DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE

MTH 31 Review Sheet


PART I:

1. Explain the concepts of function, domain and range.

2. State the definition of the derivative of a function at a point.

3. Give examples to illustrate that not all continuous functions are differentiable.

4. State the chain rule.

5. State the Intermediate Value Theorem and mention some of its main applications.

6. State the Mean Value Theorem and some of its main applications.

7. Explain how to find local and absolute maxima and minima of a function.

8. State the Fundamental Theorem of Calculus and explain why it is fundamental.

9. Name two mathematicians credited with the invention of Calculus.

PART II:

1. Find the following limits:

(a) $\displaystyle\lim_{x \to 4} \frac{x^2 + 9}{x^2 - 1}$
(b) $\displaystyle\lim_{x \to 4^-} \frac{x - 4}{|x - 4|}$
(c) $\displaystyle\lim_{\theta \to 0} \frac{\sin(2\theta)}{\tan(\theta)}$
(d) $\displaystyle\lim_{\alpha \to 0} \frac{\alpha}{\cos(\alpha)}$

2. Show that the equation $x^3 + 3x + 1 = 0$ has at least one root in the interval $[-2, 2]$.

3. Use the definition of the derivative to find $f'(3)$ if $f(x) = x^2 - 2x$.

4. Let $f(x) = ax^2 + bx + c$ for constants a, b and c.

 (a) Show that the slope of the tangent to f at $x = x_0$ is $2ax_0 + b$

 (b) Find the equation of the tangent line to the graph of f at $x = x_0$

 (c) Find the equation of the tangent line to the graph of $f(x) = x^2 - x + 16$ at $x = 4$

5. The figure shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.
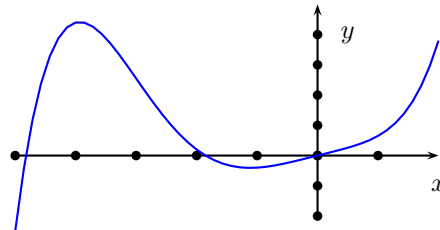
6. Find the equation of the tangent line and normal line to the graph of $y = \dfrac{x+1}{x-1}$ at $x = 2$.

7. Find $f'(x)$ if

(a) $f(x) = x\tan(x)$ (b) $f(x) = \sin(\tan(2x))$ (c) $f(x) = \left(\dfrac{x+1}{x-1}\right)^2$

8. Find $\dfrac{dy}{dx}$ if $\sin(x+y) = \tan(xy)$.

9. Use a differential to approximate $29^{1/5}$.

10. A cylindrical cup is 3 inches in diameter. If you're drinking soda from the cup through a straw at a rate of 3 cubic inches per second, how fast is the level of the soda dropping?

11. Are the following true or false?

(a) If $f''(x) \geqslant 0$ on (a,b) then $f'(x)$ is increasing on (a,b).

(b) If $f''(x) \geqslant 0$ on (a,b) then $f(x)$ is increasing on (a,b).

(c) If $f(x_0) = 0$ then x_0 is a point of inflection.

(d) If x_0 is a point of inflection then $f''(x_0) = 0$.

(e) If $f'(x)$ is decreasing on (a,b) then $f(x)$ is concave down on (a,b).

12. The derivative of the function $f(x)$ is $f'(x) = 2(x-1)^2(2x+1)$. Find all critical points of $f(x)$ and determine whether a relative maximum, relative minimum, or neither occurs at each critical point.

13. Sketch the graph of $f(x) = 2 - \dfrac{3}{x} - \dfrac{3}{x^2}$. Plot any stationary points and any points of inflection. Show any vertical and horizontal asymptotes.

14. Find the dimensions of the rectangles of maximum area which may be embedded in a right triangle with sides of length 12, 16 and 20 feet as shown in the figure.
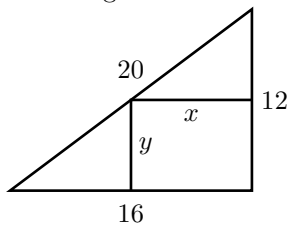
15. Evaluate the following:

(a) $\displaystyle\int \left(x^2 + 8x + \dfrac{3}{x^2}\right) dx$ (b) $\displaystyle\int \dfrac{x^2 - 4}{x^{2/3}} dx$ (c) $\displaystyle\int x\sqrt{x-5}\, dx$ (d) $\displaystyle\int x^3 \sin(x^4 + 2)\, dx$

16. Estimate the area under the curve $y = 1/x$ between $x = 1$ and $x = 2$ by partitioning this interval into 4 equal parts and obtaining the sum of inscribed rectangles.

17. Sketch the graph of $f(x) = |x-1|$, give a geometric interpretation for $\int_{-1}^{2} |x-1|\, dx$ and evaluate this definite integral.

18. Evaluate the following:

(a) $\displaystyle\int_{-1}^{2} \left(x^3 - 2 + \dfrac{1}{x^2}\right) dx$ (b) $\displaystyle\int_{\pi}^{2\pi} \sin\theta\, d\theta$ (c) $\displaystyle\int_{1}^{y} \dfrac{dx}{\sqrt{x}}$

19. The position function of a particle is given by $s = t^3 - 5t^2 + 3t$ for $t \geq 0$. Describe the motion of the particle and make a sketch.

20. Use Newton's Method to find the largest real root of $x^3 - 3x + 1 = 0$ to ten decimal places.

21. Use 3 iterations of Newton's Method to approximate a zero of $f(x) = -x^3 + x + 1$. Set $x_1 = 1.0000$ as the initial guess and round to 4 places after each iteration.

22. Use $a(t) = -32$ feet per second squared as the acceleration due to gravity. A ball is thrown vertically upward from the ground with an initial velocity of 96 feet per second. How high will the ball go?

Answers to the exercises in PART II

1. (a) 5/3, (b) -1, (c) 2, (d) 0

2. Set $f(x) = x^3 + 3x + 1$. Then $f(-1) = -3$ and $f(1) = 5$ have opposite signs and, since $f(x)$ is continuous on $[-2, 2]$, the given equation must have a root between -1 and 1 by the Intermediate Value Theorem.

3. $f'(3) = 4$

4.
(a) Slope $= \lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h} = \lim\limits_{h \to 0} \dfrac{2ax_0 h + bh + ah^2}{h} = 2ax_0 + b.$

(b) $y = (2ax_0 + b)x + (c - ax_0^2)$ (c) $y = 7x$

5.
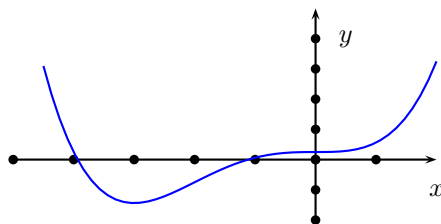
6. The equation of the tangent line is $y = -2x + 7$. The equation of the normal line is $y = x/2 + 2$.

7.
$$\text{(a)} \ \ \frac{4(x+1)}{(1-x)^3} \qquad\qquad \text{(b)} \ \ x\sec^2(x) + \tan(x) \qquad\qquad \text{(c)} \ \ 2\sec^2(2x)\cos(\tan(2x))$$

8.
$$\frac{dy}{dx} = \frac{y\sec^2(xy) - \cos(x+y)}{\cos(x+y) - x\sec^2(xy)}$$

9. Set $f(x) = x^{1/5}$, $x_0 = 32$ and $\Delta x = -3$. Then
$$f(29) = f(32) + -3 \cdot f'(32) = 32^{1/5} + -3 \cdot \frac{1}{5(32)^{4/5}} = \frac{157}{80}.$$

10. The level of the soda is dropping at the rate of 0.42 inches per second.

11. (a) True, (b) False, (c) False, (d) True

12. Solving $f'(x) = 0$ gives the critical points at $x = -1/2$, 1. Then $f'(x)$ is negative on the interval $(-\infty, -1/2)$, positive on $(-1/2, 1)$ and positive on $(1, \infty)$. There is a relative minimum of $-27/16$ at $x = -1/2$ using first derivative test and an inflection point at $x = 1$.
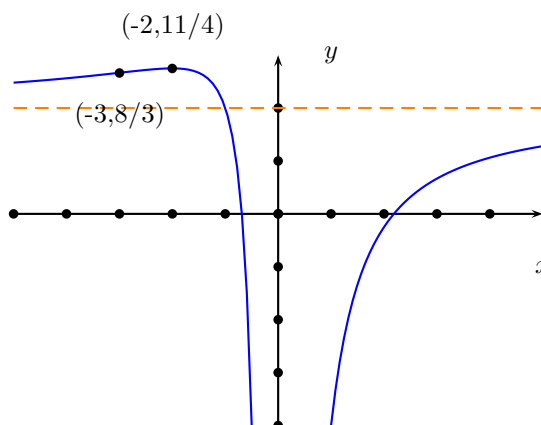
13.

$x = 0$ is a vertical asymptote

$y = 2$ is a horizontal asymptote

$(-2, 11/4)$ is a relative maximum

$(-3, 8/3)$ is an inflection point
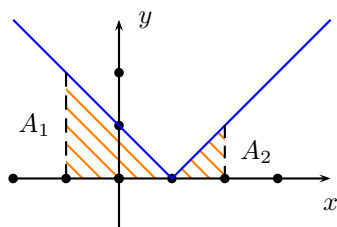

14. $x = 6$ and $y = 8$

15.

(a) $x^3 + 4x^2 - \dfrac{3}{x} + C$ (b) $\dfrac{3}{7}x^{7/3} - 12x^{1/3} + C$ (c) $\dfrac{2}{5}(x-5)^{5/2} + \dfrac{10}{3}(x-5)^{3/2} + C$ (d) $-\dfrac{1}{4}\cos(x^4 + 2) + C$

16.

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}, \quad f(x) = \frac{1}{x}, \quad c_k = \frac{5}{4}, \ \frac{3}{2}, \ \frac{7}{4}, \ 2, \quad \sum_{k=1}^{4} f(c_k)\Delta x = \left(\frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2\right)\frac{1}{4} = \frac{533}{840}.$$
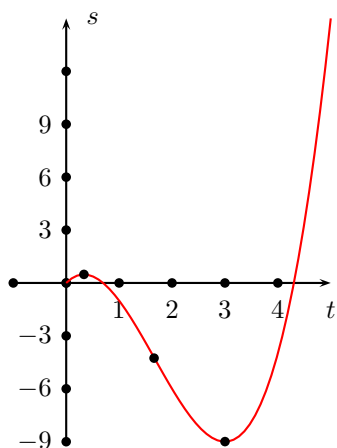
17.

$$\int_{-1}^{2} |x - 1|\, dx = A_1 + A_2 = \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1) = \frac{5}{2}.$$

18. (a) $9/4$, (b) -2, (c) $2\sqrt{y} - 2$

19.

$$s = t^3 - 5t^2 + 3t$$

$$v = 3t^2 - 10t + 3 = (3t - 1)(t - 3)$$

$$a = 6t - 10$$

Initially the particle is moving in a positive direction and slowing down. It stops at $t = 1/3$ and reverses direction. The particle's acceleration is 0 at $t = 5/3$ and after that becomes positive. The particle stops one more time at $t = 3$. After that it moves in the positive direction with increasing velocity.

20. 1.532088862   **21.** 1.3252   **22.** 144 feet

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