Tesselations of the four-dimensional space by regular polytopes

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In this paper we give a geometric proof of the fact that three out of the six regular four-dimensional polytopes tessellate the space. Also we prove that the rest three cannot tessellate the latter. By definition a regular four-dimensional polytope is made out of one and the same regular three-dimensional polytopes subject to the condition that from each edge there come out one and the same number of polytopes. Later we shall use symbols of the type $\langle m, n\rangle$ or $\langle l, m, n\rangle$, which we explain next [1].

The symbol $\langle m, n\rangle$ denotes a three-dimensional polytope made out of regular n-gonals so that m n-gonals come out of each vertex. The symbol $\langle l, m, n\rangle$ denotes a four-dimensional regular polytope made out of $\langle m, n\rangle$ polytopes so that l of the $\langle m, n\rangle$-polytopes come out of each edge. Using this notation, we can formulate the main result as

Theorem 1 *The polytopes $\langle 3,3,4\rangle$, $\langle 3,4,3\rangle$, $\langle 4,3,3\rangle$ tesselate the four-dimensional space. The regular polytopes — $\langle 3,3,3\rangle$, $\langle 3,3,5\rangle$, $\langle 5,3,3\rangle$ cannot tesselate the space.*

In order to prove the theorem, we need the following

Lemma 1 *Let $ABCO$ be a tetrahedron $|OA| = |OB| = |OC| = 1$, $\alpha = \angle BOC$, $\beta = \angle AOC$, $\gamma = \angle AOB$, and $A = \angle((AOB),(AOC))$ (see fig. 1). Then*

$$\cos A = \frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

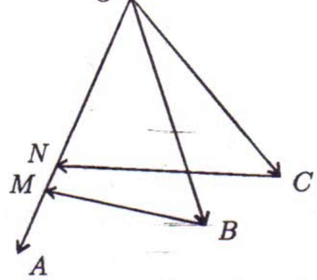

Fig. 1

Proof. Let M and N be such points on OA that BM and CN are both perpendicular to OA. Then $BM = \sin\gamma$ and $CN = \sin\beta$. The scalar product of $\overrightarrow{BM}$ and $\overrightarrow{CN}$ yields

$$\overrightarrow{BM}.\overrightarrow{CN} = BM.CN.\cos A = \cos A \sin\beta\sin\gamma$$

and

$$\begin{aligned}\overrightarrow{BM}.\overrightarrow{CN} &= (\overrightarrow{OM} - \overrightarrow{OB})(\overrightarrow{ON} - \overrightarrow{OC})\\ &= (\overrightarrow{OA}\cos\gamma - \overrightarrow{OB})(\overrightarrow{OA}\cos\gamma - \overrightarrow{OC}) = \cos\alpha - \cos\beta\cos\gamma.\end{aligned}$$

From the last two equations the lemma follows.
We shall use two simple particular cases of the lemma

- if $\beta = \gamma$, then
$$\cos A = \frac{\cos\alpha - \cos^2\beta}{\sin^2\beta}; \tag{1}$$
- if $\alpha = \beta = \gamma$, then
$$\cos A = \frac{\cos\alpha}{1+\cos\alpha}. \tag{2}$$

We shall calculate the angles between two neighbouring sides of the six regular four-dimensional polytopes. In order to do that we have to calculate the dihedral angles of each of the five Platonian polytopes. We denote by $A\langle m,n\rangle$ the angle between two neighbouring n-gonals of the $\langle m,n\rangle$ polytope. A similar meaning has the notation $A\langle l,m,n\rangle$. For $A\langle 3,3\rangle$, $A\langle 3,4\rangle$ and $A\langle 3,5\rangle$ we can use formula (2). One can consider the angles between two neighbouring sides of $\langle 4,3\rangle$ and $\langle 5,3\rangle$ as angles between the latteral sides of regular tetragonal and pentagonal pyramids respectively. In this case we use formulas (1). Thus we obtain the following table

tetrahedron	$\cos A\langle 3,3\rangle = \frac{1}{3}$	$\frac{\pi}{3} < A\langle 3,3\rangle < \frac{2\pi}{5}$
cube	$\cos A\langle 3,4\rangle = 0$	$A\langle 3,4\rangle = \frac{\pi}{2}$
octahedron	$\cos A\langle 4,3\rangle = -\frac{1}{3}$	$\frac{\pi}{2} < A\langle 4,3\rangle < \frac{2\pi}{3}$
dodecahedron	$\cos A\langle 3,5\rangle = -\frac{1}{\sqrt{5}}$	$\frac{\pi}{2} < A\langle 3,5\rangle < \frac{2\pi}{3}$
icosahedron	$\cos A\langle 5,3\rangle = -\frac{\sqrt{5}}{3}$	$A\langle 5,3\rangle > \frac{2\pi}{3}$

There are three polytopes made out of tetrahedrons — $\langle 3,3,3\rangle$, $\langle 4,3,3\rangle$, and $\langle 5,3,3\rangle$; one polytope — $\langle 3,3,4\rangle$ made out of cubes; one made out of octahedrons — $\langle 3,4,3\rangle$ and one made out of dodecahedrons — $\langle 3,5,3\rangle$. We can calculate the dihedral angles of $\langle 3,3,3\rangle$, $\langle 3,3,4\rangle$, $\langle 3,3,5\rangle$ and $\langle 3,4,3\rangle$ in the following way: Let E be one of the above polytopes. Let l be one of its line segments and α, β and γ be the three n-gonals that come out of l. Let λ be a hyperplane perpendicular to l. Then λ intersects α, β and γ in a, b, and c respectively. The angles $\angle(a,b) = \angle(\alpha,\beta)$, $\angle(b,c) = \angle(\beta,\gamma)$ and $\angle(c,a) = \angle(\gamma,\alpha)$ are known from the above table. Denote by $P_{\alpha\beta}$ the polytope determined by α and β and by $P_{\alpha\gamma}$ — the polytope determined by α and γ. Then the angle between $P_{\alpha\beta}$ and $P_{\alpha\gamma}$ is equal to the dihedral angle at the line segment a (see fig. 2). Thus we can use formula (1) from the lemma 1.

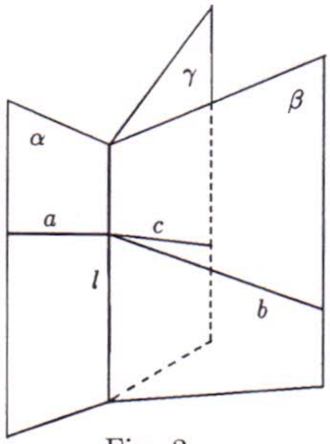

Fig. 2

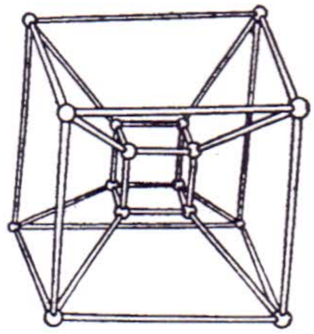

Fig. 3. $\langle 3,3,4\rangle$

The other two polytopes $\langle 4,3,3\rangle$ and $\langle 5,3,3\rangle$ are made out of tetrahedrons. Let A be a vertex of $\langle 4,3,3\rangle$. Take all the tetrahedrons that come out of A. Out of each edge there come out four tetrahedrons. So one can consider these tetrahedros as latteral sides of a regular pyramid with an octahedron as a base. Knowing $A\langle 3,3\rangle$ and $A\langle 4,3\rangle$ we can calculate $A\langle 4,3,3\rangle$. In a similar way we can calculate $A\langle 5,3,3\rangle$. Thus we have a table for the angles between two neighbouring sides of all the four-dimensional regular polytopes, showing that $\langle 3,3,3\rangle$, $\langle 3,3,5\rangle$ and $\langle 5,3,3\rangle$ cannot tesselate the space:

$$\cos A\langle 3,3,3\rangle = \frac{\cos A\langle 3,3\rangle}{1+\cos A\langle 3,3\rangle} = \frac{1}{4} \qquad \frac{2\pi}{5} < A\langle 3,3,3\rangle < \frac{\pi}{2}$$

$$\cos A\langle 3,3,4\rangle = \frac{\cos A\langle 3,4\rangle}{1+\cos A\langle 3,4\rangle} = 0 \qquad A\langle 3,3,4\rangle = \frac{\pi}{2}$$

$$\cos A\langle 3,4,3\rangle = \frac{\cos A\langle 4,3\rangle}{1+\cos A\langle 4,3\rangle} = -\frac{1}{2} \qquad A\langle 3,4,3\rangle = \frac{2\pi}{3}$$

$$\cos A\langle 3,3,5\rangle = \frac{\cos A\langle 3,5\rangle}{1+\cos A\langle 3,5\rangle} = \frac{1-\sqrt{5}}{4} \qquad A\langle 3,3,5\rangle = \frac{4\pi}{5}$$

$$\cos A\langle 4,3,3\rangle = \frac{\cos A\langle 4,3\rangle - \cos^2 A\langle 3,3\rangle}{1-\cos^2 A\langle 3,3\rangle} = -\frac{1}{2} \qquad A\langle 4,3,3\rangle = \frac{2\pi}{3}$$

$$\cos A\langle 5,3,3\rangle = \frac{\cos A\langle 5,3\rangle - \cos^2 A\langle 3,3\rangle}{1-\cos^2 A\langle 3,3\rangle} = -\frac{-1-3\sqrt{5}}{8} \qquad A\langle 5,3,3\rangle > \frac{2\pi}{3}$$

Now we are going to prove that $\langle 3,4,3\rangle$ can tessellate the four-dimensional space. By cutting and gluing two polytopes $\langle 3,3,4\rangle$ with edge 1 we can obtain a $\langle 3,4,3\rangle$ polytope with length of the edge 1. Let K_1 and K_2 be the two $\langle 3,3,4\rangle$ polytopes. Let $ABCDA_1B_1C_1D_1$ be a cube of K_1 and let O be the center of K_1 (see fig. 3). The diagonal of K_1 is $\sqrt{1^2+1^2+1^2+1^2} = 2$ long. So $OA = OB = \cdots = OD_1 = 1$. We shall call $OABCDA_1B_1C_1D_1$ a hyperpyramid with base $ABCDA_1B_1C_1D_1$. Cut K_1 into eight hyperpyramids with bases — the cubes of K_1 and common vertex at O. Glue the bases of the eight hyperpyramids on the sides of K_2 (fig. 4).

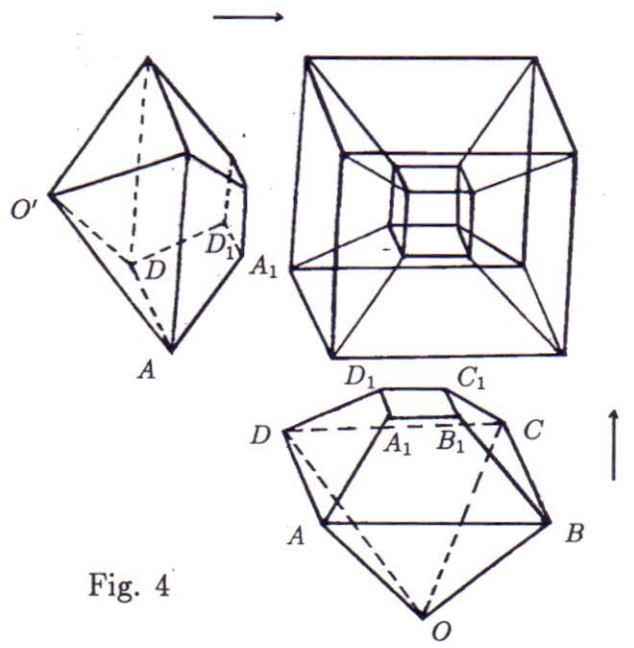

Fig. 4

We will prove that the new polytope S is $\langle 3,4,3\rangle$.

S is made only of regular quadrilateral pyramids with lateral sides equilateral triangles. Consider the hyperpyramid $ABCDA_1B_1C_1D_1O$. The angle between the base $ABCDA_1B_1C_1D_1$ and the lateral sides is $\frac{\pi}{4}$. Let O and O' be the vertices of two hyperpyramids glued to neighbouring cubes of K_2. Then obviously $OADD_1A_1O'$ is lying on one hyperplane. Let us recall that gluing the base of two regular square pyramids one obtains an octahedron. So $OADD_1A_1O'$ is an octahedron. And for each quadrilateral pyramid of S there exists another one so that both form one octahedron. Hence S is made only of octahedrons.

In the above construction there are "two kinds of edges": edges that belong to K_2 and edges that are lateral edges of the glued hyperpyramids. (OA for example.)

From an edge of K_2 there come out three squares. Considering S, a square of K_2 is a part of an octahedron and the three squares coming out of an edge of K_2 are part of three different octahedrons. So there come three octahedrons out of an edge of K_2.

From a lateral edge of a hyperpyramid glued to K_2 there come out just three pyramids (see fig. 3). For example: From OA there come out the pyramids $ABCDO$, ABB_1A_1O, AA_1D_1DO. But S is made only of octahedrons, so three pyramids are half part of three different octahedrons.

Thus, there come out just three octahedrons out of each edge. By definition it follows that S is $\langle 3,4,3\rangle$ (see fig. 5 for its projection). As a by-product we obtain that if the edge of S is 1 then the distance between two opposite vertices is 2. We will need this result later.

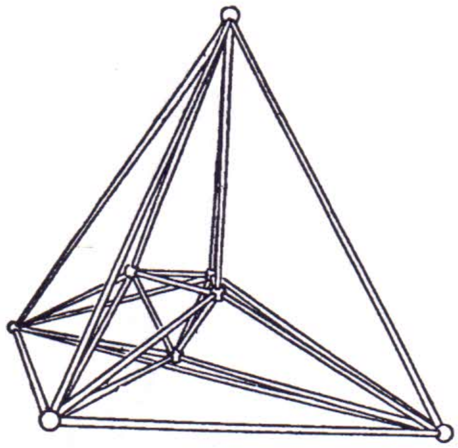

Fig. 5. $\langle 4,3,3 \rangle$

Obviously, the four-dimensional space could be tessellated by $\langle 3,3,4 \rangle$ polytopes. Take a lattice determined by $\langle 3,3,4 \rangle$. We can divide the $\langle 3,3,4 \rangle$ polytopes into two types. Choose on arbitrary $\langle 3,3,4 \rangle$ polytope to be of type I. Then each $\langle 3,3,4 \rangle$ polytope that has a common cube with a type I polytope is of type II and each polytope that has a common cube with a type II polytope is of type I. Let O_1, O_2, ... be the centers of the type II $\langle 3,3,4 \rangle$ polytopes. Cut each type II $\langle 3,3,4 \rangle$ into eight hyperpyramids with bases the cubes of the $\langle 3,3,4 \rangle$ and vertices at the center of the $\langle 3,3,4 \rangle$. And there is no hyperpyramid which has a common cube with more than one $\langle 3,3,4 \rangle$ polytope. If we consider a type I $\langle 3,3,4 \rangle$ and the eight pyramids glued to its surface as one polytope S_i, then all the S_1, S_2 etc. tessellate the space. We have already proved that S_i (i is fixed) is a $\langle 3,4,3 \rangle$ polytope. So the four-dimensional space can be tessellated by $\langle 3,4,3 \rangle$ polytopes.

In a similar way we shall prove that $\langle 4,3,3 \rangle$ polytopes (see fig. 6) can tessellate the four-dimensional space. First we shall prove that a $\langle 4,3,3 \rangle$ polytope can be made from two hyperpyramids whose bases are octahedrons and which lateral sides are tetrahedrons. Let both of the bases be $ABCA_1B_1C_1$ and the vertices be O' and O'' as it is shown on fig. 7. There come out exactly four tetrahedrons out of an lateral edge of one of the two hyperpyramids. For example: From OA' there come out the tetrahedrons $ABCO'$, ABC_1O', AB_1C_1O'. From an edge of the base of any of the pyramids, again four tetrahedrons come out. Hence the constructed polytope is $\langle 4,3,3 \rangle$.

Let us go back to the $\langle 3,4,3 \rangle$ polytope (S). We have proved that if the length of an

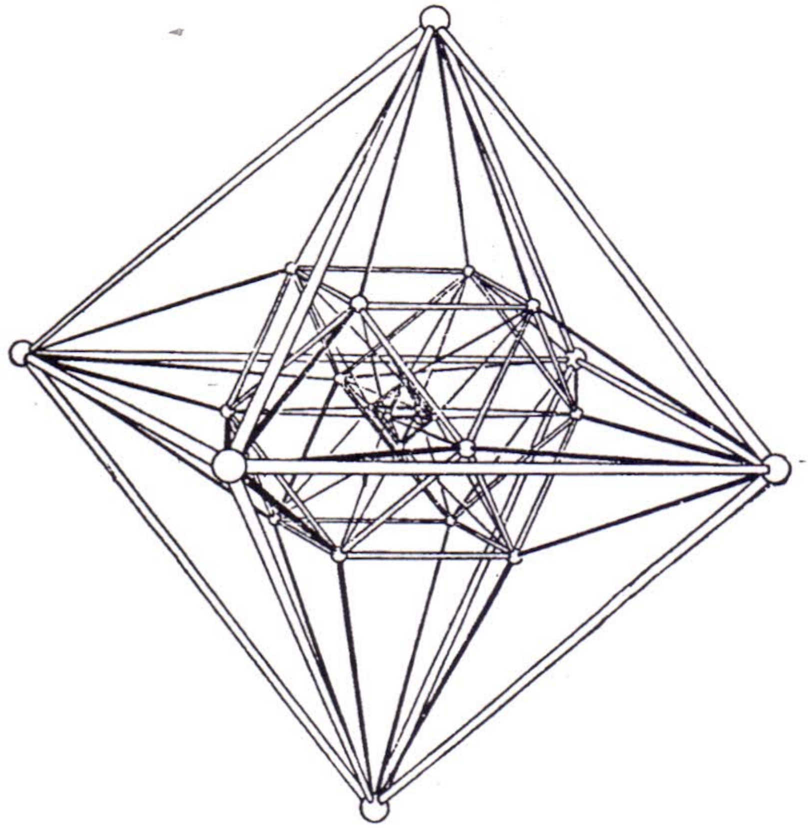

Fig. 6. ⟨3, 4, 3⟩

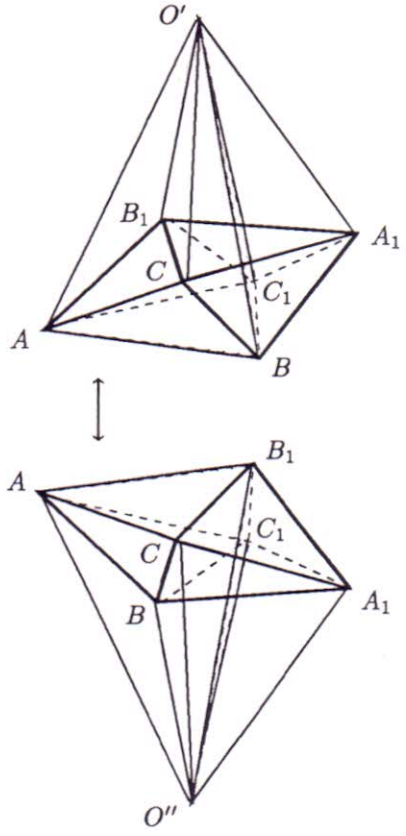

Fig. 7

edge of ⟨3, 4, 3⟩ is 1 then the distance between the opposite vertices is 2. Let O be the center of S. Then the distance between O and an arbitrary vertex of S is 1. We cut S into 24 hyperpyramids so that the bases are octahedrons and a commom vertex at O. All the edges of one of these hyperpyramids have the same length. Hence the lateral sides are regular tetrahedrons. Take a lattice made by ⟨3, 4, 3⟩ polytopes. Divide each of the polytopes into 24 hyperpyramids in such a way that the vertices of the 24 hyperpyramids are in the center of the ⟨3, 4, 3⟩ polytope. For each hyperpyramid there is exactly one hyperpyramid such that both have a common base. Obviously, both hyperpyramids form an ⟨4, 3, 3⟩ polytope. This shows that the new lattice could be made by ⟨4, 3, 3⟩ polytopes.

Thus we obtained that ⟨4, 3, 3⟩ polytopes tessellate the four-dimensional space.

Fig. 3, 5, 6 are taken from [2].

References

1. H. S. M. Coxeter. Introduction to Geometry. John Wiley.
2. D. Hilbert, S. Cohn-Vossen. Anschauliche Geometrie.