alpha)\sigma_i(\beta)) = Re(z_i\sigma_i(\beta)) > 0.$$

5.2.3 Cones and ideals

In this Subsection, we are going to examine union of cones that give a fundamental domain of the ring of integers $\mathcal{O}_K$ modulo the group of units U_K. We also examine a fundamental domain of an ideal $\mathfrak{a}$ modulo the group of units U_K.

Definition 74 *We define M as a fundamental domain of*

$$\mathcal{O}_K - \{0\} \bmod U_k.$$

For an ideal $\mathfrak{a}$, let

$$M(\mathfrak{a}) = M \cap \mathfrak{a}.$$

Lemma 75 *For any ideal $\mathfrak{a}$ the set $M(\mathfrak{a})$ can be written as a finite disjoint union of unimodular simple cones.*

Proof. It is a simple observation that $M(\mathfrak{a})$ can be written as a finite union of unimodular cones. We have to show that we can subdivide each of the unimodular cones into finite union of unimodular simple cones.

Let $\sigma_1, \ldots, \sigma_{r_1}$ be the real embeddings of the number field K and let $\sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$ be the non-conjugate complex embeddings of K. We define

$$T = \{-1, 1\}^{r_1} \times (S^1)^{r_2}.$$

Let C be an unimodular cone. We define a map J by

$$\begin{aligned} J &: C \to T \\ \alpha &\mapsto \left(\frac{\sigma_1(\alpha))}{|\sigma_1(\alpha))|}, \cdots, \frac{\sigma_{r_1+r_2}(\alpha))}{|\sigma_{r_1+r_2}(\alpha))|} \right) \end{aligned}$$

Denote it by $\bar{C}$ the closure of the image of J in T. Then one can cut the cone C into finitely many cones C_i such that for C_i and any embedding σ of K into $\mathbb{C}$, we have that $\arg(\sigma(C)) \in [\theta_0, \theta_1]$, for $\theta_1 - \theta_0 \in [0, \pi)$. Then C_i is an unimodular simple cone. Thus, the cones C_i's are finitely many unimodular simple cones, whose (disjoint) union gives the set $M(\mathfrak{a})$.

For an element α in a ring of integers $\mathcal{O}_K$, denote by (α) the principal ideal generated by α. Then $N_{K/\mathbb{Q}}((\alpha))$ denoted the norm of the principal ideal generated by α. We have that $N_{K/\mathbb{Q}}((\alpha))$ is a positive integer equal to the number of elements in the quotient $\mathcal{O}_K/(\alpha)$. Also $N_{K/\mathbb{Q}}(\alpha)$ is the norm of the algebraic number α. This is equal to the product of all of its Galois conjugates, which is an integer, possibly a negative integer. We always have that $N_{K/\mathbb{Q}}((\alpha)) = |N_{K/\mathbb{Q}}(\alpha)|$.

However, for elements of an unimodular simple cone, we can say more.

Lemma 76 *Let C be an unimodular simple cone. Then for every $\alpha \in C$, we have*

$$N_{K/\mathbb{Q}}((\alpha)) = \epsilon(C) N_{K/\mathbb{Q}}(\alpha),$$

where $\epsilon(C) = \pm 1$ depends only on the cone C, not on α.

Proof. Note that on the left we have a norm of an ideal and on the right we have a norm of a number. Since C is a simple cone, we have that for all real embeddings $\sigma : K \to \mathbb{R}$, the signs of $\sigma(\alpha)$ and $\sigma(\beta)$ are the same for all α and β in C. Let ϵ_σ be the sign of $\sigma(\alpha)$ for each real embedding σ. Then the product over all real embeddings of ϵ_σ is equal to $\epsilon(C)$.

5.3 Multiple Dedekind zeta functions

5.3.1 Dedekind polylogarithms

Let us recall the Dedekind zeta values

$$\zeta_K(m) = \sum_{\mathfrak{a} \neq (0)} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^m},$$

where $\mathfrak{a}$ is an ideal in $\mathcal{O}_K$.

We are going to express the summation over elements, which belong to a finite union of positive unimodular simple cones. We will define a Dedekind polylogarithm associates to a positive unimodular simple cone. The key result in this subsection will be that a Dedekind zeta value can be expressed as a $\mathbb{Q}$ linear combination of values of the Dedekind polylogarithms.

We also define a partial Dedekind zeta function by summing over ideals in a given ideal class $[\mathfrak{a}]$

$$\zeta_{K,[\mathfrak{a}]}(m) = \sum_{\mathfrak{b} \in [\mathfrak{a}]} N_{K/\mathbb{Q}}(\mathfrak{b})^{-m},$$

Let us consider a partial Dedekind zeta functions $\zeta_{K,[\mathfrak{a}]^{-1}}(m)$, corresponding to an ideal class $[\mathfrak{a}]^{-1}$, where $\mathfrak{a}$ is an integral ideal. For every integral ideal $\mathfrak{b}$ in the class $[\mathfrak{a}]^{-1}$, we have that

$$\mathfrak{a}\mathfrak{b} = (\alpha),$$

where $\alpha \in \mathfrak{a}$. Then

$$N_{K/\mathbb{Q}}(\mathfrak{b}) = N_{K/\mathbb{Q}}(\mathfrak{a})^{-1} N_{K/\mathbb{Q}}((\alpha)).$$

Let

$$M(\mathfrak{a}) = \bigcup_{i=1}^{n(\mathfrak{a})} C_i(\mathfrak{a}),$$

where $n(\mathfrak{a})$ is a positive integer and $C_i(\mathfrak{a})$'s are unimodular simple cones. Let α_i be an element of the intersection of $\mathcal{O}_K$ with the dual cone $C_i(\mathfrak{a})^*$, then $\alpha_i C_i(\mathfrak{a})$ is a positive unimodular simple cone (see Lemma 73).

Then,

$$\zeta_{K,[\mathfrak{a}]^{-1}}(m) = \sum_{\mathfrak{b}\in[\mathfrak{a}]^{-1}} N_{K/\mathbb{Q}}(\mathfrak{b})^{-m} =$$

$$= N_{K/\mathbb{Q}}(\mathfrak{a})^m \sum_{i=1}^{n(\mathfrak{a})} \epsilon(C_i(\mathfrak{a}))^m N(\alpha_i)^m \sum_{\alpha\in\alpha_i C_i(\mathfrak{a})} N_{K/\mathbb{Q}}(\alpha)^{-m}, \quad (5.27)$$

where $\epsilon(C_i(\mathfrak{a})) = \pm 1$, depending on the cone, $N_{K/\mathbb{Q}}(\mathfrak{a})$ is a norm of the ideal $\mathfrak{a}$ and $N(\alpha_i)$ is the norm of the algebraic integer α_i.

We are going to give an example of higher dimensional iteration in order to illustrate the usefulness of this procedure. For a positive unimodular simple cone C, we define

$$f_m(C; u_1, \dots, u_n) = \int_\infty^{u_1} \dots \int_\infty^{u_n} f_{m-1}(C; t_1, \dots, t_n) \mathrm{d}t_1 \wedge \dots \wedge \mathrm{d}t_n,$$

where $t_i \in (u_i, +\infty)$. This is an iteration, giving the simplest type of iterated integrals on a membrane. We start the induction on m from $m = 0$. Recall that f_0 was introduced in Definition 69.

Note that a norm of an algebraic number α can be expresses as a product of its embeddings in the complex numbers $\sigma_1(\alpha), \dots, \sigma_n(\alpha)$.

$$N_{K/\mathbb{Q}}(\alpha) = \sigma_1(\alpha) \dots \sigma_n(\alpha).$$

Integrating term by term, we can express f_m as an infinite sum

$$f_m(C; t_1, \dots, t_n) = \sum_{\alpha\in C} \frac{\exp(-\sum_{i=1}^n \sigma_i(\alpha) t_i)}{N_{K/\mathbb{Q}}(\alpha)^m}.$$

Note that a cone C is a linear combination of its generators so that the coefficients of the generators are positive integers. In particular, 0 is not an element of an unimodular simple cone C, since then the generators are linearly independent over $\mathbb{Q}$. Thus, there is no division by 0.

Definition 77 *We define an m-th Dedekind polylogarithm, associated to a number field K and a positive unimodular simple cone C, to be*

$$Li_m^K(C; X_1, \dots, X_n) = f_m(C; -\log(X_1), \dots, -\log(X_n)).$$

Theorem 78 *Dedekind zeta value at $s = m > 1$ can be written as a finite $\mathbb{Q}$-linear combination of Dedekind polylogarithms evaluated at $(X_1, \ldots, X_n) = (1, \ldots, 1)$.*

Proof. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ are integral ideals in $\mathcal{O}_K$, representing all the ideal classes, then using Equation (5.27), we obtain

$$\zeta_K(m) = \sum_{j=1}^{h} N_{K/\mathbb{Q}}(\mathfrak{a}_j)^m \sum_{i=1}^{n(\mathfrak{a})_j} \epsilon(C_i(\mathfrak{a}_j))^m N(\alpha_{i,j})^m f_m(\alpha_{i,j}C_i(\mathfrak{a}_j), 0, \ldots, 0),$$

where $C_i(\mathfrak{a}_j)$ are unimodular simple cones such that

$$\bigcup_{i=1}^{n(\mathfrak{a})_j} C_i(\mathfrak{a}_j) = M(\mathfrak{a}_\mathrm{j})$$

and $\epsilon(C_i(\mathfrak{a}_j)) = \pm 1$, depending on the cone (see Definition 74 and Lemma 75). Let $\alpha_{i,j} \in C_i(\mathfrak{a}_j)^* \cap \mathcal{O}_K$ be an algebraic integer in the dual cone of $C_i(\mathfrak{a}_j)$. Then by Lemma 73 we have that $\alpha_{i,j}C_i(\mathfrak{a}_j)$ is a positive unimodular simple cone. The iterated integrals are hidden in the functions f_m. Consider Definition 61 with differential forms

$$\omega_1 = f_0(\alpha_{i,j}C_i(\mathfrak{a}_j); z_1, \ldots, z_n)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

$$\omega_2 = \omega_3 = \cdots = \omega_m = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

And let g be inclusion of $(0, \infty)^n$ in $\mathbb{C}^n$. Then the corresponding iterated integral on a membrane gives

$$f_m(\alpha_{i,j}C_i(\mathfrak{a}_j); t_1, \ldots, t_n).$$

5.3.2 Multiple Dedekind zeta values

We recall an integral representation of a multiple zeta value

$$\zeta(k_1, k_2, \ldots, k_m) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}$$

has the following integral representation (see for example [**?**]):

$$\begin{aligned}\zeta(k_1, k_2, \ldots, k_m) = &\int_{0<x_1<\cdots<x_{k_1+\cdots+k_m}} \frac{dx_1}{1-x_1} \wedge \left(\frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_{k_1}}{x_{k_1}}\right) \wedge \\ &\wedge \frac{dx_{k_1+1}}{1-x_{k_1+1}} \wedge \left(\frac{dx_{k_1+2}}{x_{k_1+2}} \wedge \cdots \wedge \frac{dx_{k_1+k_2}}{x_{k_1+k_2}}\right) \wedge \\ &\cdots \\ &\wedge \frac{dx_{k_1+\cdots+k_{m-1}+1}}{1-x_{k_1+\cdots+k_{m-1}+1}} \wedge \left(\frac{dx_{k_1+\cdots+k_{m-1}+2}}{x_{k_1+\cdots+k_{m-1}+2}} \wedge \cdots \wedge \frac{dx_{k_1+\cdots+k_m}}{x_{k_1+\cdots+k_m}}\right)\end{aligned} \tag{5.28}$$

Note that there are essentially two types of differential 1-forms under the integral: $dx/(1-x)$ and dx/x. If we set $x_i = e^{-t_i}$, then we obtain the following formula, needed for the generalization to multiple Dedekind zeta values:

$$\begin{aligned}\zeta(k_1, k_2, \ldots, k_m) = \int_{t_1 > \cdots > t_{k_1+\cdots+k_m} > 0} & \frac{dt_1}{e^{t_1}-1} \wedge (dt_2 \wedge \cdots \wedge dt_{k_1}) \wedge \\ & \wedge \frac{dt_{k_1+1}}{e^{t_{k_1+1}}-1} \wedge (dt_{k_1+2} \wedge \cdots \wedge dt_{k_1+k_2}) \wedge \\ & \cdots \\ & \wedge \frac{dt_{k_1+\cdots+k_{m-1}+1}}{e^{x_{k_1+\cdots+k_{m-1}+1}}-1} \wedge \left(dt_{k_1+\cdots+k_{m-1}+2} \wedge \cdots \wedge dt_{k_1+\cdots+k_m}\right)\end{aligned} \tag{5.29}$$

Note that in Equation (5.29), we have used m copies of $dt/(e^t-1)$. To find their order, first we shuffle a set S with m elements (corresponding to m copies of $dt/(e^t-1)$) with another set S_1 consisting of $m_1 = -m+k_1+\cdots+k_m$ elements (corresponding to m_1 copies of dt). We choose a shuffle $\tau_1 \in Sh(m, m_1)$, such that $\tau_1(1) = 1$. The reason is that the first differential form in the iterated integral in Equation 5.28 has to be $dx/(1-x)$, which is needed for convergence. The corresponding 1 forms in Equation (5.29) is $dt/(e^t-1)$ The relation between the shuffle τ_1 and the set of integers $k_1, \ldots, k_m$ is the following:

$$\begin{aligned} 1 &= \tau_1(1) \\ k_1 + 1 &= \tau_1(2) \\ k_1 + k_2 + 1 &= \tau_1(3) \\ &\cdots \\ k_1 + \cdots + k_{m-1} + 1 &= \tau_1(m) \\ k_1 + \cdots + k_{m-1} + k_m &= \text{number of differential 1-forms} \end{aligned}$$

The integers $1, k_1+1, k_1+k_2+1, \ldots, k_1+\cdots+k_{m-1}+1$, are the values of the index i, where the analogue of the form $dt_i/(e^{t_i}-1)$ appears under the integral, not the form dt_i. (see Equation (5.29))

In order to define multiple Dedekind zeta values, we will use n shuffles of pairs of ordered sets, where $n = [K : \mathbb{Q}]$ is the degree on the number field.

Let $m_1, \ldots, m_n$ be positive integers. (The positive integer m_i will denote the number of times the differential form $\mathrm{d}z_i$ occurs.) We define the following ordered sets:

$$S = \{1, 2, \ldots, m\},$$

$$S_i = \{m+1, \ldots, m+m_i\},$$

Definition 79 *Denote by $Sh^1(p, q)$ the subset of all shuffles $\tau \in Sh(p, q)$ of the two sets $\{1, \ldots, p\}$ and $\{p+1, \ldots, p+q\}$ such that $\tau(1) = 1$*

For the definition of multiple Dedekind zeta values at the positive integers, we use Definition 63, where we take the n-forms to be

$$\omega_j = f_0(C_j, z_1, \ldots, z_n)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

for $j = 1, \ldots, m$, where $C_1, \ldots, C_m$ are positive unimodular simple cones, and the 1-forms to be $\mathrm{d}z_i$ on $\mathbb{C}^n$ occurring m_i times for $i = 1, \ldots, n$.

Definition 80 *(Multiple Dedekind zeta values) For each $i = 1, \ldots, n$, let $\tau_i \in Sh^1(m, m_i)$. We define the integers $k_{i,j}$ and m_i in terms of the shuffle τ_i via the following relations*

$$1 = \tau_i(1) \tag{5.30}$$

$$k_{i,1} + 1 = \tau_i(2) \tag{5.31}$$

$$k_{i,1} + k_{i,2} + 1 = \tau_i(3) \tag{5.32}$$

$$\cdots$$

$$k_{i,1} + \cdots + k_{i,m-1} + 1 = \tau_i(m) \tag{5.33}$$

$$k_{i,1} + \cdots + k_{i,m-1} + k_{i,m} = m + m_i \tag{5.34}$$

We define multiple Dedekind zeta values at the positive integers by

$$\zeta_{K;C_1,\ldots C_m}(k_{1,1}, \ldots, k_{1,m}; \ldots; k_{n,1}, \ldots, k_{n,m}) = \int_{(g,\tau)} \omega_1 \ldots \omega_m (\mathrm{d}z_1)^{m_1} \ldots (\mathrm{d}z_n)^{m_n}$$

Theorem 81 *For the general form of a multiple Dedekind zeta value, we need: a number field K; positive unimodular simple cones $C_1, \ldots, C_m$ in $\mathcal{O}_K$; elements $\alpha_j \in C_j$ for $j = 1, \ldots, m$; complex embeddings of the elements $\alpha_{i,j} = \sigma_i(\alpha_j)$. Then a multiple Dedekind zeta value has the following representation as an infinite sum*

$$\zeta_{K;C_1,\ldots C_m}(k_{1,1}, \ldots, k_{1,m}; \ldots; k_{n,1}, \ldots, k_{n,m}) =$$

$$= \sum_{\alpha_1 \in C_1} \cdots \sum_{\alpha_m \in C_m} \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_{i,1} + \cdots + \alpha_{i,j})^{-k_{i,j}}. \tag{5.35}$$

Proof. There are n different embedding $\sigma_1, \ldots, \sigma_n$ of K into $\mathbb{C}$. Given $k_{i,1}, \ldots, k_{i,m}$ we find m_i using Equation (5.34). Then we find τ_i by the values at $1, 2, \ldots, m$ obtained from Equations (5.31), (5.32), (5.33).

Now we use Definition 80 of a multiple Dedekind zeta value in terms of iterated integrals on a membrane from Definition 63. We are going to follow closely Equation (5.29). The variable $t_{1,1}$ enters as a variable of the function $f_0(C_1; \cdots)$, the variables $t_{1,2}, \ldots, t_{1,k_{1,1}}$ appear as differential 1-forms $dt_{1,2}, \ldots, dt_{1,k_{11}}$, since $\tau_1(2) = k_{1,1} + 1$ (see Equation (5.31)). Recall that $\sigma_1 : K \to \mathbb{C}$ is an embedding of K into the complex numbers and $\alpha_{1,j} = \sigma_1(\alpha_j)$. Thus integrating with respect to $t_{1,1}, \ldots, t_{1,k_{1,1}}$ gives us denominators $\alpha_{1,1}^{k_{1,1}}$, associated to each $\alpha_1 \in C_1$ Then $t_{1,k_{1,1}+1}$ enters as a variable in the function $f_0(C_2; \cdots)$, since

$\tau_1(2) = k_{1,1} + 1$. Then the variables $t_{1,k_{1,1}+2}, \ldots, t_{1,k_{1,1}+k_{1,2}}$ appear as differential 1-forms $dt_{1,k_{1,1}+2}, \ldots, dt_{1,k_{1,1}+k_{1,2}}$, since $\tau_1(3) = k_{1,1}+k_{1,2}+1$ (see Equation (5.32)). Thus integrating with respect to $t_{1,1}, \ldots, t_{1,k_{1,1}}$ gives us a denominators $\alpha_{1,1}^{k_{1,1}}(\alpha_{1,1}+\alpha_{1,2})^{k_{1,2}}$, associated to each $\alpha_1 \in C_1$ and each $\alpha_2 \in C_2$. Continuing this process to the variable $t_{1,m+m_1}$, we obtain a denominator

$$\prod_{j=1}^{m} (\alpha_{1,1} + \cdots + \alpha_{1,j})^{k_{1,j}},$$

associated to each m-tuple $(\alpha_1, \ldots, \alpha_m)$, where $\alpha_j \in C_j$. There are n different embeddings $\sigma_1, \ldots, \sigma_n$ of K into $\mathbb{C}$, where $n = [K : \mathbb{Q}]$ is the degree of the number field. So far we have considered the contribution of the first embedding. The contribution of the first and the second embedding is obtained in essentially the same way as for the first embedding. It gives a denominator

$$\prod_{j=1}^{m} (\alpha_{1,1} + \cdots + \alpha_{1,j})^{k_{1,j}} (\alpha_{2,1} + \cdots + \alpha_{2,j})^{k_{2,j}},$$

associated to each m-tuple $(\alpha_1, \ldots, \alpha_m)$, where $\alpha_j \in C_j$. Similarly, after integrating with respect to all the variables $t_{i,j}$ we obtain a denominator

$$\prod_{i=1}^{n}\prod_{j=1}^{m} (\alpha_{i,1} + \cdots + \alpha_{i,j})^{k_{i,j}},$$

associated to each m-tuple $(\alpha_1, \ldots, \alpha_m)$, where $\alpha_j \in C_j$. Then the numerators are all equal to 1 since the lower bound for the variables under the exponents in $f_0(C_j; \cdots)$ is 0. Thus, the exponents become equal to 1.

The following examples of MDZV give analogues of Dedekind zeta function and of (multiple) Eisenstein-Kronecker series.

Examples: 1. Let C be a positive unimodular simple cone in the ring of integers $\mathcal{O}_K$ of a number filed K. In $m = 1$ and if all values $k_{i,1}$ are equal to k, then

$$\zeta_{K;C}(k, \ldots, k) = \sum_{\alpha \in C} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^k}.$$

Note that the number 0 does not belong to any unimodular simple cone C.

2. Let $m = 2$, and let

$$k = k_{1,1} = \cdots = k_{n,1}$$

and

$$l = k_{1,2} = \cdots = k_{n,2}$$

be positive integers greater that 1. Finally, let C_1 and C_2 be positive unimodular simple cones in the ring of integers $\mathcal{O}_K$ of a number field K. Then the corresponding multiple Dedekind zeta value can be written both as a sum and as an integral:

$$\zeta_{K;C_1,C_2}(k,\ldots,k;l,\ldots,l)=\sum_{\alpha\in C_1,\beta\in C_2}\frac{1}{N_{K/\mathbb{Q}}(\alpha)^k N_{K/\mathbb{Q}}(\alpha+\beta)^l} \tag{5.36}$$

3. Let K be an imaginary quadratic field. Let C be a positive unimodular simple cone in $\mathcal{O}_K$. We can represent the cone C as an $\mathbb{N}$-module: $C=\mathbb{N}\{\mu,\nu\}$, for $\mu,\nu\in\mathcal{O}_K$. Put $z=\mu/\nu$. Consider then

$$\zeta_{K;C}(k,k)=\sum_{\alpha\in C}\frac{1}{N(\alpha)^k}=|\nu|^{-2k}\sum_{a,b\in\mathbb{N}}\frac{1}{|az+b|^{2k}},$$

where the last sum is a portion of the k-th Eisenstein-Kronecker series. Such series could be found in [?]

$$E_k(z)=\sum_{a,b\in\mathbb{Z};(a,b)\neq(0,0)}\frac{1}{|az+b|^{2k}}.$$

4. With the notation of Example 3, we obtain an analogue of values of multiple Eisenstein-Kronecker series

$$\zeta_{K;C,C}(k,k;l,l)=|\nu|^{-k-l}\sum_{a,b,c,d\in\mathbb{N}}\frac{1}{|az+b|^{2k}|(a+c)z+(b+d)|^{2l}}.$$

An alternative generalization was considered in [?], Section 8.2.

5.3.3 Multiple Dedekind zeta functions

We will try to give some intuition behind the integral representation of the multiple zeta functions (see in [?]). After that we will generalize the construction to define the number field analogues - multiple Dedekind zeta functions. In order to do that, we give two examples - one for $\zeta(3)$ and another for $\zeta(1,3)$.

We have

$$\begin{aligned}\zeta(3)&=\int_{0<x_1<x_2<x_3<1}\frac{\mathrm{d}x_1}{1-x_1}\wedge\frac{\mathrm{d}x_2}{x_2}\wedge\frac{\mathrm{d}x_3}{t_3}=\\&=\int_{t_1>t_2>t_3>0}\frac{\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\mathrm{d}t_3}{e^{t_1}-1}=\\&=\int_0^\infty\frac{t_1^2\mathrm{d}t_1}{\Gamma(3)(e^{t_1}-1)}.\end{aligned}$$

The first equality is due to Kontsevich. The second equality uses the change of variables $x_i=e^{-t_i}$. Both representations were examined in more details in Section 5.1. The last equality uses the following equation

$$\int_{b>t_1>t_2>\cdots>t_n>a}\mathrm{d}t_1\wedge\mathrm{d}t_2\wedge\cdots\wedge\mathrm{d}t_n=\frac{(b-a)^n}{\Gamma(n+1)}, \tag{5.37}$$

whose proof we leave for the reader.

Similarly,

$$\begin{aligned}
\zeta(1,3) &= \int_{0<x_1<x_2<x_3<x_4<1} \frac{\mathrm{d}x_1}{1-x_1} \wedge \frac{\mathrm{d}x_2}{1-x_2} \wedge \frac{\mathrm{d}x_3}{x_3} \wedge \frac{dx_4}{x_4} = \\
&= \int_{t_1>t_2>t_3>t_4>0} \frac{\mathrm{d}t_1 \wedge \mathrm{d}t_2 \wedge \mathrm{d}t_3 \wedge \mathrm{d}t_4}{(e^{t_1}-1)(e^{t_2}-1)} = \\
&= \int_{t_1>t_2>0} \frac{\mathrm{d}t_1}{\Gamma(1)(e^{t_1}-1)} \wedge \frac{t_2^{3-1}\mathrm{d}t_2}{\Gamma(3)(e^{t_2}-1)} = \\
&= \int_{(0,\infty)^2} \frac{u_1^{1-1}u_2^{3-1}\mathrm{d}u_1 \wedge \mathrm{d}u_2}{\Gamma(1)\Gamma(3)(e^{u_1+u_2}-1)(e^{u_2}-1)}.
\end{aligned}$$

The first two equalities are of the same type as in the previous example. For the third equality we use Equation (5.37). For the last equality we use the change of variable

$$\begin{aligned}
t_2 &= u_2, \\
t_1 &= u_1 + u_2,
\end{aligned}$$

where $u_1 > 0$ and $u_2 > 0$. Following [?], we can interpolate the multiple zeta values by

$$\zeta(s_1,\dots,s_d) = \Gamma(s_1)^{-1}\dots\Gamma(s_d)^{-1} \int_{(0,\infty)^d} \frac{u_1^{s_1-1}\dots u_d^{s_d-1}\mathrm{d}u_1 \wedge \dots \wedge \mathrm{d}u_d}{(e^{u_1+\dots+u_d}-1)(e^{u_2+\dots+u_d}-1)\dots(e^{u_d}-1)}.$$

If we denote by

$$f_0(\mathbb{N};t) = \sum_{a\in\mathbb{N}} e^{-at},$$

then

$$f_0(\mathbb{N},t) = \frac{1}{e^t-1}$$

and

$$\zeta(s_1,\dots,s_d) = \Gamma(s_1)^{-1}\dots\Gamma(s_d)^{-1} \int_{(0,\infty)^d} \bigwedge_{j=1}^{d} f_0(\mathbb{N};u_i+\dots+u_d)u_j^{s_j-1}\mathrm{d}u_j$$

Let $n = [K:\mathbb{Q}]$ be the degree of the number field. We recall Definition 69 of f_0,

$$f_0(C;t_1,t_2,\dots,t_n) = \sum_{\alpha\in C} e^{-\sum_{i=1}^n \sigma_i(\alpha)t_i},$$

where $\sigma_i : K \to \mathbb{C}$ run through all embeddings of the field K into the complex numbers. Let C_1 and C_2 be two unimodular simple cones. We want to raise an algebraic integer α_1 to a complex power s_1 as a portion of the multiple Dedekind zeta function. We define

$$\alpha_1^{s_1} = e^{s_1\log(\alpha_1)}$$

for one element $\alpha \in C$ and $\alpha_1 = \sigma_1(\alpha)$. Choose a branch of the logarithmic function by making a cut of the complex plane at the negative real numbers. Since C is a positive unimodular simple cone we have that $\mathbb{R}^n_> \subset C^*$, the function σ_i composed with log is well defined on a positive unimodular simple cone C.

Then, we define a double Dedekind zeta function as

$$\begin{aligned}
&\zeta_{K;C_1,C_2}(s_{1,1},\ldots,s_{n,1};s_{1,2},\ldots,s_{n,2}) = \qquad (5.38)\\
&= \Gamma(s_{1,1})^{-1}\ldots\Gamma(s_{n,2})^{-1}\int_{(0,+\infty)^{2n}} f_0(C_1;(u_{1,1}+u_{1,2}),\ldots,(u_{n,1}+u_{n,2}))\times\\
&\times f_0(C_2;(u_{1,2},\ldots,u_{n,2})\bigwedge_{i=1}^{n} u_{i,1}^{s_{i,1}-1}\mathrm{d}u_{i,1}\wedge\bigwedge_{i=1}^{n} u_{i,2}^{s_{i,2}}\mathrm{d}u_{i,2}.
\end{aligned}$$

This definition combines both double zeta function and multiple Dedekind zeta values with double iteration. More generally, we can interpolate all multiple Dedekind zeta values into multiple Dedekind zeta functions so that multiple zeta functions are particular cases. Again, we define $u_{i,j}^{s_{i,j}}$ by

$$u_{i,j}^{s_{i,j}} = e^{s_{i,j}\log(u_{i,j})},$$

along the branch of logarithm described above.

Definition 82 *(Multiple Dedekind zeta functions) Let $n = [K;\mathbb{Q}]$ be the degree of the number field. Let $C_1,\ldots,C_m$ be m positive unimodular simple cones in $\mathcal{O}_K$. Let $u_{i,j} \in (0,\infty)$ for $i = 1,\ldots,n$ and $j = 1,\ldots,m$. We define multiple Dedekind zeta functions by the integral*

$$\begin{aligned}
&\zeta_{K;C_1,\ldots,C_m}(s_{1,1},\ldots,s_{n,1};\ldots;s_{1,m},\ldots,s_{n,m}) =\\
&= \prod_{(i,j)=(1,1)}^{(n,m)} \Gamma(s_{i,j})^{-1}\times \qquad (5.39)\\
&\times\int_{(0,+\infty)^{mn}}\bigwedge_{j=1}^{m} f_0(C_j;(u_{1,j}+\cdots+u_{1,m}),\ldots,(u_{n,j}+\cdots+u_{n,m}))\bigwedge_{i=1}^{n} u_{i,j}^{s_{i,j}-1}\mathrm{d}u_{i,j},
\end{aligned}$$

when $Re(s_{i,j}) > 1$.

Theorem 83 *(Infinite Sum Representation) For the general form of a multiple Dedekind zeta function, we need: a number field K; positive unimodular simple cones C_j in $\mathcal{O}_K$, for $j = 1,\ldots,m$; elements $\alpha_j \in C_j$ for $j = 1,\ldots,m$; complex embeddings of the elements $\alpha_{i,j} = \sigma_i(\alpha_j)$; Then, a multiple Dedekind zeta function has the following infinite sum representation*

$$\begin{aligned}
&\zeta_{K;C_1,\ldots,C_m}(s_{1,1},\ldots,s_{n,1};\ldots;s_{1,m},\ldots,s_{n,m}) =\\
&= \sum_{\alpha_1\in C_1}\cdots\sum_{\alpha_m\in C_m}\prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_{i,1}+\cdots+\alpha_{i,j})^{-s_{i,j}},
\end{aligned}$$

when $Re(s_{i,j}) > 1$.

Proof. We have

$$\zeta_{K;C_1,\dots,C_m}(s_{1,1},\dots,s_{n,1};\dots;s_{1,m},\dots,s_{n,m}) =$$
$$= \prod_{(i,j)=(1,1)}^{(n,m)} \Gamma(s_{i,j})^{-1} \times$$
$$\times \int_{(0,+\infty)^{mn}} \bigwedge_{j=1}^{m} f_0(C_j;(u_{1,j}+\cdots+u_{1,m}),\dots,(u_{n,j}+\cdots+u_{n,m})) \bigwedge_{i=1}^{n} u_{i,j}^{s_{i,j}-1}\mathrm{d}u_{i,j} =$$
$$= \left(\prod_{(i,j)=(1,1)}^{(n,m)} \Gamma(s_{i,j})\right)^{-1} \sum_{\alpha_1\in C_1} \cdots \sum_{\alpha_m\in C_m} \int_{(0,+\infty)^{mn}} \bigwedge_{j=1}^{m}\bigwedge_{i=1}^{n} e^{-(\alpha_{i,1}+\cdots+\alpha_{i,j})u_{i,j}} u_{i,j}^{s_{i,j}-1} du_{i,j}$$
$$= \sum_{\alpha_1\in C_1} \cdots \sum_{\alpha_m\in C_m} \prod_{i=1}^{n}\prod_{j=1}^{m} (\alpha_{i,1}+\cdots+\alpha_{i,j})^{-s_{i,j}}.$$

The following examples give a bridge between Dedekind zeta function and values of Eisenstein series (Example 5), and between multiple Dedekind zeta function and values of multiple Eisenstein series. More about values of multiple Eisenstein series will appear in Examples 7, 8, 9 on pages 29 and 30.

Examples:

5. Let K be any number field, let $m = 1$ and let C be a positive unimodular simple cone in $\mathcal{O}_K$. Then

$$\zeta_{K;C}(s_{1,1},\dots,s_{n,1}) = \sum_{\alpha\in C} \frac{1}{\prod_{i=1}^{n} \alpha_i^{s_{i,1}}}, \tag{5.40}$$

where $\alpha_i = \sigma_i(\alpha)$ is the i-th embedding in the complex numbers. In particular, if all variables $s_{i,1}$, for $i = 1,\dots,n$ have the same value s, then

$$\zeta_{K;C}(s,\dots,s) = \sum_{\alpha\in C} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^s}. \tag{5.41}$$

6. Now, let $m = 2$. Then we have a double iteration. Let K be any number field. Let C_1 and C_2 be two positive unimodular simple cones. Then

$$\zeta_{K;C_1,C_2}(s_{1,1},\dots,s_{n,1};s_{1,2},\dots,s_{n,2}) = \sum_{\alpha\in C_1,\beta\in C_2} \frac{1}{\prod_{i=1}^{n} \alpha_i^{s_{i,1}}(\alpha_i+\beta_i)^{s_{i,2}}}. \tag{5.42}$$

In particular, if

$$s_j = s_{1,j} = \cdots = s_{n,j}$$

for $j = 1, 2$, then

$$\zeta_{K;C_1,C_2}(s_1,\dots,s_1;s_2,\dots,s_2) = \sum_{\alpha\in C_1,\beta\in C_2} \frac{1}{N_{K/\mathbb{Q}}(\alpha)^{s_1} N_{K/\mathbb{Q}}(\alpha+\beta)^{s_2}}. \tag{5.43}$$

5.4 Analytic properties and special values

Applications to multiple Eisenstein series

Assuming the analytic continuation (Theorem 85), we can consider values of the multiple Dedekind zeta functions, when one or more of the arguments are zero, which allows us to express special values of multiple Eisenstein series (see [?]) as multiple Dedekind zeta values. This is presented in the following three examples.

Examples: 7. Let K be an imaginary quadratic field. Let C be a positive unimodular simple cone in $\mathcal{O}_K$. We can represent the cone C as an $\mathbb{N}$-module: $C = \mathbb{N}\{\mu, \nu\}$, for $\mu, \nu \in \mathcal{O}_K$. Put $z = \mu/\nu$. Consider $\zeta_{K;C}(k_{1,1}, k_{2,1})$ at $k_{1,1} = k$ and $k_{2,1} = 0$.

Then

$$\zeta_{K;C}(k, 0) = \sum_{\alpha \in C} \frac{1}{\alpha_1^k} = \nu^{-k} \sum_{a,b \in \mathbb{N}} \frac{1}{(az+b)^k},$$

where the last sum is a portion of the k-th Eisenstein series.

$$E_k(\tau) = \sum_{a,b \in \mathbb{Z}; (a,b) \neq (0,0)} \frac{1}{(az+b)^k}.$$

is an analogue of Eisenstein series.

8. Let K be an imaginary quadratic field. Let C be a positive unimodular simple cone in $\mathcal{O}_K$. We can represent C as

$$C = \mathbb{N}\{\mu, \nu\} = \{\alpha \in \mathcal{O}_K \mid \alpha = a\mu + b\nu,\ a, b \in \mathbb{N}\}.$$

Put $z = \mu/\nu$. Then, we obtain a value of multiple Eisenstein series

$$\zeta_{K;C,C}(k, 0; l, 0) = \nu^{-k-l} \sum_{a,b,c,d \in \mathbb{N}} \frac{1}{(az+b)^k((a+c)z + (b+d))^l}.$$

9. Similarly, one can define analogue of values of the above Eisenstein series over real quadratic field K, by setting

$$E_k(z) = \nu^k \zeta_{K;C}(k, 0) = \sum_{\alpha \in C} \frac{1}{\alpha_1^k},$$

where $C = \mathbb{N}\{\mu, \nu\}$ is a positive unimodular simple cone in a real quadratic ring of integers $\mathcal{O}_K$.

5.4.1 Examples of Analytic continuation and Multiple Residues

The following examples of analytic continuations are based on a Theorem of Gelfand-Shilov. The constructions in example 11 is central for this Section. Using Example 11, we change the variables in a way that we can apply Gelfand-Shilov's Theorem (Theorem 84) that gives analytic continuation of MDZF.

Moreover, in Example 11 we compute a multiple residue at $(1,1,1,1)$. In Subsection 5.4.3, we generalize this method to other multiple residues and we state two conjectures - one about the values of the multiple residues, again based on Examples 10 and 11, and other about more general MDZV.

Let us recall the theorem of Gelfand-Shilov.

Theorem 84 *([?]) Let $\phi(x)$ be a test function on $\mathbb{R}$, which decreases rapidly (exponentially) when $x \to \infty$ and let*

$$x_+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Then the value of the distribution $\frac{x_+^{s-1}dx}{\Gamma(s)}$ on the test function ϕ, namely,

$$\int_{\mathbb{R}} \phi(x) \frac{x_+^{s-1}dx}{\Gamma(s)}$$

is an analytic function in the variable s.

In Examples 10 and 11, we express a multiple Dedekind zeta function (MDZF) as a test function times a distribution when $s_{i,j} > 1$ up to Γ-factors. Then Theorem 84 tells us that we have an analytic continuation of the MDZF to all complex values of $s_{i,j}$ after multiplying by a suitable Γ-factors. Using this method, we compute the multiple residue at $(1, \dots, 1)$.

Example 10. Let K be a quadratic field and let $C = \mathbb{N}\{\alpha, \beta\}$ be a positive unimodular simple cone. Put α_1, α_2 and β_1, β_2 be the images under the two embeddings into $\mathbb{C}$ of α and β, respectively. We will compute the residue of

$$\zeta_{K;C}(s_1, s_2) = \sum_{\mu \in C} \frac{1}{\mu_1^{s_1} \mu_2^{s_2}}$$

at the hyperplane $s_1 + s_2 = 2$ and evaluated at $s_1 = s_2 = 1$.

We have

$$\zeta_{K;C}(s_1, s_2) = \Gamma(s_1)^{-1}\Gamma(s_2)^{-1} \int_0^\infty \int_0^\infty \frac{t_1^{s_1-1} t_2^{s_2-1} dt_1 \wedge dt_2}{(e^{\alpha_1 t_1 + \alpha_2 t_2} - 1)(e^{\beta_1 t_1 + \beta_2 t_2} - 1)}.$$

Set $t_1 = x_1(1 - x_2)$ and $t_2 = x_1 x_2$. Then

$$\Gamma(s_1 + s_2 - 2)^{-1} \zeta_{K;C}(s_1, s_2)$$

is the value of the distribution

$$Dx = \frac{x_{1+}^{s_1+s_2-3} x_{2+}^{s_2} (1 - x_2)_+^{s_1-1} dx_1 dx_2}{\Gamma(s_1 + s_2 - 2)\Gamma(s_1)\Gamma(s_2)}$$

at the test function

$$\phi = \frac{x_1^2}{(e^{x_1(\alpha_1 + (\alpha_2 - \alpha_1)x_2)} - 1)(e^{x_1(\beta_1 + (\beta_2 - \alpha_1)x_2)} - 1)}.$$

Using Theorem 84 we obtain that $\Gamma(s_1+s_2-2)^{-1}\zeta_{K;C}(s_1,s_2)$ is an analytic function. The residue of $\zeta_{K;C}(s_1,s_2)$ at $s_1+s_2=2$ is

$$\int_0^1 \frac{1}{(\alpha_1+(\alpha_2-\alpha_1)x_2)(\alpha_1+(\beta_2-\beta_1)x_2)}\cdot\frac{x_{2+}^{s_2}(1-x_2)_+^{s_1-1}dx_1dx_2}{\Gamma(s_1)\Gamma(s_2)}.$$

Evaluating at $s_2=1$, the integral becomes

$$\int_0^1 \frac{dx_2}{(\alpha_1+(\alpha_2-\alpha_1)x_2)(\alpha_1+(\beta_2-\beta_1)x_2)}$$

Thus, the residue of $\zeta_{K;C}(s_1,s_2)$ at $s_1+s_2=2$, evaluated at $(s_1,s_2)=(1,1)$ is given by the above integral. After evaluating it, we obtain

$$(Res_{s_1+s_2=2}\zeta_{K;C}(s_1,s_2))|_{(s_1,s_2)=(1,1)}=\frac{\log\left(\frac{\alpha_2}{\alpha_1}\right)-\log\left(\frac{\beta_2}{\beta_1}\right)}{\begin{vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{vmatrix}}.$$

In particular, if $\beta=1$ we obtain

$$(Res_{s_1+s_2=2}\zeta_{K;C}(s_1,s_2))|_{(s_1,s_2)=(1,1)}=\frac{\log(\alpha_2)-\log(\alpha_1)}{\alpha_2-\alpha_1}. \tag{5.44}$$

Note that if K is a real quadratic field and α is a generator of the group of units, then

$$|\log(\alpha_2)-\log(\alpha_1)|=2|\log(\alpha_1)|$$

is two times the regulator of the number field K and $\alpha_2-\alpha_1$ is an integer multiple of the discriminant of K. For a definition of a discriminant and a regulator of a number field, one may consult [?]. Equation (5.44) is true for any quadratic field, not only for real quadratic fields.

The following Example gives key constructions needed for the proof of the analytic continuation of MDZF (Theorem 85). It is also a case study of Conjecture 87 about the multiple residue of a multiple Dedekind zeta function at $(1,\dots,1)$.

Example 11. Let K be a quadratic extension of $\mathbb{Q}$. Let $C_1=\mathbb{N}\{1,\alpha\}$ and $C_2=\mathbb{N}\{1,\gamma\}$ be two positive unimodular simple cones. Let

$$\zeta_{K;C_1,C_2}(s_1,s_2;s_1',s_2')=\sum_{\mu\in C_1;\ \nu\in C_2}\frac{1}{\mu_1^{s_1}\mu_2^{s_2}(\mu_1+\nu_1)^{s_1'}(\mu_2+\nu_2)^{s_2'}}.$$

An integral representation can be written as

$$\zeta_{K;C_1,C_2}(s_1,s_2;s_1',s_2')\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2')=$$
$$=\int_{t_1>t_1'>0;\ t_2>t_2'>0}\frac{(t_1-t_1')^{s_1-1}(t_2-t_2')_2^{s_2-1}(t_1')^{s_1'-1}(t_2')^{s_2'-1}dt_1\wedge dt_2\wedge dt_1'\wedge dt_2'}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)}. \tag{5.45}$$

We compute the residue of a double Dedekind zeta function by taking the multiple residues of six functions $\zeta^{(a)}, \ldots, \zeta^{(f)}$ and considering their sum.

We shall write the last differential form in Equation (5.46) as Dt. Then, we have

$$\zeta_{K;C_1,C_2}(s_1, s_2; s_1', s_2') = (\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2'))^{-1} \int_{t_1>t_1'>0;\ t_2>t_2'>0} Dt \quad (5.46)$$

We are going to express the above integral as a sum of six integrals, which are enumerated by all possible shuffles of $t_1 > t_1' > 0$ and $t_2 > t_2' > 0$. In other words, each of the six new integrals will be associated to each linear order among the variables t_1, t_2, t_1', t_2' that respect the above two inequalities among them. Thus, all possible cases are

(a) $t_1 > t_1' > t_2 > t_2' > 0$,
(b) $t_1 > t_2 > t_1' > t_2' > 0$,
(c) $t_1 > t_2 > t_2' > t_1' > 0$,
(d) $t_2 > t_1 > t_1' > t_2' > 0$,
(e) $t_2 > t_1 > t_2' > t_1' > 0$,
(f) $t_2 > t_2' > t_1 > t_1' > 0$.

Then the domain $t_1 > t_1' > 0;\ t_2 > t_2' > 0$ can be represented as a disjoint union of the domains of integration, given in parts (a),...,(f). Then

$$\begin{aligned}
\zeta_{K;C_1,C_2}(s_1, s_2; s_1', s_2') &= (\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2'))^{-1} \int_{t_1>t_1'>0;\ t_2>t_2'>0} Dt \\
&= (\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2'))^{-1} \times \\
&\times \left(\int_{t_1>t_1'>t_2>t_2'>0} + \int_{t_1>t_2>t_1'>t_2'>0} + \right. \\
&+ \int_{t_1>t_2>t_2'>t_1'>0} + \int_{t_2>t_1>t_1'>t_2'>0} + \\
&+ \left. \int_{t_2>t_1>t_2'>t_1'>0} + \int_{t_2>t_2'>t_1>t_1'>0} \right) Dt = \\
&= \zeta^{(a)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2') + \zeta^{(b)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2') + \\
&+ \zeta^{(c)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2') + \zeta^{(d)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2') + \\
&+ \zeta^{(e)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2') + \zeta^{(f)}_{K;C_1,C_2}(s_1, s_2; s_1', s_2').
\end{aligned}$$

We define $\zeta^{(a)}, \ldots, \zeta^{(f)}$ to be the above six integrals, corresponding to the domains of integration given by (a),...,(f). The reason for defining them is to take multiple residues of the multiple Dedekind zeta function. It is easier to work with the functions $\zeta^{(a)}, \ldots, \zeta^{(f)}$ for the purpose of proving analytic continuation and taking residues.

Thus, we compute the residue of a double Dedekind zeta function by taking the multiple residues of six functions $\zeta^{(a)}, \ldots, \zeta^{(f)}$ and considering their sum.

Consider the domain of integration

$$(a)\ t_1 > t'_1 > t_2 > t'_2 > 0.$$

We will compute the residues of

$$\zeta^{(a)}_{K;C_1,C_2}(s_1,s_2;s'_1,s'_2)\Gamma(s_1)\Gamma(s_2)\Gamma(s'_1)\Gamma(s'_2) =$$
$$= \int_{t_1>t'_1>0>t_2>t'_2>0} \frac{(t_1-t'_1)^{s_1-1}(t_2-t'_2)_2^{s_2-1}(t'_1)^{s'_1-1}(t'_2)^{s'_2-1}dt_1\wedge dt_2\wedge dt'_1\wedge dt'_2}{(e^{t_1+t_2}-1)(e^{\alpha_1 t_1+\alpha_2 t_2}-1)(e^{t'_1+t'_2}-1)(e^{\gamma_1 t'_1+\gamma_2 t'_2}-1)}.$$

Let the successive differences be $u_1 = t_1 - t'_1$, $u_2 = t'_1 - t_2$, $u_3 = t_2 - t'_2$, $u_4 = t'_2$. Their admissive values are in the interval $(0, +\infty)$. Let us make the following substitution

$$\begin{aligned}
u_1 &= x_1(1-x_2),\\
u_2 &= x_1x_2(1-x_3),\\
u_3 &= x_1x_2x_3(1-x_4),\\
u_4 &= x_1x_2x_3x_4.
\end{aligned}$$

We are going to express the above integral in terms of the variables x_1, x_2, x_3 and x_4. We have

$$\begin{aligned}
\alpha_1 t_1 + \alpha_2 t_2 &= \alpha_1(u_1+u_2+u_3+u_4) + \alpha_2(u_3+u_4) = x_1(\alpha_1+\alpha_2 x_2 x_3,)\\
t_1+t_2 &= x_1(1+x_1x_3),\\
\gamma_1 t'_1 + \gamma_2 t'_2 &= \gamma_1(u_2+u_3+u_4) + \gamma_2 u_4 = x_1x_2(1+x_3x_4),\\
t'_1+t'_2 &= x_1x_2(1+x_3x_4),\\
t_1 - t'_1 &= u_1 = x_1(1-x_2),\\
t_2 - t'_2 &= u_3 = x_1x_2x_3(1-x_4),\\
t'_1 &= u_2+u_3+u_4 = x_1x_2,\\
t'_2 &= u_4 = x_1x_2x_3x_4.
\end{aligned}$$

For the change of variables in the differential forms, we have

$$dt_1\wedge dt_2\wedge dt'_1\wedge dt'_2 = du_1\wedge du_2\wedge du_3\wedge du_4$$

and

$$\frac{du_1}{u_1}\wedge\frac{du_2}{u_2}\wedge\frac{du_3}{u_3}\wedge\frac{du_4}{u_4} = \frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4}{x_1x_2x_3x_4(1-x_2)(1-x_3)(1-x_4)}.$$

Then

$$\zeta^{(a)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2')\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2') =$$
$$= \int_{t_1>t_1'>0>t_2>t_2'>0} \frac{(t_1-t_1')^{s_1-1}(t_2-t_2')_2^{s_2-1}(t_1')^{s_1'-1}(t_2')^{s_2'-1}dt_1\wedge dt_2\wedge dt_1'\wedge dt_2'}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)}$$
$$= \int_{(u_1,\ldots,u_4)\in(0,\infty)^4} \frac{u_1^{s_1-1}u_3^{s_2-1}(u_2+u_3+u_4)^{s_1'-1}u_4^{s_2'-1}du_1\wedge du_2\wedge du_3\wedge du_4}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)}$$
$$= \int_{(u_1,\ldots,u_4)\in(0,\infty)^4} \frac{u_1^{s_1}u_3^{s_2}(u_2+u_3+u_4)^{s_1'-1}u_4^{s_2'}u_2\frac{du_1}{u_1}\wedge\frac{du_2}{u_2}\wedge\frac{du_3}{u_3}\wedge\frac{du_4}{u_4}}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)}$$
$$= \int_0^\infty\int_{(0,1)^3} \frac{[x_1(1-x_2)]^{s_1}[x_1x_2x_3(1-x_4)]^{s_2}[x_1x_2]^{s_1'}[x_1x_2x_3x_4]^{s_2'}[x_1x_2(1-x_3)]\Omega}{(e^{x_1(1+x_2x_3)-1})(e^{x_1(\alpha_1+\alpha_2x_2x_3)-1})(e^{x_1x_2(1+x_3x_4)}-1)(e^{x_1x_2(\gamma_1+\gamma_2x_3x_4)}-1)},$$

where

$$\Omega = \frac{du_1}{u_1}\wedge\frac{du_2}{u_2}\wedge\frac{du_3}{u_3}\wedge\frac{du_4}{u_4} = \frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4}{x_1x_2x_3x_4(1-x_2)(1-x_3)(1-x_4)}.$$

We can express the last integral as a distribution evaluated at a test function. Put

$$\phi(x_1,x_2,x_3,x_4) =$$
$$= \frac{x_1^4x_2^2}{(e^{x_1(1+x_2x_3)-1})(e^{x_1(\alpha_1+\alpha_2x_2x_3)-1})(e^{x_1x_2(1+x_3x_4)}-1)(e^{x_1x_2(\gamma_1+\gamma_2x_3x_4)}-1)}$$

to be a test function. Let Dx be a distribution defined by

$$Dx = \frac{x_{1+}^{s_1+s_2+s_1'+s_2'-5}}{\Gamma(s_1+s_2+s_1'+s_2'-4)}\frac{x_{2+}^{s_2+s_1'+s_2'-3}}{\Gamma(s_2+s_1'+s_2'-2)}\times \tag{5.47}$$
$$\times\frac{x_{3+}^{s_2+s_2'-1}}{\Gamma(s_2+s_2')}\frac{x_{4+}^{s_2'-1}}{\Gamma(s_2')}(1-x_2)_+^{s_1-1}(1-x_4)_+^{s_2-1}. \tag{5.48}$$

Then we have

$$\zeta^{(a)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2') = \frac{\Gamma(s_1+s_2+s_1'+s_2'-4)\Gamma(s_2+s_1'+s_2'-2)\Gamma(s_2+s_2')}{\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')}\times$$
$$\times\int\phi Dx$$

Using Theorem 84 we prove analytic continuation of $\zeta^{(a)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2')$ everywhere except at the poles of $\Gamma(s_1+s_2+s_1'+s_2'-4)\Gamma(s_2+s_1'+s_2'-2)\Gamma(s_2+s_2')$.

We will take the residue at $s_1+s_2+s_1'+s_2'=4$ and then evaluate at $(s_1,s_2,s_1',s_2')=(1,1,1,1)$. Note that

$$\phi(0,x_2,x_3,x_4) = \frac{1}{(1+x_2x_3)(\alpha_1+\alpha_2x_2x_3)(1+x_3x_4)(\gamma_1+\gamma_2x_3x_4)}$$

Then we can compute

$$(Res_{s_1+s_2+s_1'+s_2'=4}\zeta^{(a)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2'))|_{(1,1,1,1)} =$$
$$= \int_{(0,1)^3} \frac{x_3dx_2\wedge dx_3\wedge dx_4}{(1+x_2x_3)(\alpha_1+\alpha_2x_2x_3)(1+x_3x_4)(\gamma_1+\gamma_2x_3x_4)}.$$

Note that we cannot take a double residue at the point $(1,1,1,1)$. Also, the value that we obtain is a period.

Consider the domain in integration

$$\text{(b) } t_1 > t_2 > t_1' > t_2' > 0$$

We will compute the residues of

$$\zeta^{(b)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2')\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2') =$$
$$= \int_{t_1>t_2>t_1'>t_2'>0} \frac{(t_1-t_1')^{s_1-1}(t_2-t_2')^{s_2-1}(t_1')^{s_1'-1}(t_2')^{s_2'-1}dt_1\wedge dt_2\wedge dt_1'\wedge dt_2'}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)}.$$

Let the successive differences be

$$\begin{aligned}
u_1 &= t_1-t_2,\\
u_2 &= t_2-t_1',\\
u_3 &= t_1'-t_2',\\
u_4 &= t_2'.
\end{aligned}$$

Their admissive values are $(0,+\infty)$. Let

$$\begin{aligned}
u_1 &= x_1(1-x_2),\\
u_2 &= x_1x_2(1-x_3),\\
u_3 &= x_1x_2x_3(1-x_4),\\
u_4 &= x_1x_2x_3x_4.
\end{aligned}$$

We will express the above integral in terms of $x_1,\ldots,x_4$. We have

$$\begin{aligned}
\alpha_1t_1+\alpha_2t_2 &= \alpha_1(u_1+u_2+u_3+u+4)+\alpha_2(u_2+u_3+u_4) = x_1(\alpha_1+\alpha_2x_2),\\
t_1+t_2 &= x_1(1+x_2),\\
\gamma_1t_1'+\gamma_2t_2' &= \gamma_1(u_3+u_4)+\gamma_2u_4 = x_1x_2x_3(\gamma_1+\gamma_2x_4),\\
t_1'+t_2' &= x_1x_2x_3(1+x_4),\\
t_1-t_1' &= u_1+u_2 = x_1(1-x_2x_3),\\
t_2-t_2' &= u_2+u_3 = x_1x_2(1-x_3x_4),\\
t_1' &= u_3+u_4 = x_1x_2u_3,\\
t_2' &= u_4 = x_1x_2x_3x_4.
\end{aligned}$$

For the differential forms, we have

$$dt_1 \wedge dt_2 \wedge dt_1' \wedge dt_2' = du_1 \wedge du_2 \wedge du_3 \wedge du_4$$
$$\frac{du_1}{u_1} \wedge \frac{du_2}{u_2} \wedge \frac{du_3}{u_3} \wedge \frac{du_4}{u_4} = \frac{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}{x_1x_2x_3x_4(1-x_2)(1-x_3)(1-x_4)}.$$

Then

$$\begin{aligned}
&\zeta^{(b)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2')\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')\Gamma(s_2') = \\
&= \int_{t_1>t_2>t_1'>t_2'>0} \frac{(t_1-t_1')^{s_1-1}(t_2-t_2')^{s_2-1}(t_1')^{s_1'-1}(t_2')^{s_2'-1}dt_1\wedge dt_2\wedge dt_1'\wedge dt_2'}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)} \\
&= \int_{(u_1,\dots,u_4)\in(0,\infty)^4} \frac{(u_1+u_2)^{s_1-1}(u_2+u_3)^{s_2-1}(u_3+u_4)^{s_1'-1}u_4^{s_2'-1}du_1\wedge du_2\wedge du_3\wedge du_4}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)} \\
&= \int_{(u_1,\dots,u_4)\in(0,\infty)^4} \frac{(u_1+u_2)^{s_1-1}(u_2+u_3)^{s_2-1}(u_3+u_4)^{s_1'-1}u_4^{s_2'}u_1u_2u_3\frac{du_1\wedge du_2\wedge du_3\wedge du_4}{u_1u_2u_3u_4}}{(e^{t_1+t_2}-1)(e^{\alpha_1t_1+\alpha_2t_2}-1)(e^{t_1'+t_2'}-1)(e^{\gamma_1t_1'+\gamma_2t_2'}-1)} \\
&= \int_0^\infty \int_{(0,1)^3} \frac{[x_1(1-x_2x_3)]^{s_1-1}[x_1x_2(1-x_3x_4)]^{s_2-1}[x_1x_2x_3]^{s_1'-1}[x_1x_2x_3x_4]^{s_2'}X\Omega}{(e^{x_1(1+x_2x_3)}-1)(e^{x_1(\alpha_1+\alpha_2x_2x_3)}-1)(e^{x_1x_2(1+x_3x_4)}-1)(e^{x_1x_2(\gamma_1+\gamma_2x_3x_4)}-1)},
\end{aligned}$$

where

$$X = x_1^3x_2^2x_3(1-x_2)(1-x_3)(1-x_4)$$

and

$$\Omega = \frac{du_1\wedge du_2\wedge du_3\wedge du_4}{u_1u_2u_3u_4} = \frac{dx_1\wedge dx_2\wedge dx_3\wedge dx_4}{x_1x_2x_3x_4(1-x_2)(1-x_3)(1-x_4)}$$

Now we can express the last integral as a distribution evaluated at a test function. Put

$$\phi(x_1,x_2,x_3,x_4) = \frac{x_1^4x_2^2x_3^2}{(e^{x_1(1+x_2)}-1)(e^{x_1(\alpha_1+\alpha_2x_2)}-1)(e^{x_1x_2x_3(1+x_4)}-1)(e^{x_1x_2x_3(\gamma_1+\gamma_2x_4)}-1)},$$

to be a test function. Let Dx be a distribution defined by

$$Dx = \frac{x_{1+}^{s_1+s_2+s_1'+s_2'-5}}{\Gamma(s_1+s_2+s_1'+s_2'-4)}\frac{x_{2+}^{s_2+s_1'+s_2'-3}}{\Gamma(s_2+s_1'+s_2'-2)}\times \tag{5.49}$$
$$\times\frac{x_{3+}^{s_2+s_2'-3}}{\Gamma(s_2+s_2'-2)}\frac{x_{4+}^{s_2'-1}}{\Gamma(s_2')}(1-x_2x_3)_+^{s_1-1}(1-x_3x_4)_+^{s_2-1} \tag{5.50}$$

Then we have

$$\begin{aligned}\zeta^{(b)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2') =& \frac{\Gamma(s_1+s_2+s_1'+s_2'-4)\Gamma(s_2+s_1'+s_2'-2)\Gamma(s_2+s_2'-2)}{\Gamma(s_1)\Gamma(s_2)\Gamma(s_1')}\times \\ &\times\int\phi Dx\end{aligned}$$

Using Theorem 84 we prove analytic continuation of $\zeta^{(b)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2')$ everywhere except at the poles of $\Gamma(s_1+s_2+s_1'+s_2'-4)\Gamma(s_2+s_1'+s_2'-2)\Gamma(s_2+s_2'-2)$.

We will take the residues at $s_1+s_2+s_1'+s_2'=4$ and at $s_1'+s_2'=2$. And then, we will evaluate at $(s_1,s_2,s_1',s_2')=(1,1,1,1)$. Note that

$$\phi(0,x_2,0,x_4)=\frac{1}{(1+x_2)(\alpha_1+\alpha_2x_2)(1+x_4)(\gamma_1+\gamma_2x_4)}$$

Then we can compute

$$\begin{aligned}&(Res_{s_1'+s_2'=2}Res_{s_1+s_2+s_1'+s_2'=4}\zeta^{(b)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2'))|_{(1,1,1,1)}=\\&=\int_{(0,1)^2}\frac{dx_2\wedge dx_4}{(1+x_2)(\alpha_1+\alpha_2x_2)(1+x_4)(\gamma_1+\gamma_2x_4)}=\\&=\frac{\log\left(\frac{2\alpha_1}{\alpha_1+\alpha_2}\right)}{\alpha_1-\alpha_2}\cdot\frac{\log\left(\frac{2\gamma_1}{\gamma_1+\gamma_2}\right)}{\gamma_1-\gamma_2}.\end{aligned}$$

For the cases (c), (d) and (e), we obtain

$$\begin{aligned}&(Res_{s_1'+s_2'=2}Res_{s_1+s_2+s_1'+s_2'=4}\zeta^{(c)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2'))|_{(1,1,1,1)}=\\&=\int_{(0,1)^2}\frac{dx_2\wedge dx_4}{(1+x_2)(\alpha_1+\alpha_2x_2)(x_4+1)(\gamma_1x_4+\gamma_2)}=\\&=-\frac{\log\left(\frac{2\alpha_1}{\alpha_1+\alpha_2}\right)}{\alpha_1-\alpha_2}\cdot\frac{\log\left(\frac{2\gamma_2}{\gamma_1+\gamma_2}\right)}{\gamma_1-\gamma_2}.\end{aligned}$$

$$\begin{aligned}&(Res_{s_1'+s_2'=2}Res_{s_1+s_2+s_1'+s_2'=4}\zeta^{(d)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2'))|_{(1,1,1,1)}=\\&=\int_{(0,1)^2}\frac{dx_2\wedge dx_4}{(x_2+1)(\alpha_1x_2+\alpha_2)(1+x_4)(\gamma_1+\gamma_2x_4)}=\\&=-\frac{\log\left(\frac{2\alpha_2}{\alpha_1+\alpha_2}\right)}{\alpha_1-\alpha_2}\cdot\frac{\log\left(\frac{2\gamma_1}{\gamma_1+\gamma_2}\right)}{\gamma_1-\gamma_2}.\end{aligned}$$

$$\begin{aligned}&(Res_{s_1'+s_2'=2}Res_{s_1+s_2+s_1'+s_2'=4}\zeta^{(e)}_{K;C_1,C_2}(s_1,s_2;s_1',s_2'))|_{(1,1,1,1)}=\\&=\frac{\log\left(\frac{2\alpha_2}{\alpha_1+\alpha_2}\right)}{\alpha_1-\alpha_2}\cdot\frac{\log\left(\frac{2\gamma_2}{\gamma_1+\gamma_2}\right)}{\gamma_1-\gamma_2}.\end{aligned}$$

Case (f) is similar to case (a), namely, there is no double residue at the point

$(1, 1, 1, 1)$. Thus, we obtain

$$Res_{s'_1+s'_2=2}Res_{s_1+s_2+s'_1+s'_2=4}\zeta_{K;C_1,C_2}(s_1, s_2; s'_1, s'_2))|_{(1,1,1,1)} =$$
$$= Res_{s'_1+s'_2=2}Res_{s_1+s_2+s'_1+s'_2=4}(\zeta^{(b)}_{K;C_1,C_2}(s_1, s_2; s'_1, s'_2) + \zeta^{(c)}_{K;C_1,C_2}(s_1, s_2; s'_1, s'_2)+$$
$$+ \zeta^{(d)}_{K;C_1,C_2}(s_1, s_2; s'_1, s'_2) + \zeta^{(e)}_{K;C_1,C_2}(s_1, s_2; s'_1, s'_2))|_{(1,1,1,1)} =$$
$$= \frac{\log(\alpha_2) - \log(\alpha_1)}{\alpha_2 - \alpha_1} \cdot \frac{\log(\gamma_2) - \log(\gamma_1)}{\gamma_2 - \gamma_1}.$$

Note that if K is a real quadratic field and α is a generator of the group of units, then

$$|\log(\alpha_2) - \log(\alpha_1)| = 2\log|\alpha_1|$$

is two times the regulator of the number field K and

$$\alpha_2 - \alpha_1$$

is an integer multiple of the discriminant of the field K. For a definition of a discriminant and a regulator of a number field one may consult with [**?**]. The above formula is true for any quadratic field, not necessarily for a real quadratic field.

5.4.2 Analytic Continuation of Multiple Dedekind Zeta Functions

Theorem 85 *Multiple Dedekind zeta functions*

$$\zeta_{K;c_1,\ldots,C_m}(s_{1,1}, \ldots, s_{n,1}; \ldots; s_{1,m}, \ldots, s_{n,m})$$

have an analytic continuation from the region $Re(s_{i,j}) > 1$ for all i and j to $s_{i,j} \in \mathbb{C}$ with exception of hyperplanes. The hyperplanes are defined by sum several of the variables $s_{i,j}$ without repetitions being set equal to an integer.

Proof. Recall that $f_0(C_j; t_{1,j}, \ldots, t_{n,j})$ is used to define the multiple Dedekind zeta functions, where the domain of integration is

$$D = \{(t_{i,j}) \in \mathbb{R}^{mn} | t_{i,1} > t_{i,2} > \cdots > t_{n,j} > 0\}.$$

$$J_i = \{k_{i,1}, \ldots, k_{i,m}\}.$$

Note that there are n sets $J_1, \ldots, J_n$, and each of them has m elements, $|J_i| = m$. Let τ run through all the shuffles of the ordered sets $J_1, \ldots, J_n$,

$$\tau \in Sh(J_1, \ldots, J_n).$$

Let $t_1, t_2 \ldots, t_{mn}$ be the variables $t_{1,1}, \ldots, t_{n,m}$, written in decreasing order. There are finitely many ways of arranging the variables in decreasing order. More precisely, the number of such arrangements is equal to the number of shuffles in $Sh(J_1, \ldots, J_n)$. We need to consider all such shuffles in order to express the multiple Dedekind zeta function as a sum of partial multiple Dedekind zeta functions, corresponding to each shuffle τ (see Example 11). Let $u_k = t_k - t_{k+1}$, for $k = 1, \dots, mn-1$ and $u_{mn} = t_{mn}$. Let

$$\begin{aligned} u_1 &= x_1(1-x_2) \\ u_2 &= x_1 x_2 (1-x_3) \\ &\dots \\ u_{mn-1} &= x_1 \dots x_{mn-1}(1-x_{mn}) \\ u_{mn} &= x_1 \dots x_{mn}. \end{aligned}$$

Then each of the linear factors in the denominator of $f_0(C_j;\cdots)$ can be written as

$$x_1 \dots x_k g_{j,l,\tau}.$$

for some positive integer k, $k \leq mn$ and a polynomial $g_{j,l,\tau}$, in the variables $x_1, \dots, x_{mn}$, not vanishing at the origin. Also, the indices i and j of the polynomial $g_{j,l,\tau}$ are associated to the l-th generator of the cone C_j, and τ is a shuffle of ordered sets $Sh(J_1, \dots, J_n)$. Example of polynomials $g_{j,l,\tau}$, can be found in Equations (5.48) and (5.50). The integral representation in terms of $x_1, \dots, x_{mn}$ (similar to the ones in Example 11, giving $\zeta^{(a)}$), are a type of zeta function that we call *partial MDZF* times Γ-factors. Using Theorem 84, we find that the partial MDZF together with the Γ-factors is an analytic function. The Γ-factors give hyperplanes where the poles of the partial MDZF occur. Expressing a MDZF as a finite sum of partial MDZF we obtain the analytic continuation from the domain $Re(s_{i,j}) > 1$ to $s_{i,j} \in \mathbb{C}$ with poles along hyperplanes coming from Γ-factors.

5.4.3 Remarks

In this final Subsection, we proof that certain multiple residue of a multiple Dedekind zeta functions is a period in the sense of algebraic geometry. Based on Theorem 86, we state two conjectures. One of the conjectures is about the exact values of the multiple residue and the other conjecture is about values of the multiple Dedekind zeta functions at other integers.

Theorem 86 *The multiple residue of a multiple Dedekind zeta function at the point*

$$(s_{1,1}, \dots, s_{n,1}; \dots; s_{1,m}, \dots, s_{n,m}) = (1, \dots, 1)$$

is a period over $\mathbb{Q}$.

Proof. We use the notation introduced in the proof of Theorem 85 and of Example 11 in Subsection 5.4.1.

The m-fold residue at $(1, \dots, 1)$ can be computed via an integral of a rational function, which is a product of the functions representing the hyperplanes, where f_0 vanishes, expressed in terms of the variables x_i. The value is a period. In general, after we take the multiple residues at $(1, \dots, 1)$, we obtain an integral of a rational function, (which is a product of $g_{j,l,\tau}$ over j and l, for $j = 1, \dots, m$, where l signifies the l-generator of the cone C_j). Note that τ is a shuffle. So that different shuffles τ correspond to different partial MDZF. The boundaries of the integral (after taking the multiple residues) form a unit cube. Therefore, the value of the multiple residue at $(1, \dots, 1)$ of a multiple Dedekind zeta function is a period.

For a more precise interpretation see Conjecture 87 and Examples 10 and 11.

From Examples 10 and 11, we know that a multiple residue of multiple Dedekind zeta function is a product of residues of partial Dedekind zeta functions, for quadratic fields and double iteration. For unimodular simple cones $C_1, \dots, C_m$, we consider a multiple Dedekind zeta function

$$\zeta_{K;C_1,\dots,C_m}(s_1, \dots, s_d) = \sum_{\alpha_1 \in C_1, \dots \alpha_d \in C_m} \frac{1}{N(\alpha_1)^{s_1} N(\alpha_1 + \alpha_2)^{s_2} \cdots N(\alpha_1 + \cdots + \alpha_m)^{s_m}}.$$

We expect that

Conjecture 87 *The multiple residue of $\zeta_{K;C_1,\dots,C_m}(s_1, \dots, s_m)$ at the point $(1, \dots, 1)$, namely*

$$Res_{s_m=1} \dots Res_{s_1+\dots s_m=m} \zeta_{K;C_1,\dots,C_m}(s_1, \dots, s_m) = \prod_{j=1}^{m} Res_{s=1} \zeta_{K;C_j}(s).$$

The conjecture is proven for a quadratic fields K and double iteration in Examples 10 and 11.

We do expect that multiple Dedekind zeta values should be periods over $\mathbb{Q}$.

Conjecture 88 *Let K be a number field. For any choice of unimodular simple cones $C_1, \dots, C_m$, in the ring of integers of a number field K, we have that the multiple Dedekind zeta values (see Definition 80)*

$$\zeta_{K;C_1,\dots,C_m}(k_{1,1}, \dots, k_{n,1}; \dots; k_{1,m}, \dots, k_{n,m})$$

are periods over $\mathbb{Q}$, when the $k_{1,1}, \dots, k_{m,n}$ are natural numbers greater than 1.

The reasons for this conjecture are the following:

1. We have that multiple zeta values are periods;
2. Dedekind zeta values are periods;
3. From Theorem 86, we have that the multiple residue of a multiple Dedekind zeta function at $(1, \dots, 1)$ is a period.
4. The main reason is the representation of multiple Dedekind zeta values as iterated integrals on membranes. We will give a semi-algebraic relations among the variables in such integrals.

We use Equations (5.25) and (5.26). Recall that

$$y_j = \prod_{i=1}^{n} \exp(-\sigma_i(e_j)z_i).$$

If we set

$$x_i = \exp(-z_i),$$

then a multiple Dedekind zeta value is an iterated integral on a membrane of the n-form

$$\prod_j \frac{dy_j}{1-y_j}$$

and the 1-forms

$$\frac{dx_i}{x_i},$$

which mostly resembles polylogarithms. However, the relations between x_i's and y_j's are (semi)-algebraic, namely,

$$\frac{dy_j}{y_j} = \sum_{i=1}^{n} \sigma_i(e_j)\frac{dx_i}{x_i},$$

which are not algebraic. Explicitly, they are given by

$$\log(y_j) = \sum_{i=1}^{n} \sigma_i(e_j)\log(x_i),$$

which involves the logarithmic function. Note that the logarithmic function is a homotopy invariant function on a path space. In this setting, the above logarithmic functions can be considered as a function on the path space of an affine n-space without some divisors. One may take a simplicial scheme as a model of the path space so that it restricts well onto the loop space of a scheme as a simplicial scheme. Hopefully, that would interpret the above (semi)-algebraic relations in terms of logarithms in an algebraic context.

Chapter 6

Non-commutative modular symbols

Modular symbols were defined by Birch and by Manin [**?**] for certain congruence subgroups of $SL_2(\mathbb{Z})$. A modular symbol $\{p, q\}$ is associated to a pair of cusp points $p, q \in \mathbb{P}^1(\mathbb{Q})$ on the completed upper half plane $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. One can think of the modular symbol $\{p, q\}$ as a homology class of the geodesic connecting p and q, in $H_1(X_\Gamma, \{cusps\})$, where X_Γ is the modular curve associated to a congruence subgroup of $SL_2(\mathbb{Z})$. One can pair $\{p, q\}$ with a cusp form f by

$$\{p, q\} \times f \mapsto \int_p^q f dz$$

If f is a cusp form of weight 2 then it can be considered as cohomology class in $H^1(X_\Gamma)$. This gives a pairing between homology and cohomology. Modular symbols are a useful tool applied to L-functions and computation of cohomology groups. For a review of such topics, one can consult [**?**]

Manin's non-commutative modular symbol [**?**] is a generalization of both the classical modular symbol and of multiple zeta values in terms of Chen's iterated integral theory in the holomorphic setting. In this paper we generalize Manin's non-commutative modular symbol to non-commutative Hilbert modular symbol. We also compute explicit integrals in the non-commutative Hilbert modular symbol and present similar formulas for the recently defined multiple Dedekind zeta values (see [**?**]).

Let us recall the non-commutative modular symbol of Manin [**?**]. Let $\nabla = d - \sum_{i=1}^m X_i f_i dz$ be a connection on the upper half plane, where $f_1, \ldots, f_m$ are cusp forms and $X_1, \ldots, X_m$ are formal variables. One can think of $X_1, \ldots, X_m$ as constant square matrices of the same size.

Let J_a^b be the parallel transport of the identity matrix 1 at the point a to the matrix J_a^b at the point b. Alternatively, J_b^a can be written as a generating series of iterated path integrals of the forms $f_1 dz, \ldots, f_m dz$, (see [**?**] and [**?**]), namely,

$$J_a^b = 1 + \sum_{i=1}^m X_i \int_a^b f_i dz + \sum_{i,j=1}^m X_i X_j \int_a^b f_i dz \cdot f_j dz + \cdots$$

Then $J_a^b J_b^c = J_a^c$. This property leads to the 1-cocycle $c_a^1(\gamma) = J_{\gamma a}^a$, which is the non-commutative modular symbol (see [?] and Section 1 of this paper).

Let us recall Riemann zeta values and multiple zeta values (MZVs). The Riemann zeta values are defined as

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k},$$

where n is an integer. MZVs are defined as

$$\zeta(k_1, \ldots, k_m) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1} \ldots n_m^{k_m}},$$

where $n_1, \ldots, n_m$ are integers. The above MZV is of depth m. Riemann zeta values $\zeta(k)$ and MZV $\zeta(k_1, \ldots, k_m)$ were defined by Euler [?]. The common feature of MZVs and the non-commutative modular symbol is that they both can be written as iterated path integrals (see [?], [?]). Moreover, Manin's non-commutative modular symbol resembles the generating series of MZV, which is the Drinfeld associator. Let us recall that the Drinfeld associator is a generating series of iterated integrals of the type J_a^b associated to the connection

$$\nabla = d - A\frac{dx}{x} - B\frac{dx}{1-x}$$

on $Y_{\Gamma(2)} = \mathbb{P}^1 - \{0, 1, \infty\}$. One can think of $Y_{\Gamma(2)}$ as the modular curve associated to the congruence subgroup $\Gamma(2)$ of $SL_2(\mathbb{Z})$. Then the differential forms $\frac{dx}{x}$ and $\frac{dx}{1-x}$ are Eisenstein series of weight 2 on the modular curve $Y_{\Gamma(2)}$.

In Subsection 3.5, we explore similar relations between the non-commutative Hilbert modular symbols and multiple Dedekind zeta values (see [?]). Let us recall multiple Dedekind zeta values. Let each of $C_1, \ldots, C_m$ be a suitable subset of the ring of integers $\mathcal{O}_K$ of a number field K. We call each of $C_1, \ldots, C_m$ a cone. Then **multiple Dedekind zeta values** are defined as

$$\zeta_{K;C_1,\ldots,C_m}(k_1, \ldots, k_m) = \sum_{\alpha_i \in C_i \text{ for } i=1,\ldots,m} \frac{1}{N(\alpha_1)^{k_1} N(\alpha_1+\alpha_2)^{k_2} \cdots N(\alpha_1 + \cdots + \alpha_m)^{k_m}}.$$

The connection between non-commutative Hilbert modular symbols and multiple Dedekind zeta values is both in similarities in the infinite sum formulas and in the definition in terms of iterated integrals over membranes (see [?]).

Relations between MZV and modular forms have been examined by many authors. For example, Goncharov has considered a mysterious relation between MZV (multiple zeta values) of given weight and depth 3 and cohomlogy of

$GL_3(\mathbb{Z})$ (see [?] and [?]), which is closely related to the cohomology of $SL_3(\mathbb{Z})$. In pursue for such a relation in depth 4, Goncharov has suggested and the author has computed the group cohomology of $GL_4(\mathbb{Z})$ with coefficients in a family of representations, [?]. Another relation between modular forms and MZV is presented in [?].

In this paper, we construct both commutative and non-commutative modular symbols for the Hilbert modular group $SL_2(\mathcal{O}_K)$. For the Hilbert modular group, one may consult [?] and [?]. It is not possible to repeat Manin's constructions for the non-commutative modular symbols, since the integration domain is two-dimensional over the complex numbers. Instead, we develop a new approach (Section 2), which we call iterated integrals over membranes. This is a higher dimensional analogue of iterated path integrals. In Subsection 3.5, we explore similar relations between non-commutative Hilbert modular symbols and multiple Dedekind zeta values (see [?]).

In Section 3, we associate modular symbols for $SL_2(\mathcal{O}_K)$ to geodesic triangles and geodesic diangles (2-cells whose boundary has two vertices and two edges, which are geodesics.) We are going to explain how the geodesic triangles and the geodesic diangles are constructed. Consider 4 cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. We can map every three of them to 0, 1 and ∞ with a linear fractional transformation $\gamma \in GL_2(K)$. There is a diagonal map $\mathbb{H}^1 \to \mathbb{H}^2$, whose image Δ contains 0, 1 and ∞. We can take a pull-back of Δ with respect to the map γ in order to obtain a holomorphic (or anti-holomorphic) curve that passes through the given three points. If $\det\gamma$ is totally positive or totally negative the $\gamma^*\Delta$ is a holomorphic curve in $\mathbb{H}^2$. If $\det\gamma$ is not totally positive or totally negative (that is in one of the real embedding it is positive and in the other it is negative) then $\gamma^*\Delta$ is anti-holomorphic. This means that it is holomorphic in $\mathbb{H}^2$ if we conjugate the complex structure in one of the copies of $\mathbb{H}^1$. The same type of change of the complex structure is considered in [?].

On each holomorphic, or anti-holomorphic curve $\gamma^*\Delta$, there is a unique geodesic triangle connecting the three given points. However, if we take two of the points, we see that they belong to two geodesic triangles. Thus they belong to two holomorphic, (or anti-holomorphic) curves. Therefore, there are two geodesic connecting the two points - each lying on different holomorphic (or anti-holomorphic) curves, as faces of the corresponding geodesic triangles, defining the curves. There are two pairings that we consider: the first one is an integral of a cusp form over a geodesic triangle and the second one is an integral of a cusp form over a geodesic diangle. If we integrate a holomorphic 2-form coming from a cusp form over a geodesic triangle, we obtain 0, if the triangle lies on an holomorphic curve. Thus the only non-zero pairings come from integration of a cusp form over a diangle or over a triangle, lying on an anti-holomorphic curve.

Now let us look again at the four cusp points together with the geodesics that we have just described. We obtain four geodesic triangles, corresponding to each triple of points among the four points, and six diangles, corresponding to the six "edges" of a tetrahedron with vertices the four given points. Thus, we obtain a "tetrahedron" with thickened edges. We will use tetrahedrons with thickened edges in order to prove analogues of a non-commutative 2-coclycle relation (see Theorem 123) for the non-commutative Hilbert modular symbol, which are analogues of Manin's non-coomutative 1-cocycle relation for the non-commutative modular symbol.

Usually, the four vertices are treated as a tetrahedton and a 2-cocycle is functional on the faces, considered as 2-chains. The boundary is defined as a sum of the 2-cocycles on each of the faces (which are triangles). The boundary of the tetrahedron gives a boundary relation of a 2-cycle.

In our case the analogue of a 2-cocycle is a functional on diangles and on triangles. And the boundary map is a sum over the faces of the thickened tetrahedron. Thus, the faces of the thickened tetrahedron are four triangles and six diangles, corresponding to the six edges of a tetrahedron.

We show that the geodesics on the boundary of a diangle or of a geodesic triangle lie on a holomorphic curves $\gamma^*\Delta$ for various elements γ with totally positive or totally negative determinant. This implies that when we take the quotient by a Hilbert modular group the holomorphic curve $\gamma^*\Delta$ becomes Hirzebruch-Zagier divisor [**?**]. Then we prove that the commutative Hilbert modular symbols paired with a cusp forms of weight $(2, 2)$ give periods in the sense of [**?**].

In order to construct a non-commutative Hilbert modular symbol, first we define a suitable generalization of iterated path integrals, which we call iterated integrals over membranes (see Section 2). We choose the word "membrane" since such integrals are invariant under suitable variation of the domain of integration.

There is a topological reason for considering non-commutative Hilbert modular symbol as opposed to only commutative one. Let us first make such comparison for the case of $SL_2(\mathbb{Z})$. The commutative modular symbol captures $H_1(X_\Gamma)$, while the non-commutative symbol captures the rational homotopy type of the modular curve X_Γ. Now, let $\tilde{X}$ be a smooth Hilbert modular surface, by which we mean the minimal desingularization of the Borel-Baily compactification due to Hirzebruch. Then the rational fundamental group of a Hilbert modular surface vanishes, $\pi_1(\tilde{X})_\mathbb{Q} = 0$, (see [**?**]). The non-commutative Hilbert modular symbol is an attempt to capture more from the rational homotopy type compared to what $H_2(\tilde{X})$ captures.

For convenience of the reader, first we define type **a** iterated integrals over membranes (Definition 99). They are simpler to define. However, they have do not have enough properties. (For example, they do not have an integral shuffle relation.) Then we define the type **b** iterated integrals over membranes (Definition 100), which involves two permutations. Type **b** has integral shuffle relation (Theorem 112 (i)) and type **a** is a particular case of type **b**.

We are mostly interested in iterated integrals of type **b**. If there is no index specifying the type of iterated integral over membranes, we assume that it is of type **b**.

Similarly to Manin's approach, we define a generating series of iterated integrals over membrane of type **b** over U, which we denote by $J(U)$. We also define a shuffle product of generating series of iterated integrals over membranes of type **b** (see Theorem 112 part (iii)),

$$\phi(J(U_1) \times_{Sh} J(U_2)) = J(U_1 \cup U_2)$$

for disjoint manifolds with corners of dimension 2, U_1 and U_2, as subsets of $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, (see [**?**]). This shuffle product generalizes the composition of generating series of iterated path integrals, namely, $J_a^b J_b^c = J_a^c$, to dimension 2. Note that similar definition is also possible in higher dimensions. Also $J(U)$ is invariant under homotopy. This allows to consider cocycles and coboundaries, where the relations use homotopy invariance and the values at the different cells can be composed via the shuffle product.

We define non-commutative Hilbert modular symbols c^1 and c^2. The cocycle c^1 is the functional J on certain geodesic diangles. The cocycle c^2 is the functional J on geodesic triangles. Then c^1 is a 1-cocycle such that if we change the base point of c^1 then c^1 is modified by a coboudary. Similarly, c^2 is a 2-cocycle up to finitely many multiples of different values of c^1. Also, if we change the base point of c^2 then c^2 is modified by a coboundary up to a finitely many multiples of different values of c^1.

In the last Subsection, we define a multiple L-values associated to cusp forms and we compare them to multiple Dedekind zeta values. First, we explore a modular symbol over one diangle and compare it with (multiple) Dedekind zeta values with summation over one discrete cone [**?**]. However, then the two series look very different. We obtain that the multiple L-values are non- commutative modular symbols defined as J evaluated at an infinite union of diangles. We obtain that such L-values are very similar to the sum of multiple Dedekind zeta values, in the same way as the integrals in the Manin's non-commutative modular symbol are similar to the multiple zeta values (MZV). Then the sum of the multiple Dedekind zeta values is over an infinite union of cones. The idea to consider cones originated in [**?**] and more generally in [**?**].

Classical or commutative modular symbols for $SL_3(\mathbb{Z})$ and $SL_4(\mathbb{Z})$ were constructed by Ash and Gunnels [**?**]. For $GL_2(\mathcal{O}_K)$, where K is a real quadratic field, Gunnels and Yasaki have defined a modular symbol based on Voronoi decomposition of a fundamental domain, in order to compute the 3-rd cohomlogy group of $GL_2(\mathcal{O}_K)$, (see [**?**]). (For the Hilbert modular group, $SL_2(\mathcal{O}_K)$, one may consult [**?**] and [**?**].) In contrast, here we use a geodesic triangulation of $\mathbb{H}^2/SL_2(\mathcal{O}_K)$. We are interested mostly in 2-cells, whose boundaries are geodesics. One of the (commutative) symbols that we define here, resembles combinatorially the symplectic modular symbol of Gunnells, [**?**]. However, the meanings of the two type of symbols and their approaches are different.

6.1 Manin's non-commutative modular symbol

6.1.1 Iterated path integrals

In this section we would like to recall the definition and the main properties of the Manin's non-commutative modular symbol, (see [**?**]). In his paper [**?**], Manin uses iterated path integrals on a modular curve and on its universal cover - the upper half plane. Using a generalization of iterated path integrals to higher dimensions, which we call iterated integrals over membranes, we are going to make analogous constructions to Manin's approach suitable for Hilbert modular surfaces (without the cusps) and for its universal cover - a product of two upper half planes.

Here we recall iterated path integrals (see also [?], [?], [?], [?]). In the next Section, we are going to generalize them to iterated integrals over membranes.

Definition 89 *Let $\omega_1, \ldots, \omega_m$ be m holomorphic 1-forms on the upper half plane together with the cusps, $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. Let*

$$g : [0,1] \to \mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q}),$$

be a piece-wise smooth path. We define an iterated integral

$$\int_g \omega_1 \ldots \omega_m = \int \ldots \int_{0<t_1<t_2\cdots<t_m<1} g^*\omega_1(t_1) \wedge \cdots \wedge g^*\omega_m(t_m).$$

Let $X_1, \ldots, X_m$ be formal variables. Consider the differential equation

$$dF(\Omega) = F(\Omega)(X_1\omega_1 + \ldots X_n\omega_m) \tag{6.1}$$

with values in the ring of formal power series in the variables $X_1, \ldots, X_m$ over the ring of holomorphic functions on the upper half plane $\mathcal{O} << X_1, \ldots, X_m >>$. There is a unique solution with initial condition $g(0) = 1$. Then F at the end of the path, that is at $g(1)$, has the value

$$F_g(\Omega) = 1 + \sum_{i=1}^{m} X_i \int_g \omega_i + \sum_{i,j=1}^{m} X_iX_j \int_g \omega_i\omega_j + + \sum_{i,j,k=1}^{m} X_iX_jX_k \int_g \omega_i\omega_j\omega_k + \ldots \tag{6.2}$$

Using the Solution (6.2) to Equation (6.1), we prove the following theorem.

Theorem 90 *Let g_1 and g_2 be two paths such that the end of g_1, $g_1(1)$ is equal to the beginning of g_2, $g_2(0)$. Let g_1g_2 denote the concatenation of g_1 and g_2. Then*

$$F_{g_1g_2}(\Omega) = F_{g_1}(\Omega)F_{g_2}(\Omega).$$

Proof. The left hand side is the value of the solution of the linear first order ordinary differential equation at the point $g_2(1)$. From the uniqueness of the solution, we have that the solution along g_2 gives the same result, when the initial condition at $g_2(0)$ is $F_{g_1}(\Omega)$. That result is $F_{g_1}(\Omega)F_{g_2}(\Omega)$.

The same result can be proven via product formula for iterated integrals. We need this alternative proof in order to use it for generalization to higher dimensions.

Lemma 91 *(Product Formula) Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and g_1, g_2 be two paths such that the end of g_1 is the beginning of g_2, that is*

$g_1(1) = g_2(0)$. As before we denote by g_1g_2 the concatenation of the path g_1 and g_2. Then

$$\int_{g_1g_2} \omega_1 \cdots \omega_m = \sum_{i=1}^{m} \int_{g_1} \omega_1 \cdots \omega_i \int_{g_2} \omega_{i+1} \cdots \omega_m.$$

Definition 92 *The set of all shuffles $sh(i,j)$ is a subset of all permutations of the set $\{1, 2, \ldots, i+j\}$ such that*

$$\sigma(1) < \cdots < \sigma(i)$$

and

$$\sigma(i+1) < \cdots < \sigma(i+j).$$

Such a permutation σ is called a shuffle.

Lemma 93 *(Shuffle Relation) Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and let g be a path. Then*

$$\int_g \omega_1 \cdots \omega_i \int_g \omega_{i+1} \cdots \omega_m = \sum_{\sigma \in sh(i,m-i)} \int_g \omega_{\sigma(1)} \cdots \omega_{\sigma(m)},$$

where $sh(i,j)$ is the set of shuffles from Definition 92.

Construction of Manin's non-commutative modular symbol

Now let g be a geodesic connecting two cusps a and b in the completed upper half plane $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ be a finite set of holomorphic cusp forms with respect to a congruence subgroup Γ of $SL_2(\mathbb{Z})$. Let

$$J_a^b = F_g(\Omega).$$

As a reformulation of Theorem 90, we obtain the following.

Lemma 94

$$J_a^b J_b^c = J_a^c.$$

We give a direct consequence of it.

Corollary 95

$$J_b^a = (J_a^b)^{-1}.$$

Now we are ready to define Manin's non-commutative modular symbol. Let Π be a subgroup of the invertible elements $\mathbb{C} << X_1, \ldots, X_m >>$ with constant term 1.

Following Manin, we present the key Theorem for and Definition of the non-commutative modular symbol.

Theorem 96 *Put*

$$c_a^1(\gamma) = J_{\gamma a}^a.$$

Then c_a^1 represent a cohomology class in $H^1(\Gamma, \Pi)$ independent of the base point a

Proof. First c_a^1 is a cocycle:

$$dc_a^1(\beta,\gamma) = J^a_{\beta a}(\beta \cdot J^a_{\gamma a})(J^a_{\beta\gamma a})^{-1} = J^a_{\beta a} J^{\beta a}_{\beta\gamma a} J^{\beta\gamma a}_a = 1.$$

Second, c_a^1 and c_b^1 are homologous:

$$c_a^1(\gamma) = J^a_{\gamma a} = J^a_b J^b_{\gamma b} J^{\gamma b}_{\gamma a} = J^a_b c_b^1(\gamma)(\gamma \cdot J^a_b)^{-1}.$$

Definition 97 *A non-commutative modular symbol as a non-abelian cohomology class in $H^1(\Gamma, \Pi)$, with representative*

$$c_a^1(\gamma) = J^a_{\gamma a},$$

NC modular symbol and multiple zeta values

Non-commutative 1-cocycle and non-commutative 1-coboundary

6.2 Non-commutative Hilbert modular symbols

6.2.1 Iterated integrals over membranes, revisited

Iterated integrals over membranes are a higher dimensional analogue of iterated path integrals. This technical tool was used in [**?**] for constructing multiple Dedekind zeta values and in [**?**] for proving new and classical reciprocity laws on algebraic surfaces. It appeared first in the author preprint [**?**] for the purpose of non-commutative Hilbert modular symbols.

Let $\mathbb{H}^1$ be the upper half plane. Let $\mathbb{H}^2$ be a product of two upper half planes. We are interested in the action of $GL_2(K)$, where K is a real quadratic field. This group acts on $\mathbb{H}^2$ by linear fractional transforms. It is convenient to introduce cusp points $\mathbb{P}^1(K)$ as boundary points of $\mathbb{H}^2$.

Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$, which are continuous at the cusps $\mathbb{P}^1(K)$. Let

$$g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a continuous map, which is smooth almost everywhere. Denote by F^1 and F^2 the following coordinate-wise foliations: For any $a \in [0,1]$, define the leaves

$$F^1_a = \{(t_1,t_2) \in [0,1]^2 \mid t_1 = a\}.$$

and

$$F^2_a = \{(t_1,t_2) \in [0,1]^2 \mid t_2 = a\}.$$

Definition 98 *We call the above map $g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ a membrane if it is continuous and piecewise differentiable map such that $g(F^1_a)$ and $g(F^2_a)$ belong to a finite union of holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ for all constants a.*

We define two types of iterated integrals over membranes - type **a** and type **b**. Type **a** consists of linear iteration and type **b** is more general and involves permutations. Type **a** is less general, more intuitive. The advantage of type **b** is that we have integral shuffle relation (Theorem 112). In other words a product of two integrals of type **b** can be expresses as a finite sum of iterated integrals over membranes of type **b**. However, one might not be able to express a product of two integrals of type **a** as a sum of finitely many integrals of type **a**.

Definition 99 *(Type* **a***, ordered iteration over membranes)*

$$\int_g \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^{m} g^*\omega_i(t_{1,j}, t_{2,j}),$$

where

$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \ \mid\ 0 < t_{1,1} < \cdots < t_{1,m} < 1,\ \ 0 < t_{2,1} < \cdots < t_{2,m} < 1\}$$

Definition 100 *(Type* **b***, 2 permutations) Let ρ_1, ρ_2 be two permutations of the set $\{1, 2 \dots, m\}$.*

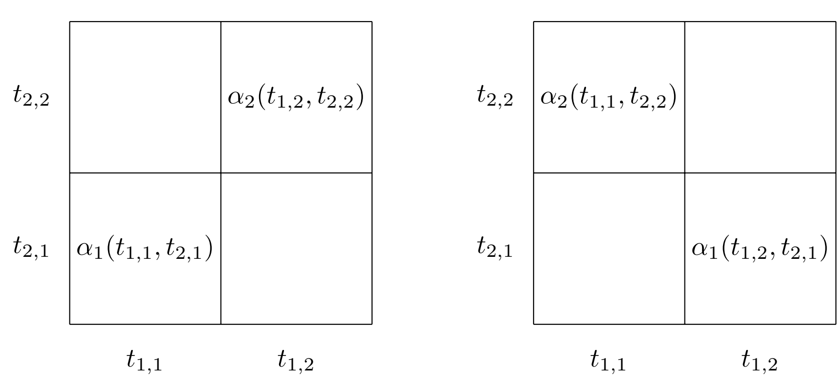
$$\int_g^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^{m} g^*\omega_j(t_{1,\rho_1(j)}, t_{2,\rho_2(j)}),$$

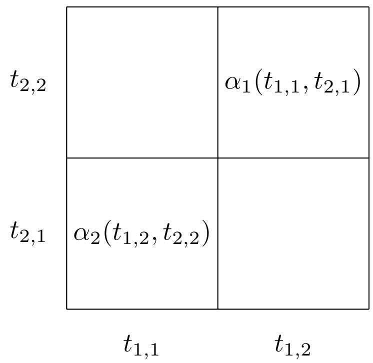
where

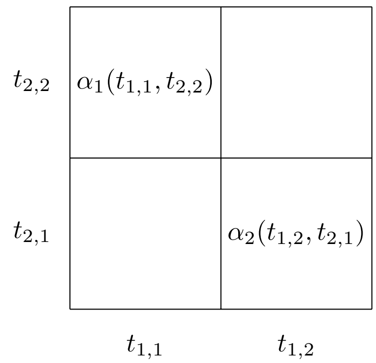
$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \ \mid\ 0 < t_{1,1} < \cdots < t_{1,m} < 1,\ \ 0 < t_{2,1} < \cdots < t_{2,m} < 1\}$$

Examples of iterated integral of type b: Let $\alpha_i(t_1, t_2) = g^*\omega_i(t_1, t_2)$. Denote by (1) the trivial permutation and by (12) the permutation exchanging 1 and 2.

1. The following 4 diagrams





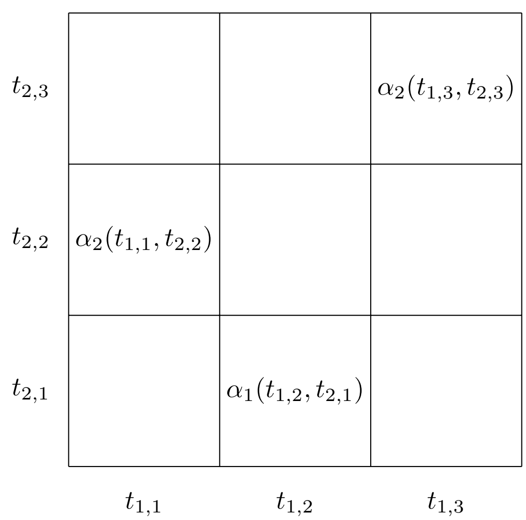


correspond, respectively, to the integrals

$$\int_g^{(1),(1)} \omega_1 \cdot \omega_2, \qquad \int_g^{(12),(1)} \omega_1 \cdot \omega_2,$$

$$\int_g^{(12),(12)} \omega_1 \cdot \omega_2, \qquad \int_g^{(1),(12)} \omega_1 \cdot \omega_2,$$

2. The following diagram



corresponds to the integral

$$\int_g^{(12),(1)} \omega_1 \cdot \omega_2 \cdot \omega_3.$$

Remark 101 *Let us give more intuition about Definition 100. Each of the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ has two arguments. Consider the set of first* *arguments for each of the differential forms $g^*\omega_1, \dots, g^*\omega_m$. They are ordered as follows*

$$0 < t_{1,1} < t_{1,2} < \dots < t_{1,m} < 1, \tag{6.3}$$

(They are some of the coordinates of the domain D.) Since $g^*\omega_j$ depends on $t_{1,\rho_1(j)}$, we have that $t_{1,k}$ is an argument of $g^*\omega_{\rho_1^{-1}(k)}$, where $k = \rho_1(j)$. Then we can order the differential forms $g^*\omega_1, \dots, g^*\omega_m$ according to the order of their first arguments given by the Inequalities (6.3)*, which is*

$$g^*\omega_{\rho_1^{-1}(1)}, g^*\omega_{\rho_1^{-1}(2)}, \dots, g^*\omega_{\rho_1^{-1}(m)}$$

Similarly, we can order the differential forms $g^*\omega_1, \dots, g^*\omega_m$, with respect to the order of their second arguments

$$g^*\omega_{\rho_2^{-1}(1)}, g^*\omega_{\rho_2^{-1}(2)}, \dots, g^*\omega_{\rho_2^{-1}(m)}$$

We call the first ordering **horizontal** *and the second ordering* **vertical***.*

Now we are going to examine homotopy of a domain of integration and how that reflects on the integral. Let $g_s : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ be a family of membranes such that $g_s(0,0) = \infty$ and $g_s(1,1) = 0$. Assume that the parameter s is in the interval $[0,1]$.

Put $h(s, t_1, t_2) = g_s(t_1, t_2)$ to be a homotopy between g_0 and g_1. Let

$$G_s : [0,1]^{2m} \to (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m,$$

be the map

$$G_s(t_{1,1}, \dots, t_{2,m}) = \left(g_s\left(t_{1,\sigma_1(1)}, t_{2,\sigma_2(1)}\right), g_s\left(t_{1,\sigma_1(2)}, t_{2,\sigma_2(2)}\right), \dots, g_s\left(t_{1,\sigma_1(m)}, t_{2,\sigma(m)}\right)\right).$$

Let H be the induces homotopy between G_0 and G_1, defined by

$$H(s, t_{1,1}, \dots, t_{2,m}) = G_s(t_{1,1}, \dots, t_{2,m}).$$

We define diagonals in $(\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$. For $k = 1, \dots, m-1$, let $H_{1,k} = H|_{z_{1,k} = z_{1,k+1}}$. Let also, $H_{1,0} = H|_{z_{1,1}=0}$ and $H_{1,m} = H|_{z_{1,m}=1}$. Similarly, for $k = 1, \dots, m-1$, let $H_{2,k} = H|_{z_{2,k}=z_{2,k+1}}$. Let also, $H_{2,0} = H|_{z_{2,1}=0}$ and $H_{2,m} = H|_{z_{2,m}=1}$.

Also, we define diagonals in $[0,1]^{2m}$. We define $D_{1,k}$ for $k = 0, \dots, m$ as $D_{1,0} = D|_{t_{1,1}=0}$, $D_{1,k} = D|_{t_{1,k}=t_{1,k+1}}$, for $k = 1, \dots, m-1$ and $D_{1,m} = D|_{t_{1,m}=1}$. Similarly, we define $D_{2,k}$ for $k = 0, \dots, m$ as $D_{2,0} = D|_{t_{2,1}=0}$, $D_{2,k} = D|_{t_{2,k}=t_{2,k+1}}$, for $k = 1, \dots, m-1$ and $D_{2,m} = D|_{t_{2,m}=1}$.

Theorem 102 *(Homotopy Invariance Theorem I) The iterated integrals on membranes from Definition 100 (of type* **b***) are homotopy invariant, when the homotopy preserves the boundary of the membrane.*

Proof. Let

$$\Omega = \bigwedge_{j=1}^{m} \omega_j(z_{1,\sigma_1(j)}, z_{2,\sigma_2(j)}).$$

Note that Ω is a closed form, since ω_i is a form of top dimension. By Stokes Theorem, we have

$$0 = \int_{s=0}^{s=1} \int_D H^* d\Omega = \tag{6.4}$$

$$= \int_D G_1^*\Omega - \int_D G_0^*\Omega \pm \tag{6.5}$$

$$\pm \int_{s=0}^{s=1} \sum_{k=1}^{m-1} \left(\int_{D_{1,k}} \pm \int_{D_{2,k}} \right) H^*\Omega \tag{6.6}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,0}} \pm \int_{D_{2,0}} \right) H^*\Omega \tag{6.7}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,2m}} \pm \int_{D_{2,2m}} \right) H^*\Omega \tag{6.8}$$

We want to show that the difference in the terms in (6.5) is zero. It is enough to show that each of the terms (6.6), (6.7) and (6.8) are zero. If $z_k = z_{k+1}$, then the wedge of the corresponding differential forms will vanish. Thus the terms in (6.6) are zero. If $z_1 = 0$ then $dt_1 = 0$, defined via the pull-back H^*. Then the terms (6.7) are equal to zero. Similarly, we obtain that the last integral (6.8) vanishes.

Let A be a manifold with corners of dimension 2 in $[0,1]^2$. We recall the domain of integration

$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \mid 0 < t_{1,1} < \dots < t_{1,m} < 1,\ 0 < t_{2,1} < \dots < t_{2,m} < 1\}.$$

Let us define

$$A^D = \{(t_{1,1}, \dots, t_{2,m}) \in D \mid (t_{1,i}, t_{2,i}) \in A \text{ for } i = 1, \dots, m\}$$

Let ρ_1 and ρ_2 be two permutations of m elements. We define

$$G(t_{1,1}, \dots, t_{2,m}) = \left(g\left(t_{1,\sigma_1(1)}, t_{2,\sigma_2(1)}\right), g\left(t_{1,\sigma_1(2)}, t_{2,\sigma_2(2)}\right), \dots, g\left(t_{1,\sigma_1(m)}, t_{2,\sigma(m)}\right)\right).$$

as a function on A^D. Recall

$$\Omega = \bigwedge_{j=1}^{m} \omega_j(z_{1,\sigma_1(j)}, z_{2,\sigma_2(j)}).$$

Definition 103 *With the above notation, we define an iterated integral over a membrane restricted to a domain U, where $U = g(A)$ as*

$$\int_{g,U}^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_{A^D} G^*\Omega.$$

Let A_1 and A_2 be two manifolds with corners, with a common component of the boundary as subsets of $[0,1]^2$. Let $A = A_1 \cup A_2$. Let s be a map of sets with values 1 or 2,

$$s : \{1, \ldots, m\} \to \{1, 2\}.$$

We define a certain set A_s^D as a subset of A^D in the following way: Consider the image of the map G. It has m coordinates. The first coordinate, $g\left(t_{1,\sigma_1(1)}, t_{2,\sigma_2(1)}\right)$, will be restricted to the set $A_{s(1)}$. The second coordinate, $g\left(t_{1,\sigma_1(2)}, t_{2,\sigma_2(2)}\right)$, will be restricted to $A_{s(2)}, \ldots$ and the last m coordinate $g\left(t_{1,\sigma_1(m)}, t_{2,\sigma(m)}\right)$ will be restricted to $A_{s(m)}$. Formally, this can be written as

$$A_s^D = \{t_{1,1}, \ldots, t_{2,m}) \in A^D \mid (t_{1,\rho_1(i)}, t_{2,\rho_2(i)}) \in A_{s(i)} \text{ for } i = 1, \ldots, m\}.$$

Note that the image of the map s is 1 or 2.

Definition 104 *With the above notation, we define an iterated integral over two domains U_1 and U_2, where $U_i = g(A_i)$ and $U = U_1 \cup U_2$ by*

$$\int_{g,U,s}^{\rho_1\rho_2} \omega_1 \cdots \omega_m = \int_{A_s^D} G^* \Omega. \tag{6.9}$$

Again we examine homotopy of iterated integrals over membranes. Now we restrict the domain of integration to a manifold with corners A as a subset of $[0,1]^2$. Assume that for the boundary of a domain A, denoted by ∂A, we have that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$. We call a *complex boundary* of $g(\partial A)$ the minimal union of complex analytic (holomorphic) curves such that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$.

Theorem 105 *(Homotopy Invariance Theorem II) Iterated integrals over membranes are homotopy invariant with respect to a homotopy that changes the boundary ∂U of the domain of integration U, so that the boundary varies on a finite union of complex analytic curves.*

Proof. Assume that $g_0(\partial A)$ and $g_1(\partial A)$ have the same complex boundary. Let h be a homotopy between g_0 and g_1, such that for each value of s we have that $h(s, \partial A)$ has the same complex boundary as $h(0, \partial A) = g_0(\partial A)$. Let $A \subset B$ be a strict inclusion of disks. Identify $B - A^\circ$ with $A \times [0,1]$. Let $i : B - A^\circ \to [0,1] \times \partial A$. Here A° is the interior of A and ∂A is the boundary of A. Let $\tilde{g}_0$ be a map from B to $\mathbb{H}^2$ so that $\tilde{g}_0(a) = g_0(a)$ for $a \in A$ and $\tilde{g}_0(b) \in h(i(b))$. Since the restriction of pull-back $(\tilde{g}_0^* \omega_i)|_{B-A} = 0$ is mapped to a finite union of complex curves, then it vanishes. Therefore

$$\int_A g_0^* \Omega = \int_B \tilde{g}_0^* \Omega. \tag{6.10}$$

Let $\tilde{g}_1$ be a membrane from B defined by $\tilde{g}_1(a) = g_1(a)$ for $a \in A$ and $\tilde{g}_1(b) = \tilde{g}_1(a)$ for i(b)=(s,a). (Note that $i(b) \in [0,1] \times \partial A$.) Again

$$\int_A g_1^* \Omega = \int_B \tilde{g}_1^* \Omega. \tag{6.11}$$

However, the boundary of B is mapped to the same set (point-wise) by both $\tilde{g}_0$ and $\tilde{g}_1$. Moreover, the homotopy between g_0 and g_1 extends to a homotopy between $\tilde{g}_0$ and $\tilde{g}_1$ that respects the inclusion into the complex boundary. Thus by Theorem 102, we have that

$$\int_B \tilde{g}_0^*\Omega = \int_B \tilde{g}_1^*\Omega.$$

Using Equations (6.10) and (6.11), we complete the proof of this theorem.

We are going to define two types of generating series - type **a** and type **b**, corresponding to the iterated integrals of membranes of type **a** and type **b**.

Let A be a domain in $\mathbb{R}^2$. Let g be a membrane. Let $U = g(A)$. Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$. We define a generating series of type a by

$$J^a(U) = 1 + \sum_{k=1}^{\infty} \sum_{c:\{1,\ldots,k\}\to\{1,\ldots,m\}} X_{c(1)} \otimes \cdots \otimes X_{c(k)} \int_{g,U} \omega_{c(1)} \ldots \omega_{c(k)}.$$

Let $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ be a map of sets and let ρ_1, ρ_2 be permutations of $\{1, 2, \ldots, k\}$. Two triples (c', ρ_1', ρ_2') and $(c'', \rho_1'', \rho_2'')$ are equivalent if they are in the same orbit of the permutation group S_k. That is, $(c'', \rho_1'', \rho_2'') \sim (c', \rho_1', \rho_2')$ if for some $\tau \in S_k$ we have $c'' = c'\tau^{-1}$, $\rho_1'' = \rho_1'\tau^{-1}$ and $\rho_2'' = \rho_2'\tau^{-1}$. Then for each equivalence class of a triple (c, ρ_1, ρ_2), we can associate a unique pair $(c \circ \rho_1, c \circ \rho_2)$, (which are precisely the indices of the X-variables and Y-variables in (6.12) and (6.13), respectively.) The reason for using such an equivalence is that the integral in (6.13) is invariant by the above action of $\tau \in S_k$ on the triple (c, ρ_1, ρ_2). The values of the generation series is in the ring of formal power series

$$R = \mathbb{C} << X_1, Y_1, \ldots, X_m, Y_m >> .$$

Definition 106 *We define the generating series of type* **b** *by*

$$J^b(U) = 1 + \sum_{k=1}^{\infty} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1))} \otimes \cdots \otimes X_{c(\rho_1^{-1}(k))} \otimes \tag{6.12}$$

$$\otimes Y_{c(\rho_2^{-1}(1))} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(k))} \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \ldots \omega_{c(k)}, \tag{6.13}$$

where the second summation is over all maps of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and all permutations ρ_1, ρ_2 of k elements, up to the above equivalence.

Definition 107 *We define a generating series of iterated integrals over two disjoint domain U_1 and U_2 (see Definition 104). Let $U_i = g(A_i)$.*

$$J(U_1, U_2) = 1 + \sum_{k=1}^{\infty} \sum_{s:\{1,\ldots,k\}\to\{1,2\}} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1)),s(1)} \otimes \cdots \otimes X_{c(\rho_1^{-1}(k)),s(k)} \otimes \tag{6.14}$$

$$\otimes Y_{c(\rho_2^{-1}(1)),s(1)} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(k)),s(k)} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \ldots \omega_{c(k)}, \tag{6.15}$$

The generating series takes values in the ring of formal power series

$$R' = \mathbb{C} << X_{1,1}, X_{1,2}, Y_{1,1}, Y_{1,2}, \ldots, X_{m,1}, X_{m,2}, Y_{m,1}, Y_{m,2} >> .$$

Lemma 108 *Let $\phi : R' \to R$ be a homomorphism of rings defined by $\phi(X_{i,1}) = \phi(X_{i,2}) = X_i$ and $\phi(Y_{i,1}) = \phi(Y_{i,2}) = Y_i$. Then*

$$\phi(J(U_1, U_2)) = J(U),$$

where $U = U_1 \cup U_2$.

Proof. After applying the homomorphism ϕ the formal variables on the left hand side become independent of the map s. Therefore, we have to examine what happens when we sum over all possible maps s. The map $s(i)$ is 1 or 2. Their meaning is the following: If $s(i) = 1$ then we restrict the form $g^*\omega_{c(i)}$ to A_1 (instead of to A). Similarly, if $s(i) = 2$, we restrict $g^*\omega_{c(i)}$ to A_2. If we add both choices, restriction to A_1 and restriction to A_2, then we obtain restriction of $g^*\omega_{c(i)}$ to $A = A_1 \cup A_2$. Thus, we obtain the formula

$$\sum_{s:\{1,\ldots,k\}\to\{1,2\}} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \ldots \omega_{c(k)} = \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \ldots \omega_{c(k)}.$$

We do the same for every monomial in R. That proves the above Lemma for the generating series.

The regions of integration that we are mostly interested in will be diangles, that is, a 2-cell whose boundary has two vertices and two edges. All other regions that we will deal with are going to be a finite union of diangles. The first type of decomposition is based on a union of two diandles with a common vertex. The second type of decomposition will be based on two diangles with a common edge.

Let g_1 and g_2 be two membranes. Let $P = (0,0)$ and $Q = (1,1)$ be the vertices of a diangle A and $Q = (1,1)$ and $R = (2,2)$ are the two points of a diangle B as subsets of $\mathbb{R}^2$. Assume that A lies within the rectangle with vertices $(0,0), (0,1), (1,1), (1,0)$. Similarly, assume that B lies within the rectangle $(1,1), (1,2), (2,2), (2,1)$. Let $U = g(A)$ and $V = g(B)$.

Theorem 109 *(i)*

$$\int_{g,U\cup V} \omega_1 \ldots \omega_m = \sum_{j=0}^{m} \int_{g,U} \omega_1 \ldots \omega_j \int_{g,V} \omega_{j+1} \ldots \omega_m;$$

(ii)

$$J^a(g; A \cup B; \Omega) = J^a(g; A; \Omega) J^a(g; B; \Omega).$$

The proof of the first statement is essentially the same as the combinatorial proof for composition of path, when one considers iterated path integrals (see Lemma 93). The second statement is combining all compositions into generating series, resembling the Manin's approach for non-commutative modular symbol.

For generating series of type b, we have a similar statement.

Definition 110 *Let ρ' and ρ'' be two permutations of the sets $\{1,\dots,i\}$, $\{i+1,\dots,i+j\}$, respectively. We define the permutation $\rho'^{-1}\cup\rho''^{-1}$ of $\{1,\dots,i+j\}$, which acts on $\{1,\dots,i\}$ as ρ'^{-1} and on $\{i+1,\dots,i+j\}$ as ρ''^{-1}. We define the set of shuffles of two given permutations, denoted by $sh(\rho',\rho'')$, as the set of all permutations ρ of the set $\{1,2,\dots,i+j\}$ such that ρ^{-1} is the composition of a shuffle of sets $\tau\in sh(i,j)$ (see Definition 92) and with $\rho'^{-1}\cup\rho''^{-1}$. That is,*

$$\rho^{-1}=\tau\circ(\rho'^{-1}\cup\rho''^{-1}).$$

Definition 111 *We define a shuffle of two monomials*

$$M'=X_{c'(\rho_1'^{-1}(1))}\otimes\cdots\otimes X_{c'(\rho_1'^{-1}(i))}\otimes Y_{c'(\rho_2'^{-1}(1))}\otimes\cdots\otimes Y_{c'(\rho_2'^{-1}(i))}\int_{g,U'}^{\rho_1',\rho_2'}\omega_{c'(1)}\dots\omega_{c'(i)}$$

and

$$M''=X_{c''(\rho_1''^{-1}(1))}\otimes\cdots\otimes X_{c''(\rho_1''^{-1}(j))}\otimes Y_{c''(\rho_2''^{-1}(1))}\otimes\cdots\otimes Y_{c''(\rho_2''^{-1}(j))}\int_{g,U''}^{\rho_1'',\rho_2''}\omega_{c''(i+1)}\dots\omega_{c''(i+j)},$$

where ρ_1' and ρ_2' are permutations of $\{1,\dots,i\}$ and c' is a map of sets $c':\{1,\dots,i\}\to\{1,\dots,m\}$, and ρ_1'' and ρ_2'' are permutations of $\{i+1,\dots,i+j\}$ and c'' is a map of sets $c'':\{i+1,\dots,i+j\}\to\{1,\dots,m\}$. By a shuffle product of the monomials M' and M'', we mean the following sum

$$\begin{aligned}M'\times_{Sh}M''=&\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}X_{c(\rho_1^{-1}(1)),s(1)}\otimes\cdots\otimes X_{c(\rho_1^{-1}(i+j)),s(i+j)}\otimes\\&\otimes Y_{c(\rho_2^{-1}(1)),s(1)}\otimes\cdots\otimes Y_{c(\rho_2^{-1}(i+j)),s(i+j)}\int_{g,U'\cup U''}^{\rho_1,\rho_2}\omega_{c(1)}\dots\omega_{c(i+j)},\end{aligned}$$

where $c:\{1,\dots,i+j\}\to\{1,\dots,m\}$ such that the map c restricted to the first i elements is c' and c restricted to the last j elements is c''. Here the maps s takes the value 1 on the set $c^{-1}\{1,\dots,i\}=c'^{-1}\{1,\dots,i\}$ and it takes the value 2 on the set $c^{-1}\{i+1,\dots,i+j\}=c''^{-1}\{i+1,\dots,i+j\}$.

Theorem 112 *(Shuffle product) For iterated integrals of type* **b** *and the corresponding generating series, we have the following shuffle relations: (i)*

$$\int_{g,U}^{\rho_1',\rho_2'}\omega_1\dots\omega_j\int_{g,U}^{\rho_1'',\rho_2''}\omega_{j+1}\dots\omega_m=\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}\int_{g,U}^{\rho_1,\rho_2}\omega_1\cdots\omega_m \tag{6.16}$$

(ii)

$$\int_{g,U'}^{\rho_1',\rho_2'}\omega_1\dots\omega_j\int_{g,U''}^{\rho_1'',\rho_2''}\omega_{j+1}\dots\omega_m=\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}\int_{g,U,s}^{\rho_1,\rho_2}\omega_1\cdots\omega_m, \tag{6.17}$$

where s is a map from $\{1,\dots,m\}$ to $\{1,2\}$ so that $\{1,\dots,j\}$ are mapped to 1 *and the remaining elements are mapped to* 2.

(iii)

$$\phi(J^b(U')\times_{Sh}J^b(U''))=J^b(U'\cup U'') \tag{6.18}$$

Proof. For part (i), it is useful to consider the two orders of differential forms, given in Remark 101. Note that we need to order the forms both horizontally and vertically in the terminology of Remark 101. Let us consider first the horizontal order. That is the order with respect to the first variables of the differential forms $g^*\omega_{\rho_1'^{-1}(1)},\dots,g^*\omega_{\rho_1'^{-1}(j)}$ and $g^*\omega_{\rho_1''^{-1}(j+1)},\dots,g^*\omega_{\rho_1''^{-1}(m)}$, corresponding to the two integrals on the left hand side of Equation (6.16). In order to arrange both of the above orderings in one sequence of increasing first arguments, we need to shuffle them (similarly to a shuffle of a deck of cards.) That leads to $\rho_1\in sh(\rho_1',\rho_1'')$ (see Definition 110). We proceed similarly, with the second arguments and the permutations ρ_2',ρ_2'' and ρ_2.

For part (ii) apply the equality from part (i) when the differential forms $g^*\omega_1,\dots,g^*\omega_j$ are multiplied by the function $\mathbf{1_{A'}}$ defined by

$$\mathbf{1_{A'}}(x)=\begin{cases}1 & \text{for } x\in A'\\ 0 & \text{for } x\notin A'\end{cases}$$

and the differential forms $g^*\omega_{j+1},\dots,g^*\omega_m$ are multiplied by $\mathbf{1_{A''}}$. For part (iii), we are going to establish similar relation among generating series as elements of R'. Applying the homomorphism $\phi:R'\to R$ from Lemma 108, we obtain desired equality. Every monomial from $J(U_1)$ is of the form

$$M'=X_{c'(\rho_1'^{-1}(1))}\otimes\cdots\otimes X_{c'(\rho_1'^{-1}(i))}\otimes Y_{c'(\rho_2'^{-1}(1))}\otimes\cdots\otimes Y_{c'(\rho_2'^{-1}(i))}\int_{g,U'}^{\rho_1',\rho_2'}\omega_{c'(1)}\dots\omega_{c'(i)}$$

and similarly every monomial from $J(U_2)$ is of the form

$$\begin{aligned}M''=&X_{c''(\rho_1''^{-1}(1))}\otimes\cdots\otimes X_{c''(\rho_1''^{-1}(j))}\otimes\\ &\otimes Y_{c''(\rho_2''^{-1}(1))}\otimes\cdots\otimes Y_{c''(\rho_2''^{-1}(j))}\int_{g,U''}^{\rho_1'',\rho_2''}\omega_{c''(i+1)}\dots\omega_{c''(i+j)},\end{aligned}$$

where ρ_1' and ρ_2' are permutations of $\{1,\dots,i\}$ and c' is a map of sets $c':\{1,\dots,i\}\to\{1,\dots,m\}$, and ρ_1'' and ρ_2'' are permutations of $\{i+1,\dots,i+j\}$ and c'' is a map of sets $c'':\{i+1,\dots,i+j\}\to\{1,\dots,m\}$. We take the shuffle product of the monomials M' and M'' (see Definition 111)

$$\begin{aligned}M'\times_{Sh}M''=&\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}X_{c(\rho_1^{-1}(1)),s(1)}\otimes\cdots\otimes X_{c(\rho_1^{-1}(i+j)),s(i+j)}\otimes\\ &\otimes Y_{c(\rho_2^{-1}(1)),s(1)}\otimes\cdots\otimes Y_{c(\rho_2^{-1}(i+j)),s(i+j)}\int_{g,U,s}^{\rho_1,\rho_2}\omega_{c(1)}\dots\omega_{c(i+j)},\end{aligned}$$

where the map s takes the value 1 on the set $c^{-1}\{1,\dots,i\}$ and takes the value 2 on the set $c^{-1}\{i+1,\dots,i+j\}$. It determines the map s uniquely.

In order to complete the proof, we have to show that every monomial in $J(U_1, U_2)$ can be obtained in exact one way as a result (on the right hand side) of a shuffle product of a pair of monomials (M_1, M_2) from $J(U_1)$ and $J(U_2)$. Every monomial from $J(U_1, U_2)$ is characterized by two permutation ρ_1, ρ_2, and two maps of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and $s : \{1, \ldots, k\} \to \{1, 2\}$. Let i be the number of elements in $s^{-1}(1)$ and j be the number of elements in $s^{-1}(2)$. Then $i + j = k$. Then i is the number of differential forms among $g^*\omega_{c(1)}, \cdots, g^*\omega_{c(k)}$, which are restricted to the set A_1. The remaining j differential forms are restricted to A_2. Also, every permutation ρ_1 can be written in an unique way as a composition of a shuffle $\tau_1 \in sh(i, j)$ and two disjoint permutations ρ_1' and ρ_1'' of i and of j elements, respectively (see Definition 110). Similarly, ρ_2 can be written in a unique way as a product of a shuffle $\tau_2 \in sh(i, j)$ and two disjoint permutation ρ_2' and ρ_2''. The map of sets c_1 is defined as a restriction of the map c to the image of ρ_1'. Similarly, the map c_2 is defined as a restriction of the map c to the image of ρ_1''. Now we can define the monomials M' and M'' in $J(U_1)$ and $J(U_2)$, based on the triples ρ_1', ρ_2', c' and ρ_1'', ρ_2'', c'', respectively. Such monomials are unique. One can show that the shuffle product of M' and M'' contains the monomial in $J(U_1, U_2)$, that we started with, exactly once. The proof of part (iii) is completed after applying Lemma 108.

In this Section, we recall the Hilbert modular group and its action on the product of two upper half planes. Then we define commutative Hilbert module symbol, (Subsection 3.1) and its pairing with the cohomology of the Hilbert modular surface, (Subsection 3.2). In Subsections 3.3 and 3.4, we define the non-commutative Hilbert module symbols (Definition 121) as a generating series of iterated integrals over membranes of type **b**. We also examine relations among the non-commutative Hilbert modular symbols (Theorem 120), which we interpret as cocycle conditions or as a difference by a coboundary (Theorem 122). In Subsection 3.5, we make explicit computations and compare them to computations for multiple Dedekind zeta values.

6.2.2 Commutative Hilbert modular symbols

In this Subsection, we define a commutative Hilbert modular symbol, using geodesics, geodesic triangles and geodesic diangles. We prove certain relations among the non-commutative Hilbert modular symbols. We are going to use that geometric constructions here in order to define a non-commutative Hilbert modular symbol in Subsection 3.4.

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic extension of $\mathbb{Q}$. Then the ring of integers in K is

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{for } d = 1 \mod 4, \\ \\ \mathbb{Z}[\sqrt{d}] & \text{for } d = 2, 3 \mod 4. \end{cases}$$

Then $\Gamma = SL_2(\mathcal{O}_K)$ is called a Hilbert modular group. Let $\gamma \in \Gamma$. We recall the action of γ on a product of two upper half planes, $\mathbb{H}^2$. Let

$$\gamma = \gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Let a_2, b_2, c_2, d_2 be the Galois conjugate of a_1, b_1, c_1, d_1, respectively. Let us define γ_2 by

$$\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Let $z = (z_1, z_2)$ be any point of the product of two upper half planes $\mathbb{H}^2$. We define

$$\gamma z = (\gamma_1 z_1, \gamma_2 z_2),$$

where

$$\gamma_1 z_1 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1} \text{ and } \gamma_2 z_2 = \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}$$

are linear fractional transforms.

For an element $\gamma \in GL_2(K)$, we define the following action. If $\det\gamma$ is totally negative, that is, $\det\gamma_1 < 0$ and $\det\gamma_2 < 0$, then we define

$$\gamma z = \left(-\frac{a_1 \overline{z}_1 + b_1}{c_1 \overline{z}_1 + d_1}, -\frac{a_2 \overline{z}_2 + b_2}{c_2 \overline{z}_2 + d_2} \right).$$

Similarly if one of the embeddings of $\det\gamma$ is positive and the other is negative, for example, $\det\gamma_1 > 0$ and $\det\gamma_2 < 0$, such as $\det\gamma = \sqrt{d}$, then

$$\gamma z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, -\frac{a_2 \overline{z}_2 + b_2}{c_2 \overline{z}_2 + d_2} \right).$$

We add cusp points $\mathbb{P}^1(K)$ to $\mathbb{H}^2$. We are going to examine carefully geodesics joining the cusps 0, 1 and ∞.

Let $\gamma_0, \gamma_1, \gamma_\infty$ be four group elements of the Hilbert modular group. Let $z_i = \gamma_i(\infty)$ Then there exists an element of $\gamma \in GL_2(K)$ such that $\gamma(z_0) = 0$, $\gamma(z_1) = 1$ and $\gamma(z_\infty) = \infty$. Namely,

$$\gamma = \begin{pmatrix} z_1 - z_\infty & -z_0(z_1 - z_\infty) \\ z_1 - z_0 & -z_\infty(z_1 - z_0) \end{pmatrix}$$

Let

$$i : \mathbb{H} \to \mathbb{H}^2$$

$$i(x) = (x, x)$$

be the diagonal map. Let Δ be the image of the diagonal map. Consider the Hirzebruch-Zagier divisor $X = \gamma^*\Delta$. It is an analytic curve that passes through the points z_0, z_1 and z_∞. Then X is a holomorphic (or anti-holomorphic) curve in $\mathbb{H}^2$ if $\det\gamma$ is totally positive or totally negative. If $\det\gamma$ is not totally positive or totally negative, then X is holomorphic curve in $\mathbb{H}^1 \times \overline{\mathbb{H}^1} \cup \mathbb{P}^1(K)$, in other words it is anti-holomorphic curve in $\mathbb{H}^2$, such as $z_1 = -\overline{z}_2$. Let $\Delta_X = \gamma^*\Delta$

be the pull-back of the geodesic triangle Δ between the points $0, 1, \infty$ in the analytic curve X.

Given four points on the boundary in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, we are tempted to consider them a vertices of a geodesic tetrahedron in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, whose faces are triangles of the type Δ_X. However, there is one problem that we encounter: Two distinct cusps could be connected by two different geodesics in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. In particular, two triangles from the faces of the "tetrahedron" might not have a common edge, but only two common vertices. Thus, we are led to consider a *thickened tetrahedron* with two types of faces on the boundary: the first type is an ideal triangle that we have just defined and the other type is an ideal diangle - a unionn of geodesics connecting two fixed points, which has the homotopy type of a disc with two vertices and two edges. The two edges of an ideal diangle in the boundary of a thickened tetrahedron correspond to the two geodesics connecting the same two cusps, where two geodesics belong to the geodesic triangles that have the two cusps in common.

Lemma 113 *(i) Each geodesic triangle lies either on a holomorphic curve or on an anti-holomorphic curve.*

(ii) Each geodesic in a geodesic triangle Δ_X belongs both to a holomorphic curve and to an anti-holomorphic curve.

Part (i) follows from the construction of a geodesic triangle before the lemma. For part (ii), consider the following: Let $\Delta(0, 1, \infty)$ be the geodesic triangle in the diagonal of $\mathbb{H}^2$ connecting the points 0 ,1 and ∞. It is a holomorphic curve. Thus, a geodesic $\{(it, it) \in \mathbb{H}^2 \mid t > 0\}$, connecting the points 0 and ∞ as a face of the geodesic triangle $\Delta(0, 1, \infty)$ lies on a holomorphic curve. Now consider the geodesic triangle $D(0, \sqrt{d}, \infty)$. It lies on an anti-holomorphic curve in $\mathbb{H}^2$, by which we mean a complex curve in $\mathbb{H}^2$, where we have taken the complex conjugate complex structure in one of the upper half planes. Since, the linear fractional transform that sends $D(0, \sqrt{d}, \infty)$ to $D(0, 1, \infty)$ does not have totally positive (or totally negative) determinant. Explicitly, the linear fractional transform that sends $(0, \sqrt{d}, \infty)$ to $(0, 1, \infty)$ is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}$$

Then

$$(\gamma_1, \gamma_2) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \right)$$

We have $\gamma_1(it) = \frac{1}{\sqrt{d}} it$ and $\gamma_2(it) = -\frac{1}{\sqrt{d}} \overline{it} = \gamma_1(it)$. Then the same geodesic (it, it) belongs to the anti-holomorphic curve given by the pull-back of the diagonal with respect to the linear fractional map γ. Thus, we obtain that the geodesic (it, it), connecting 0 and ∞, belongs to both a holomorphic curve and an anti-holomorphic curve. Similarly, any translate of the geodesic (it, it) via a linear fractional map from $GL_2(K)$ would belong to both a holomorphic curve and an anti-holomorphic curve. That proves part (ii).

Definition 114 *Let p_1, p_2, p_3, p_4 be cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. To each triple of points p_1, p_2, p_3, we associate the geodesic triangle $\{p_1, p_2, p_3\}$ with coefficient 1 as an element of the singular chain complex in $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$. Also, to each quadruple of points p_1, p_2, p_3, p_4, we associate the geodesic diangle between the two geodesic connecting p_1 and p_2 so that the first geodesic is a face of the geodesic triangle $\{p_1, p_2, p_3\}$ and the second geodesic is a face of the geodesic triangle $\{p_1, p_2, p_4\}$. We denote such diangle by $\{p_1, p_2; p_3, p_4\}$. We call the geodesic triangle $\{p_1, p_2, p_3\}$ and the geodesic diangle $\{p_1, p_2; p_3, p_4\}$, considered as elements of $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$,* **commutative Hilbert modular symbols***.*

Theorem 115 *The commutative Hilbert modular symbols satisfy the following properties:*

1. If σ is a permutation of the set $\{1, 2, 3\}$ then

$$\{p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}\} = sign(\sigma)\{p_1, p_2, p_3\}.$$

2. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve then

$$\{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} = \{p_1, p_2, p_4\} + \{p_1, p_3, p_4\}.$$

To each four points p_1, p_2, p_3, p_4, we associate a diangle with vertices p_1 and p_2. Let $\{p_1, p_2; p_3, p_4\}$ be the corresponding symbol.

3. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve then

$$0 = \{p_1, p_2; p_3, p_4\}.$$

4. For every district four points p_1, p_2, p_3, p_4, we have the following relations based on the orientation of the domain

$$\{p_2, p_1; p_3, p_4\} = \{p_1, p_2; p_4, p_3\} = -\{p_2, p_1; p_4, p_3\} = -\{p_1, p_2; p_3, p_4\}.$$

5. For every five points p_1, p_2, p_3, p_4, p_5, we have

$$\{p_1, p_2; p_3, p_4\} + \{p_1, p_2; p_4, p_5\} = \{p_1, p_2; p_3, p_5\}.$$

6. We also have relation between the two types of commutative Hilbert modular symbols. For every four distinct points $p_1, p_2; p_3, p_4$, we have

$$\begin{aligned} 0 =& \{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} - \\ &- \{p_1, p_2, p_4\} - \{p_1, p_3, p_4\} + \\ &+ \{p_1, p_2; p_3, p_4\} + \{p_2, p_3; p_1, p_4\} + \{p_3, p_1; p_2, p_4\} + \\ &+ \{p_3, p_4; p_1, p_2\} + \{p_1, p_4; p_2, p_3\} + \{p_2, p_4; p_3, p_1\}. \end{aligned}$$

Proof. Part 1 follows from orientation of the simplex in singular homology. Part 2 is an equality induced by two different triangulations on a holomorphic (or anti-holomoprhic) curve with 4 vertices. In that setting the diangles are trivial, which proves Part 3. Part 4 follows from orientation of the diangle.

Part 5 corresponds to a union of two geodesic diangles with a common face, given by a third geodesic diangle. Part 5 will be used for a non-commutative 1-cocycle relation for the non-commutative Hilbert modular symbol (see Theorem 122). Part 6 is a boundary relation for the boundary of a thickened tetrahedron. By a thickened tetrahedron we mean a union of four geodesic triangles corresponding to each triple of points among the four points p_1, p_2, p_3, p_4 together with six geodesic diangles that correspond to the area between the faces of the geodesic triangles. They correspond exactly to the thickening of the six edges of a tetrahedron. Part 6 will be used for establishing a non-commutative 2-cocycle relation for the non-commutative Hilbert modular symbol (see Theorem 123).

6.2.3 Pairing of the modular symbols with cohomology

In this subsection, we consider pairings between commutative Hilbert modular symbols and cusp forms. In some cases, we prove that such pairings give periods in the sense of [**?**].

We are interested in holomorphic cusp forms with respect to Γ. Equivalently, we can consider the holomorphic 2-forms on $\tilde{X}$, which is the minimal smooth algebraic compactification of X. At this point we should distinguish between geodesic triangles p_1, p_2, p_3 that lie on a holomorphic curve or on anti-holomorphic curve. The reason for distinguishing is that a holomorphic 2-form restricted to a holomorphic curve vanishes. The way to distinguish the two type of geodesic triangles is the following: Let γ be a linear fractional transform that sends the points p_1, p_2, p_3 to $0, 1, \infty$. If $\det\gamma$ is totally positive or totally negative then the geodesic triangle p_1, p_2, p_3 lies on a holomorphic curve. If $\det\gamma$ is not totally positive nor totally negative then the geodesic triangle p_1, p_2, p_3 lies on an anti-holomorphic curve.

Definition 116 *Let $M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$ be the span of the modular symbols $\{p_1, p_2, p_3\}$ and $\{p_1, p_2; p_3, p_4\}$ as a subspace of the singular chain $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$. We define the following pairing*

$$<,>: M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K)) \times S_{2,2}(\Gamma) \to \mathbb{C},$$

by setting

$$< \{p_1, p_2, p_3\}, f dz_1 \wedge dz_2 >= \int_{\{p_1,p_2,p_3\}} f dz_1 \wedge dz_2$$

for geodesic triangles and

$$< \{p_1, p_2; p_3, p_4\}, f dz_1 \wedge dz_2 >= \int_{\{p_1,p_2;p_3,p_4\}} f dz_1 \wedge dz_2$$

for geodesic diangles.

Theorem 117 *The image of the above pairing is a period over a number field, when the cusp form is represented by a rational differential 2-form on the Hilbert modular surface.*

Proof. The boundary of the geodesic triangles of the diangles are geodesics that lie on holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. Therefore, in the quotient by the congruence group Γ, the geodesic lie in Hirzebruch-Zagier divisor on the Hilbert modular surface. Thus, we integrate a closed rational differential 2-form on the Hilbert modular surface with boundaries on 2 (for a diangle) or 3 (for a triangle) Hirzebruch-Zagier divisors.

Conjecture 118 *Let $f \in S_{k,k}(\Gamma)$ be a normalized cusp Hecke eigenform of weight (k,k). Then the image of $< , f dz_1 \wedge dz_2 >$ evaluated on any geodesic triangle or any geodesic diangle is a period.*

Theorem 117 is a proof of Conjecture 118 for the case of cusp form of weight $(2,2)$.

6.2.4 Generating series and relations

In this Subsection we examine the generating series of iterated integrals over membranes (of type **b**), J, evaluated at geodesic triangles and geodesic diangles. We prove relations among them. Most importantly, the generating series J will be used in Subsection 3.4 for Defining non-commutative Hilbert modular symbols. Moreover, the relations that we prove in this Section, will be interpreted as coccyges or as coboundaries of the the non-commutative Hilbert modular symbols satisfy in Subsection 3.4.

Definition 119 *Let $f_1, \ldots, f_m$ be m cusp forms with respect to a Hilbert modular group Γ. Let $f_1 dz_1 \wedge dz_2, \ldots, f_m dz_1 \wedge dz_2$ be the corresponding differential forms, defining the generating series. Let $J(p_1, p_2, p_3)$ be the generating series J evaluated at the geodesic triangle with vertices p_1, p_2, p_3. Let $J(p_1, p_2; p_3, p_4)$ be the generating series J evaluated at the geodesic diangle $\{p_1, p_2; p_3, p_4\}$.*

Both $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ will be called non-commutative Hilbert modular symbols after the acting of the arithmetic group is included.

Theorem 120 *The generating series $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ satisfy the following relations:*

1. If σ is a permutation of the set $\{1, 2, 3\}$ then

$$J(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = J^{sign(\sigma)}(p_1, p_2, p_3\}).$$

2. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve then

$$\begin{aligned} 1 = & J(p_1, p_2, p_3) J(p_2, p_3, p_4) \\ & J(p_2, p_1, p_4) J(p_1, p_4, p_3) \end{aligned}$$

and

3. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve then

$$1 = J(p_1, p_2; p_3, p_4).$$

4. For every four points p_1, p_2, p_3, p_4, we have the following relation based on the orientation of the domain

$$\begin{aligned} J(p_2, p_1; p_3, p_4) =& J(\{p_1, p_2; p_4, p_3) = \\ =& J^{-1}(p_2, p_1; p_4, p_3) = \\ =& J^{-1}(p_1, p_2; p_3, p_4). \end{aligned}$$

5. For every five points p_1, p_2, p_3, p_4, p_5, we have

$$J(p_1, p_2; p_3, p_4)J(p_1, p_2; p_4, p_5) = J(p_1, p_2; p_3, p_5).$$

We also have relation between the two types of generating series. For every four distinct points $p_1, p_2; p_3, p_4$, we have

6. For every four points p_1, p_2, p_3, p_4, we have the following relation, based on the boundary of a thickened tetrahedron,

$$\begin{aligned} 1 =& J(p_1, p_2, p_3)J(p_2, p_3, p_4) \\ & J(p_2, p_1, p_4)J(p_1, p_4, p_3) \\ & J(p_1, p_2; p_3, p_4)J(\{p_2, p_3; p_1, p_4)J(\{p_3, p_1; p_2, p_4) \\ & J(p_3, p_4; p_1, p_2)J(\{p_1, p_4; p_2, p_3)J(\{p_2, p_4; p_3, p_1). \end{aligned}$$

Proof. for part 1, let σ be an odd permutation. Let U be an union of two triangles along the edges. Let the first triangle be with vertices p_1, p_2, p_3 and the second triangle be with vertices p_3, p_2, p_1 with the opposite orientation. From the product formula Thereom 112 (iii), it follows that $J(U) = J(p_1, p_2, p_3)J(p_3, p_2, p_1)$. From the second homotopy invariance theorem (Theorem 105) it follows that the generating series $J(U)$ depends on U up to homotopy, which keeps the boundary components on a fixed union holomorphic curves. Since U has no boundary, we have that U is contractible (to a point) so that the contracting homotopy "keeps the boundary components" on a fixed union of holomorphic curves. Therefore, $J(U) = J(point) = 1$.

Parts 2, 4 and 5 can be proven similarly.

For part 3. If p_1, p_2, p_3, p_4 belong to the same holomorphic (or anti-holomorphic) curve then the corresponding diangle has no interior, since the two edges will coincide. Recall that the edges of the diangle are defined via unique geodesic triangles lying on a holomorphic (or anti-holomorphic) curve.

The proof of part 6 is essentially the same as the one for part 1; however, we will prove it independently, since it is a key property of the non-commutative Hilbert modular symbol. Consider a thickened tetrahedron with vertices p_1, p_2, p_3, p_4. The faces of the thickened tetrahedron are precisely the ones listed in the product of property 6. The whole product is equal to $J(V)$, where $V =$ *union of all faces of the thickened tetrahedron.* From the second homotopy invariance theorem it follows that the generating series $J(V)$ depends on V up to homotopy, which keeps the boundary components on a fixed union holomorphic curves. Since V bounds a contractible 3-dimensional region (a thickened tetrahedron), from Theorem 105, it follows that $J(V) = J(point) = 1$.

6.2.5 Construction of non-commutative Hilbert modular symbols

In this Subsection we define non-commutative Hilbert modular symbols. They are analogues of Manin's non-commutative Hilbert modular symbol (see [**?**]), applicable to the Hilbert modular group. In stead of iterated path integrals that Manin uses, we use a higher dimensional analogue, which we call iterated integrals over membranes.

The non-commutative Hilbert modular symbols will be defined as certain cocycles in terms of the generating series of iterated integrals over membranes of type **b**, (see Definition 100). The Hilbert modular symbols satisfy non-commutative 1-cocycle and 2-cocycle conditions. Unlike a non-commutative group, where the multiplication is linear, non-commutative 2-cocycles (in terms of J) multiply according to the relative locations of the 2-dimensional regions of integration via shuffle product. The relations among the symbols are based on two properties: composition via shuffle product and the homotopy invariance. Note that from the shuffle product from Theorem 112 (iii), we have that the product uses the location of the regions of integration. In particular, the shuffle product from Theorem 112 (iii) provides a proof of Lemma 91, which generalizes compositions of iterated path integrals over a concatenation of paths, which is a key ingredient in the non-commutative 1-cocycle that Manin defines to be the non-commutative modular symbol.

Definition 121 *We define non-commutative Hilbert modular symbols as*

$$c^1_{p_1,p_2;p_3}(\gamma) = J(p_1, p_2; p_3, \gamma p_3).$$

and

$$c^2_p(\beta, \gamma) = J(p, \beta p, \beta\gamma p),$$

where p, p_1, p_2, p_3 are coup points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ and $\gamma \in \Gamma$ is an element of a congruence subgroup Γ of the Hilbert modular group $SL_2(\mathcal{O}_K)$.

We are going to show that the non-commutative Hilbert modular symbols satisfy cocycle relations. Moreover, when we choose different base points, the corresponding change in the cocycle(s) is a coboundary.

Theorem 122 *The non-commutative Hilbert modular symbol $c^1_{p_1,p_2;p_3}$ is a 1-cycle. Moreover, if we change the point p_3 to q_3, then the cocycle changes by a coboundary.*

Proof. Property 5 can be interpreted as a 1-cocycle relation. We define the following action of $\beta \in \Gamma$ on $c^1_{p_1,p_2;p_3}(\gamma)$ as the action of β on p_3 and on γp_3. That is

$$\beta * c^1_{p_1,p_2;p_3}(\gamma) = \beta * J(p_1, p_2; p_3, \gamma p_3) = J(p_1, p_2; \beta p_3, \beta\gamma p_3).$$

Then

$$dc^1_{p_1,p_2;p_3}(\beta,\gamma) = J(p_1,p_2;p_3,\beta p_3)J^{-1}(p_1,p_2;p_3;\beta\gamma p_3)(\beta\cdot J(p_1,p_2;p_3,\gamma p_3)) \tag{6.19}$$
$$= J(p_1,p_2;p_3,\beta p_3)J^{-1}(p_1,p_2;p_3;\beta\gamma p_3)J(p_1,p_2;\beta p_3,\beta\gamma p_3) \tag{6.20}$$
$$= 1. \tag{6.21}$$

If we change p_3 to q_3 then the cocycle changes by a coboundary. Let $b^0 = J(p_1,p_2;p_3,q_3)$ be a 0-cochain. Then

$$c^1_{p_1,p_2;q_3}(\gamma) = J(p_1,p_2;p_3,\gamma p_3) \tag{6.22}$$
$$= J(p_1,p_2;p_3,q_3)J(p_1,p_2;q_3,\gamma q_3)J(p_1,p_2;\gamma q_3,\gamma p_3) \tag{6.23}$$
$$= J(p_1,p_2;p_3,q_3)J(p_1,p_2;q_3,\gamma q_3)(\gamma * J(p_1,p_2;p_3,q_3))^{-1} \tag{6.24}$$
$$= b^0 c^1_{p_1,p_2;q_3}(\gamma)(\gamma * b^0)^{-1} \tag{6.25}$$

Theorem 123 *The non-commutative Hilbert modular symbol $c^2_p(\beta,\gamma)$ satisfies a 2-cocycle relation. Moreover, if we change the point p to q, then the cocycle changes by a coboundary up to terms involving c^1.*

Proof. Recall
$$c^2_p(\beta,\gamma) = J(p,\beta p,\beta\gamma p).$$
Then c^2_p satisfies a 2-cocycle condition up to a multiple of the 1-cocycle $c^1_{q_1,q_2,q_3}$ for various points q_1,q_2,q_3. For the 2-cocycle relation, we compute $dc^2_p(\beta,\gamma,\delta)$.

$$dc^2_p(\beta,\gamma,\delta) = c^2_p(\beta,\gamma)c^2(\beta,\gamma\delta)(c^2(\beta\gamma,\delta))^{-1}(\beta\cdot c^2(\gamma,\delta))^{-1} = \tag{6.26}$$
$$= J(p,\beta p,\beta\gamma p)J(p,\beta p,\beta\gamma\delta p)\times$$
$$\times J(p,\beta\gamma p,\beta\gamma\delta p)^{-1}J(\beta p,\beta\gamma p,\beta\gamma\delta p)^{-1} \tag{6.27}$$

In order to have $dc^2_p(\beta,\gamma,\delta) = 1$, we must multiply by suitable values of c^1, corresponding to edges of a certain thickened tetrahedron. Then

$$\begin{aligned}
dc^2_p(\beta,\gamma,\delta)\times{}&[c^1_{p,\beta p;\beta\gamma p}((\beta\gamma)\delta(\beta\gamma)^{-1})c^1_{\beta p,\beta\gamma p;p}(\beta\gamma\delta)c^1_{\beta\gamma p,p;\beta p}((\beta)\gamma\delta\beta^{-1})\times\\
&\times c^1_{\beta\gamma p,\beta\gamma\delta p;p}(\beta)c^1_{p,\beta\gamma\delta p;\beta p}(\beta\gamma\beta^{-1})c^1_{\beta p,\beta\gamma\delta p;\beta\gamma p}((\beta\gamma)^{-1})] =\\
={}&[c^2_p(\beta,\gamma)c^2(\beta,\gamma\delta)(c^2(\beta\gamma,\delta))^{-1}(\beta\cdot c^2(\gamma,\delta))^{-1}]\times\\
&\times [c^1_{p,\beta p;\beta\gamma p}((\beta\gamma)\delta(\beta\gamma)^{-1})c^1_{\beta p,\beta\gamma p;p}(\beta\gamma\delta)c^1_{\beta\gamma p,p;\beta p}((\beta)\gamma\delta\beta^{-1})\times\\
&\times c^1_{\beta\gamma p,\beta\gamma\delta p;p}(\beta)c^1_{p,\beta\gamma\delta p;\beta p}(\beta\gamma\beta^{-1})c^1_{\beta p,\beta\gamma\delta p;\beta\gamma p}((\beta\gamma)^{-1})] =\\
={}&[J(p,\beta p,\beta\gamma p)J(p,\beta p,\beta\gamma\delta p)\times\\
&\times J(p,\beta\gamma p,\beta\gamma\delta p)^{-1}J(\beta p,\beta\gamma p,\beta\gamma\delta p)^{-1}]\times\\
&\times [J(p,\beta p;\beta\gamma p,\beta\gamma\delta p)J(\beta p,\beta\gamma p;p,\beta\gamma\delta p)J(\beta\gamma p,p;\beta p,\beta\gamma\delta p)\times\\
&\times J(\beta\gamma p,\beta\gamma\delta p;p,\beta p)J(p,\beta\gamma\delta p;\beta p,\beta\gamma p)J(\beta p,\beta\gamma\delta p;\beta\gamma p,p)] =\\
={}&1.
\end{aligned}$$

The first equality follows from Equation (6.26). The second equality follows from the definition of the symbols. And the last equality follows from Property 6 with $(p_1, p_2, p_3, p_4) = (p, \beta p, \beta\gamma p, \beta, \gamma\delta p)$. Therefore, we obtain that $dc_p^2(\beta, \gamma, \delta)$ is 1 up to values of the 1-cocycle c^1.

Now we are going to show that c_p^2 and c_q^2 are homologous. Before we proceed, we would like to make an analogy between 1-dimensional and 2-dimensional cocycles. For the 1-dimensional cocycle, the property that it is a cocycle has the geometry of a triangle where the faces of the triangle are essentially the 1-cocycle. We want commutativity of the triangular diagram. We think of the commutativity of the diagram as follows: consider the interior of the triangle as a homotopy of paths and we think of the 1-cocycle as a homotopy invariant function. For the 2-cocycle, the 2-cocycle relation is represented by the faces of a tetrahedron. By a 'commutativity' of the diagram, we mean a homotopy invariant 2-cocycle and a homotopy from one of the faces to the union of the other three faces.

The comparison that $c^1_{p_1,p_2;p_3}$ and $c^1_{p_1,p_2;q_3}$ are homologous is given by a square-shaped diagram. The analogy if dimension 2 is that the cocycles c_p^2 and c_q^2 are two faces of an octahedron. The vertices associated to $c_p^2(\beta, \gamma)$ are $(p, \beta p, \beta\gamma p)$ and the vertices associated to c_q^2 are $(q, \beta q, \beta\gamma q)$. The two faces will be opposite to each other on the octahedron Oct so that the three pair of opposite vertices are $(p, \beta\gamma q)$, $(\beta p, q)$ and $(\beta\gamma p, \beta q)$. The remaining 6 faces are combined into two triples. Each of them corresponds to a coboundary of a 1-chain.

Let

$$b^1_{p,q}(\beta) = [J(p, q, \beta p)J(q, \beta q, \beta p)][J(q, \beta p; p; \beta q)].$$

Consider the action of $\gamma \in \Gamma$ on b^1, by action on each point in the argument of J, denoted as before by $\gamma \cdot b^1$. Then

$$db^1_{p,q}(\beta, \gamma) = b^1_{p,q}(\beta)[\beta \cdot b^1_{p,q}(\gamma)][b^1_{p,q}(\beta\gamma)]^{-1},$$

where $\beta \cdot b^1_{p,q}(\gamma) = [J(\beta p, \beta q, \beta\gamma p)J(\beta q, \beta\gamma q, \beta\gamma p)][J(\beta q, \beta\gamma p; p; \beta\gamma q)]$

Lemma 124 *With the above notation,, we have that c_p^2 and c_q^2 are homologous*

$$c_p^2(\beta, \gamma) = c_q^2(\beta, \gamma)[db^1_{pq}(\beta, \gamma)] \prod_i J(D_i),$$

up to a product of $J(D_i)$,where D_i are geodesic diangles.

Proof. Consider the above octahedron Oct. Remove from it the tetrahedron T with vertices $(p, q, \beta\gamma q, \beta\gamma p)$. Then the triangles of the remaining geometric figure are precisely the triangles in the definitions of $c_p^2(\beta, \gamma)$, $c_q^2(\beta, \gamma)$ and $db^1_{p,q}(\beta, \gamma)$. Now, consider thickening of the edges, which are common for two triangles. It can be done in the following way. Instead of any triangle, we can take a geodesic triangle. The two triangles that had a common edge might have only two common vertices. Then the region between the two geodesic, one for each of the geodesic triangles, forms the induced diangle. Take J of the induces diangles from the octahedron Oct and J^{-1} of the induced diangles from the tetrahedron T. Their product gives $\prod_i J(D_i)$. The equality holds because we apply J to the union of the faces of the thickened $Oct - T$, which gives 1.

6.2.6 Non-commutative 2-cocycle and non-commutative 2-coboundary

6.2.7 NC Hilbert Modular Symbol and Multiple Dedekind Zeta Values

In this Subsection, we make explicit commutations of some ingredients in the non-commutative Hilbert modular symbol. In [**?**], Manin compares explicit formulas of integrals in the non-commutative modular symbol to multiple zeta values. The similarities are both in terms of infinite series formulas and in terms of formulas via iterated path integrals. Here we compare certain integrals in the non-commutative Hilbert modular symbol to multiple Dedkeind zeta values. Again the similarities are both in terms of infinite series formulas and in terms of formulas via iterated integrals over membranes.

We are going to consider the Fourier expansion of two Hilbert cusp forms f and g. Let $\omega_f = f dz_1 \wedge dz_1$, $\omega_g = g dz_1 \wedge dz_1$ and $\omega_0 = dz_1 \wedge dz_1$. We are going to associate L-values to iterated integrals of the forms ω_f and ω_g. The L-values would be iterated integrals over an union of diangles. One can think of a diangle connecting 0 and ∞ as a segment or a real cone. The union will be a disjoint union of all such real cones connecting 0 and ∞ or simply $\mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$. We also recall the definition of a multiple Dedekind zeta values via (discrete) cone. Finally, we show analogous formulas for an iterated L-values associated to Hilbert cusp forms and multiple Dedekind zeta values, where we consider an union of discrete cones.

We will be mostly interested in the modular symbol associated to a diangle. Let us recall what we mean by a diangle.

Let p_1, p_2, p_3, p_4 be four cusp points. Let $\gamma_1 \in GL_2(K)$ be a linear fractional transform that sends $\gamma_1(p_1) = 0$, $\gamma_1(p_2) = \infty$, $\gamma_1(p_3) = 1$. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0, \infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_1(p_1, p_2) = \gamma_1^{-1} \lambda(0, \infty)$$

be the pull-back of the geodesic λ to a geodesic connecting p_1 and p_2.

Now consider the triple p_1, p_2 and p_4. Let $\gamma_2 \in GL_2(K)$ be a linear fractional transform that sends $\gamma_2(p_1) = 0$, $\gamma_2(p_2) = \infty$ and $\gamma_2(p_4) = 1$. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0, \infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_2(p_1, p_2) = \gamma_2^{-1} \lambda(0, \infty)$$

be the pull-back of the geodesic λ to a geodesic connecting p_1 and p_2.

By a diangle, we mean a region in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ of homotopy type of a disc, bounded by the geodesics $\lambda_1(0,\infty)$ and $\lambda_2(0,\infty)$.

We are going to present a computation for the diangle D_u defined by the points $(0,\infty,u^1,u^{-1})$, where u is a generator for the group of units modulo ± 1 in K. Let (1) be the trivial permutation.

Lemma 125 *Let u be a totally positive unit. Then*

$$\int\int_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1+\alpha_2 z_2)} dz_1 \wedge dz_2 = \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)}$$

Proof. Let u_1 and u_2 be the two embeddings of u into $\mathbb{R}$. Then $(0,\infty,u)$ can be mapped to $(0,\infty,1)$ by $\gamma_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. The geodesic $\lambda(0,\infty)$ can be parametrized by (it,it) for $t \in \mathbb{R}$. Then the geodesic $\lambda_1(0,\infty)$ on the geodesic triangle $(0,\infty,u)$ can be parametrized by $\{(iu_1 t, iu_2 t) \mid t > 0\}$. Similarly, the geodesic $\lambda_2(0,\infty)$ on the geodesic triangle $(0,\infty,u^{-1})$ can be parametrized by $\{(iu_2 t, iu_1 t) \mid t > 0\}$. Then the diangle D_u can be parametrized by

$$D_u = \{(z_1,z_2) \in \mathbb{H}^2 \,|\, Re(z_1) = Re(z_2) = 0, Im(z_1) \in \left(\frac{u_1}{u_2}t, \frac{u_2}{u_1}t\right), Im(z_2) = t \in (0,\infty)\}.$$

Then we have

$$\begin{aligned}
\int\int_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1+\alpha_2 z_2)} dz_1 \wedge dz_2 &= \int_\infty^0 \left(\int_{\frac{u_2}{u_1}t}^{\frac{u_1}{u_2}t} e^{2\pi i(\alpha_1 z_1 + \alpha_2 t)} dz_1\right) dt = \\
&= \frac{1}{2\pi i \alpha_1} \int_\infty^0 \left(e^{\alpha_1 \frac{u_1}{u_2} t + \alpha_2 t} - e^{\alpha_1 \frac{u_2}{u_1} t + \alpha_2 t}\right) dt = \\
&= \frac{1}{(2\pi i)^2} \frac{1}{\alpha 1} \left(\frac{1}{\alpha_1 \frac{u_1}{u_2} + \alpha_2} - \frac{1}{\alpha_1 \frac{u_2}{u_1} + \alpha_2}\right) = \\
&= \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)}
\end{aligned}$$

Therefore, one term of the Fourier expansion of a Hilbert cusp form paired with a symbol given by one diangle does not resemble a norm of an algebraic integer. However, if we integrate over an infinite union of diangles then a similarity with Dedekind zeta and with multiple Dedekind zeta occurs.

Consider the limit when $n \to \infty$ of D_{u^n}. It is the product of the two imaginary axes of the two upper half planes. Denote by

$$\mathrm{Im}(\mathbb{H}^2) = \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H}).$$

One can think of this region as an infinite union of diangles.

Denote by αz the sum of products $\alpha_1 z_1 + \alpha_2 z_2$. Using the methods of [**?**], Section 1, we obtain

$$\frac{(2\pi i)^{-2}}{N(\alpha)N(\alpha+\beta)} = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i \alpha z} dz_1 \wedge dz_2 \cdot e^{2\pi i \beta z} dz_1 \wedge dz_2$$

and

$$\frac{1}{(2\pi i)^2}\frac{1}{N(\alpha)^3 N(\alpha+\beta)^2} = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i\alpha z}dz_1\wedge dz_2\cdot(dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)\cdot e^{2\pi i\beta z}dz_1\wedge dz_2\cdot(dz_1\wedge dz_2)$$

Let f and g be two cusp form of wights $(2k,2k)$ and $(2l,2l)$, respectively. Consider the Fourier expansion of both of the cusp forms. Let

$$f = \sum_{\alpha>>0} a_\alpha e^{2\pi i\alpha z}$$

and let

$$g = \sum_{\beta>>0} b_\beta e^{2\pi i\beta z}.$$

Since f is of weight $(2k,2k)$, we have that $a_{u\alpha} = a_\alpha$, where u is a unit. For such a modular form the modular factor with respect to the transformation $z \to uz$ is 1. The L-values of f is

$$L_f(n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{\alpha\in\mathcal{O}_K^+/U^+} a_\alpha e^{2\pi i\alpha z}dz_1\wedge dz_2\cdot(dz_1\wedge dz_2)^{\cdot(n-1)} = \frac{1}{(2\pi i)^{2n}}\sum_{\alpha\in\mathcal{O}_K^+/U^+}\frac{a_\alpha}{N(\alpha)^n}.$$

Here $\mathcal{O}_K^+$ denotes the totally positive algebraic integers in K and U^+ denotes the totally positive units.

We recall some of the definitions from [**?**]. We fix a positive cone C in $\mathcal{O}_K$, by which we mean

$$C = \mathbb{N}\cup\{\alpha\in\mathcal{O}_K \mid a+b\epsilon,\, a,b\in\mathbb{N}\},$$

where ϵ is a generator of the group of totally positive units. By $\epsilon^k C$, we mean the collection of products $\epsilon^k\alpha$, where α varies in the cone C.

The following infinite sum is an example of a multiple Dedekind zeta value

$$\zeta_{K;C,\epsilon^k C}(m,n) = \sum_{\alpha\in C}\sum_{\beta\in\epsilon^k C}\frac{1}{N(\alpha)^m N(\alpha+\beta)^n}.$$

Let $Z(m,n) = \sum_{k\in\mathbb{Z}}\zeta_{K;C,\epsilon^k C}(m,n)$, where C is any set representing the totally positive algebraic integers $\mathcal{O}_K^+$ modulo totally positive units U^+.

Lemma 126 *The values $Z(m,n)$ are finite for $m>n>1$.*

Proof. Let ϵ be a generators of the group of totally positive units U^+ in K. For the two real embeddings ϵ_1 and ϵ_2 of ϵ, we can assume that $\epsilon_1 > 1 > \epsilon_2$. Otherwise we can take its reciprocal.

$$Z(m,n) = \sum_{k\in Z}\sum_{\alpha,\beta\in C} \frac{1}{N(\alpha)^m N(\alpha+\epsilon^k\beta)^n} < \tag{6.28}$$

$$< \sum_{\alpha,\beta\in C} \frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n} + \right. \tag{6.29}$$

$$\left. + \sum_{k=1}^{\infty} \frac{2^n}{\epsilon_1^k}\left(\frac{1}{\alpha_1^n\beta_2^n} + \frac{1}{\alpha_2^n\beta_1^n}\right)\right) <$$

$$< \sum_{\alpha,\beta\in C} \frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n} + \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{2}{\epsilon_1^k}\left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n}\right)\right) = \tag{6.30}$$

$$= \sum_{\alpha,\beta\in C} \frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n} + \right. \tag{6.31}$$

$$\left. + \frac{2}{\epsilon_1 - 1}\left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n}\right)\right) =$$

$$= \sum_{\alpha,\beta\in C} \frac{1}{N(\alpha)^m N(\alpha+\beta)^n} + \tag{6.32}$$

$$+ \frac{2}{\epsilon_1 - 1}\frac{1}{N(\alpha)^n} - \frac{2}{N(\alpha)^{m-n}N(\alpha+\beta)^n} =$$

$$= \zeta_K(C;m,n) - \frac{2}{\epsilon_1 - 1}\left(\zeta_K(C;n) + \zeta_K(C;m-n,n)\right). \tag{6.33}$$

Equation (6.28) is the definition. Inequality (6.29) is based on the following: $\epsilon_2 < 1$ is replaced with 1 then $k > 0$. For $k < 0$ we use $\epsilon_2^k = \epsilon_1^{-k}$. We put 1 for ϵ_1^k for $k < 0$. The case $k = 0$ is treated separately. Finally we group the terms with equal powers of ϵ_1. In the Inequality (6.30) we estimate the mixed terms in the brackets. In Equation (6.31) we take the sum of the geometric series in ϵ_1^{-1}. Then in Equation (6.32) we open the brackets. And finally, in Equation (6.33), we express the sums as a finite linear combinations of a Dedekind zeta value and multiple Dedekind zeta values.

Definition 127 *For a pair of Hilbert cusp forms f and g with Fourier expansion*

$$f = \sum_{\alpha>>0} a_\alpha e^{2\pi i\alpha z} \text{ and } g = \sum_{\beta>>0} b_\beta e^{2\pi i\beta z},$$

we define the following iterated L-values

$$L_{f,g}(m,n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} (a_\alpha e^{2\pi i\alpha z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(m-1)}\cdot$$

$$\cdot (b_\beta e^{2\pi i\beta z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(n-1)}.$$

Theorem 128 *Using the above definition we have*

$$L_{f,g}(m,n) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in C, \beta\in\epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}$$

Proof.

$$\begin{aligned} L_{f,g}(m,n) = &\int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} (a_\alpha e^{2\pi i\alpha z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(m-1)}\cdot \\ &\cdot (b_\beta e^{2\pi i\beta z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(n-1)} = \\ = &\sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n} = \\ = &\sum_{k\in\mathbb{Z};\alpha,\beta\in C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\epsilon^k\beta)^n} = \\ = &\sum_{k\in\mathbb{Z},} \sum_{\alpha\in C} \sum_{\beta\in\epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}. \end{aligned}$$

We would like to bring to the attention of the reader the Definition 127 of the multiple L- values. More specifically, we would like to point out that the region of integration is an infinite union of diangles, (or equivalently an infinite union of real cones; see the beginning of this Section.) Note also that in Theorem 128 the values of the multiple L-functoins are expressed as an infinite sums over different discrete cones, namely, over $\epsilon^k C$, for $k\in\mathbb{Z}$. However, a single real cone D_u as in Lemma 126, does not correspond to a single discrete cone. Only a good union of real cones $\mathrm{Im}(\mathbb{H})\times\mathrm{Im}(\mathbb{H})$ corresponds to a good union of discrete cones $\bigcup_{k\in\mathbb{Z}}(C,\epsilon^k C)$ as a fundamental domain of $(\mathcal{O}_K^+,\mathcal{O}_K^+)/U^+$.

Bibliography

[1] Rektorys, K., *Variational methods in Mathematics, Science and Engineering*, D. Reidel Publishing Company, Dordrecht-Hollanf/Boston-U.S.A., 2th edition, 1975

[2] Bertóti, E.: *On mixed variational formulation of linear elasticity using nonsymmetric stresses and displacements*, International Journal for Numerical Methods in Engineering., **42**, (1997), 561-578.

[3] Szeidl, G.: *Boundary integral equations for plane problems in terms of stress functions of order one*, Journal of Computational and Applied Mechanics, **2**(2), (2001), 237-261.

[4] Carlson D. E.: *On Günther's stress functions for couple stresses*, Quart. Appl. Math., **25**, (1967), 139-146.

Afterword