WORKBOOK. MATH 30. PRE-CALCULUS MATHEMATICS.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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Contents

1. Basics of Functions and their Graphs

(1) What is a set? Give three examples of finite sets, and give three examples of infinite sets.

(2) What is a relation?

(3) Construct three examples of relations using the finite sets you presented in question (1).

(4) Construct three examples of relations using the infinite sets you presented in question (1).

(5) What is the domain of a relation? What is the domain of each of the six examples of relations you constructed in questions (3) and (4)?

(6) What is the range of a relation? What is the range of each of the six examples of relations you constructed in questions (3) and (4)?

(7) What is a function?

(8) Construct three examples of functions using the finite sets you presented in question (1).

(9) Construct three examples of functions using the infinite sets you presented in question (1).

(10) What is the domain of a function? What is the domain of each of the six examples of functions you constructed in questions (8) and (9)?

(11) What is the range of a function? What is the range of each of the six examples of functions you constructed in questions (8) and (9)?

(12) Is every function a relation?

(13) Is every relation a function? Was every relation you constructed in questions (3) and (4) also a function?

(14) The set of natural numbers is denoted by $\mathbb{N}$. Given values that a function $f : \mathbb{N} \to \mathbb{N}$ takes on certain numbers, write an equation that describes f.

$f : \mathbb{N} \to \mathbb{N}$	$f : \mathbb{N} \to \mathbb{N}$	$f : \mathbb{N} \to \mathbb{N}$	$f : \mathbb{N} \to \mathbb{N}$
$1 \mapsto 1$	$1 \mapsto 4$	$1 \mapsto 3$	$1 \mapsto 3$
$2 \mapsto 4$	$2 \mapsto 7$	$2 \mapsto 6$	$2 \mapsto 12$
$3 \mapsto 9$	$3 \mapsto 12$	$3 \mapsto 9$	$3 \mapsto 27$
$4 \mapsto 16$	$4 \mapsto 19$	$4 \mapsto 12$	$4 \mapsto 48$
$f(x) =$	$f(x) =$	$f(x) =$	$f(x) =$

(15) The set of integers is denoted by $\mathbb{Z}$. Given values that a function $f : \mathbb{Z} \to \mathbb{Z}$ takes on certain numbers, write an equation that describes f.

$\begin{array}{lll} f:\mathbb{Z} & \to & \mathbb{Z} \\ 0 & \mapsto & 0 \\ 1 & \mapsto & -1 \\ -2 & \mapsto & -4 \\ 3 & \mapsto & -9 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{Z} & \to & \mathbb{Z} \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ -2 & \mapsto & -2 \\ 3 & \mapsto & 3 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{Z} & \to & \mathbb{Z} \\ 1 & \mapsto & -5 \\ 2 & \mapsto & -10 \\ 3 & \mapsto & -15 \\ 4 & \mapsto & -20 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{Z} & \to & \mathbb{Z} \\ 0 & \mapsto & -7 \\ 1 & \mapsto & -6 \\ -2 & \mapsto & -9 \\ 3 & \mapsto & -4 \\ f(x) & = & \end{array}$

(16) The set of real numbers is denoted by $\mathbb{R}$. Given values that a function $f : \mathbb{Z} \to \mathbb{Z}$ takes on certain numbers, write an equation that describes f.

$\begin{array}{lll} f:\mathbb{R} & \to & \mathbb{R} \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ -2 & \mapsto & 2 \\ 3 & \mapsto & 3 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ 1 & \mapsto & 1 \\ -2 & \mapsto & -\frac{1}{2} \\ 3 & \mapsto & \frac{1}{3} \\ \frac{1}{2} & \mapsto & 2 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{R} & \to & \mathbb{R} \\ 0 & \mapsto & 0 \\ 1 & \mapsto & 1 \\ -8 & \mapsto & -2 \\ 27 & \mapsto & 3 \\ f(x) & = & \end{array}$	$\begin{array}{lll} f:\mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ 1 & \mapsto & 1 \\ -2 & \mapsto & \frac{1}{4} \\ 3 & \mapsto & \frac{1}{9} \\ \frac{1}{2} & \mapsto & 4 \\ f(x) & = & \end{array}$

When $f(x) = y$, we say "f takes value y at x." The variable x is the **independent** variable, while the variable y is the **dependent** variable.

(17) The equation $y = \dfrac{(x^3 + 5)}{2}$ represents a function from $\mathbb{R}$ to $\mathbb{R}$ because

(18) Give five examples of equations in x and y which do not represent functions. In each case explain why the equation does not represent a function.

(19) Give five examples of equations in x and y which represent functions.

(20) Given function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = 5x - 4$ find

- $h(3)$
- $h(0)$
- $h(-4)$
- $h(-1)$
- $h(a)$
- $h(3a)$
- $h(x + 2)$
- $h(-x)$

(21) Given function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^2 - 5x + 4$ find

- $f(3)$
- $f(0)$
- $f(-4)$
- $f(-1)$

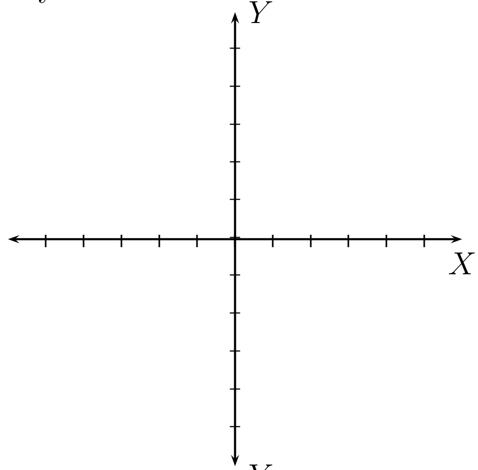
- $f(2a)$

- $f(x+a)$

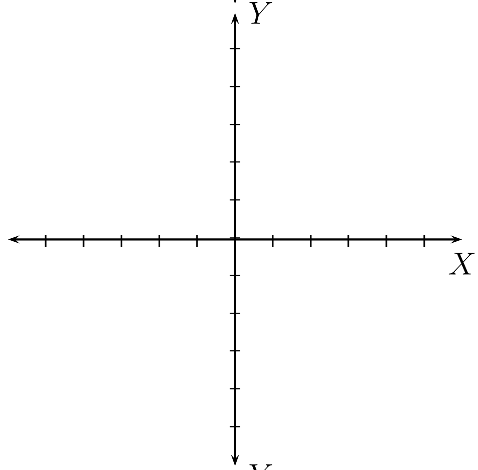
- $f(-x)$

(22) Graph the following functions (plot at least five points). In each case state the **domain** and **range**. What are the x and y intercepts in each case, if any.

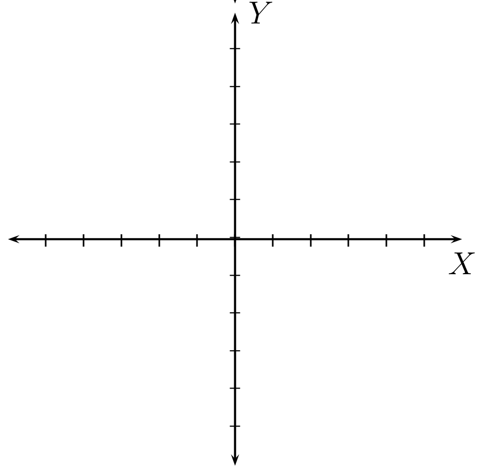
(a) $f(x) = x$



(a1) $g(x) = x - 3$

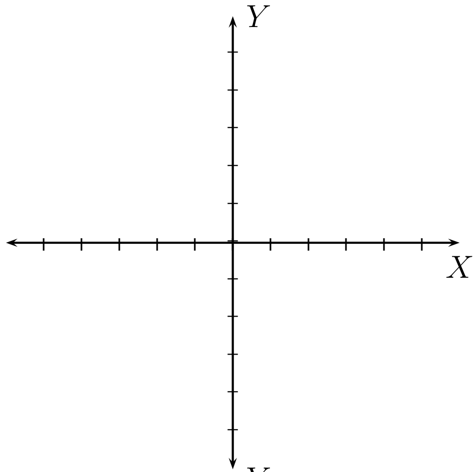


(a2) $h(x) = x + 5$

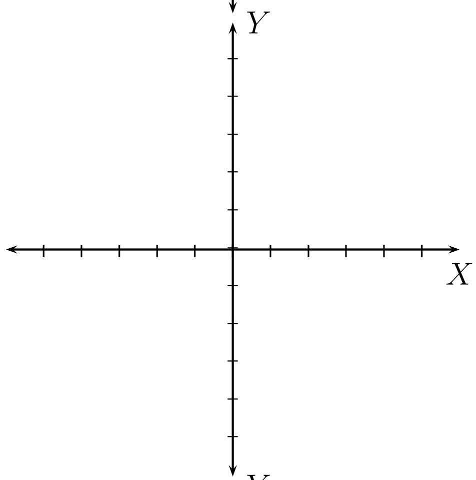


How are the graphs (a1) and (a2) related to (a)? Use words such as "shifting" up, down, right, or left by specific number of units.

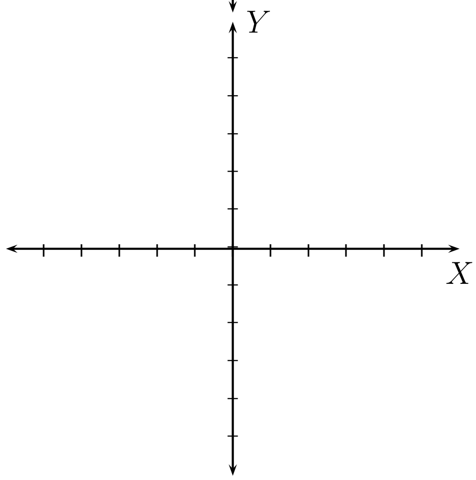
(b) $f(x) = x^2$



(b1) $g(x) = (x-3)^2$

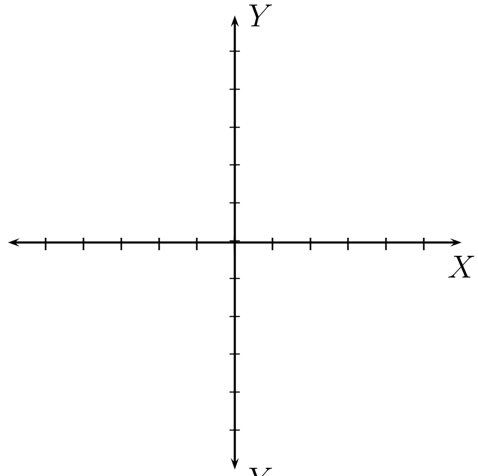


(b2) $h(x) = (x+5)^2$

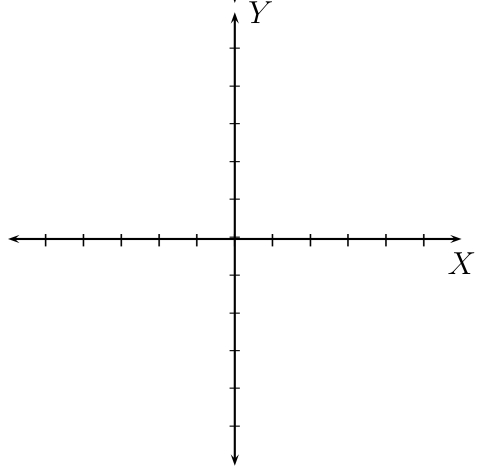


How are the graphs (b1) and (b2) related to (b)? Use words such as "shifting" up, down, right, or left by specific number of units.

(b3) $k(x) = x^2 - 3$

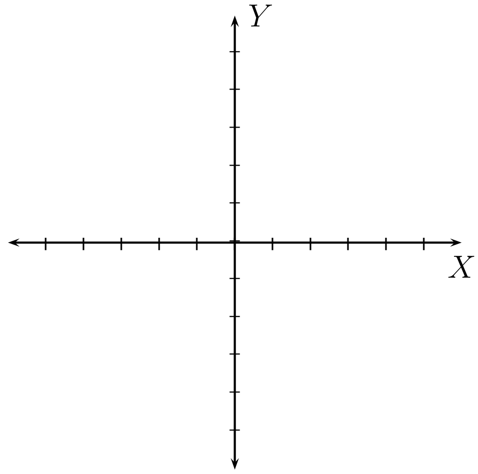


(b4) $m(x) = x^2 + 5$

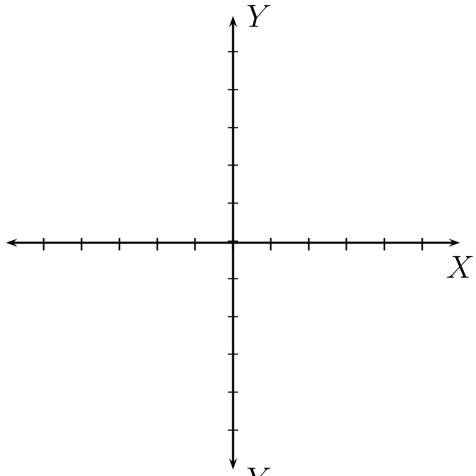


How are the graphs (b3) and (b4) related to (b)? Use words such as "shifting" up, down, right, or left by specific number of units.

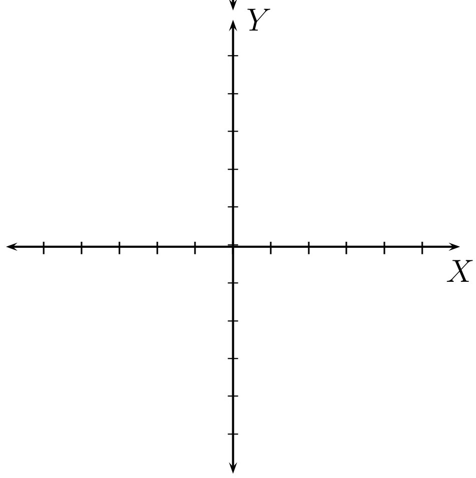
(c) $f(x) = x^3$



(c1) $h(x) = x^3 - 2$

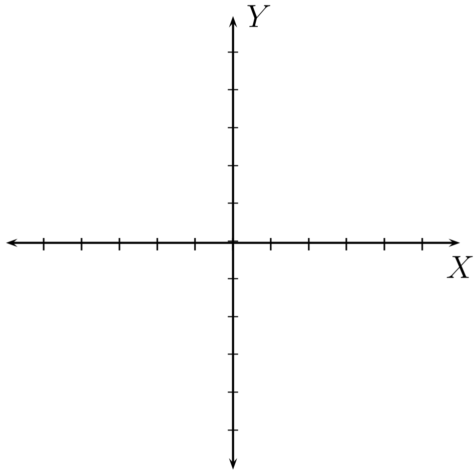


(c2) $k(x) = x^3 + 4$

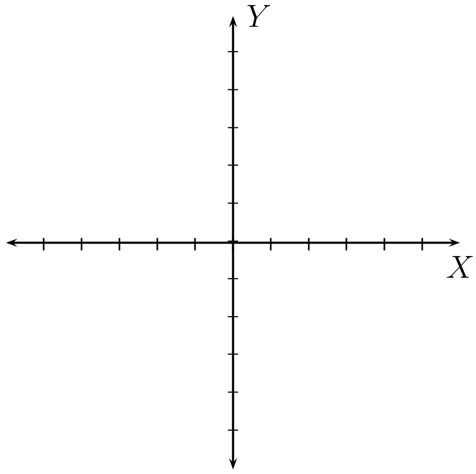


How are the graphs (c1) and (c2) related to (c)? Use words such as "shifting" up, down, right, or left by specific number of units.

(c3) $l(x) = (x - 2)^3$

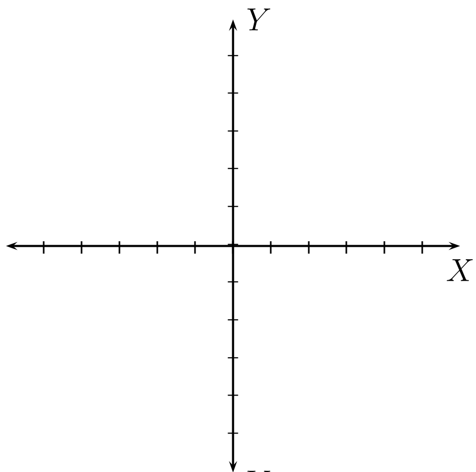


(c4) $m(x) = (x+4)^3$

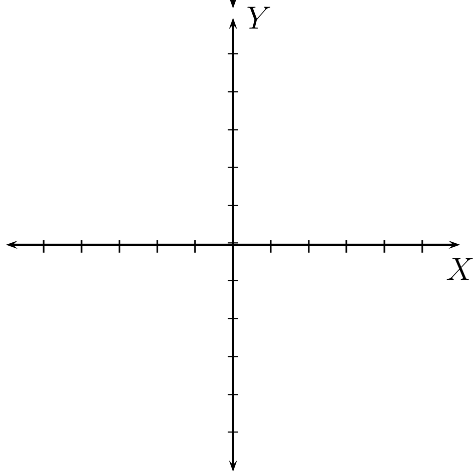


How are the graphs (c3) and (c4) related to (c)?

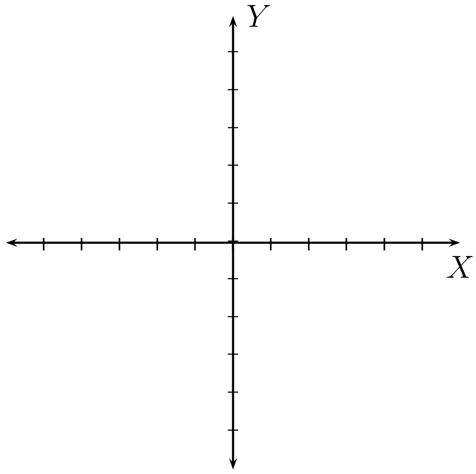
(d) $f(x) = |x|$



(d1) $g(x) = |x-3|$

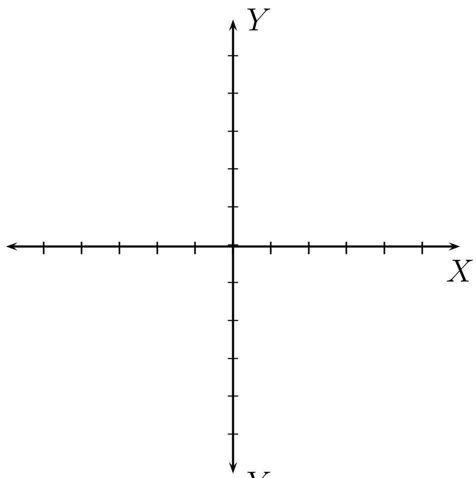


(d2) $k(x) = |x+4|$

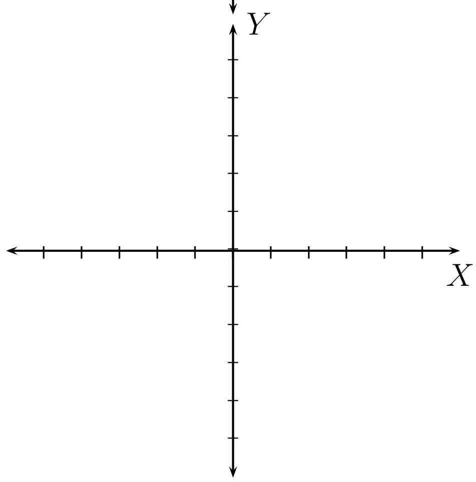


How are the graphs (d1) and (d2) related to (d)?

(d3) $l(x) = |x| - 3$

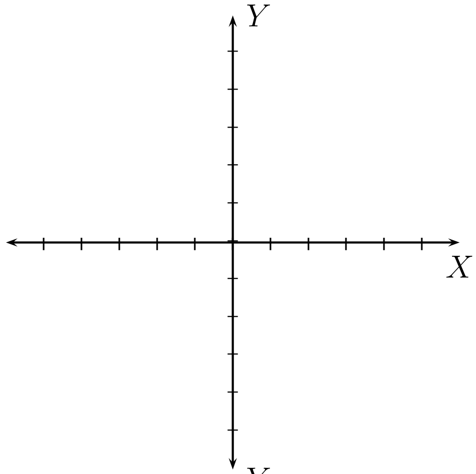


(d4) $m(x) = |x| + 4$

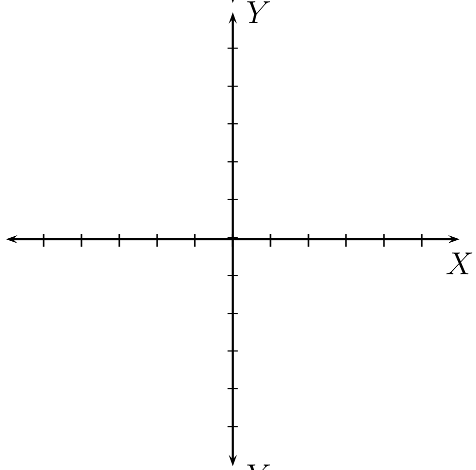


How are the graphs (d1) and (d2) related to (d)?

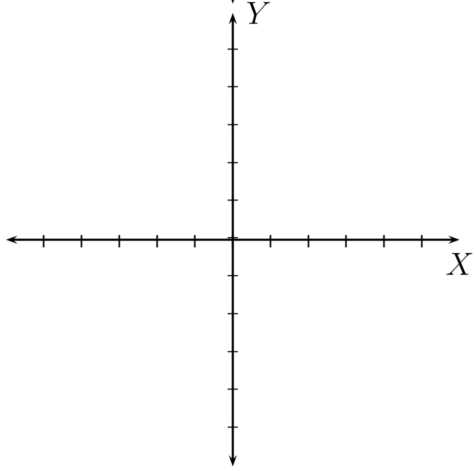
(e) $f(x) = \sqrt{x}$



(e1) $g(x) = \sqrt{x-3}$

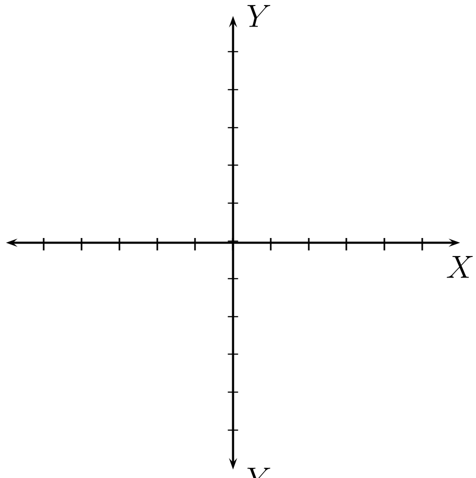


(e2) $h(x) = \sqrt{x+4}$

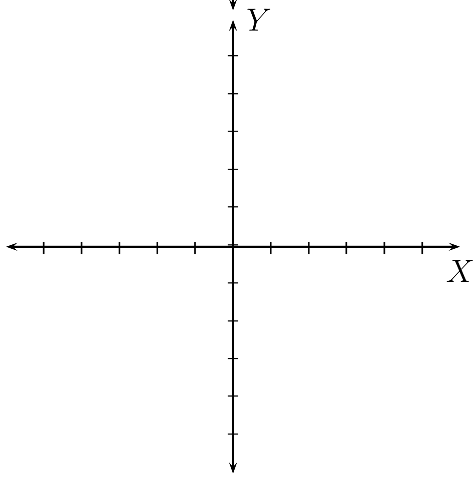


How are the graphs (e1) and (e2) related to (e)?

(e3) $k(x) = \sqrt{x} - 3$

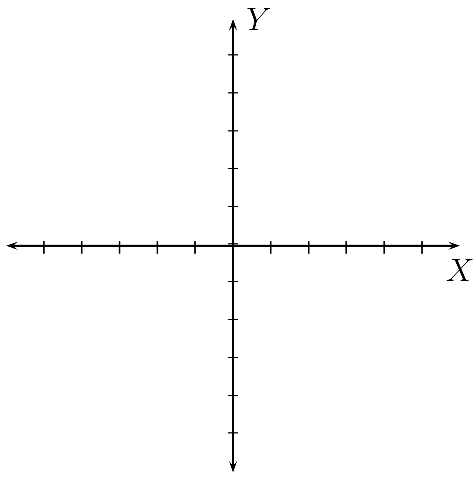


(e4) $l(x) = \sqrt{x} + 4$

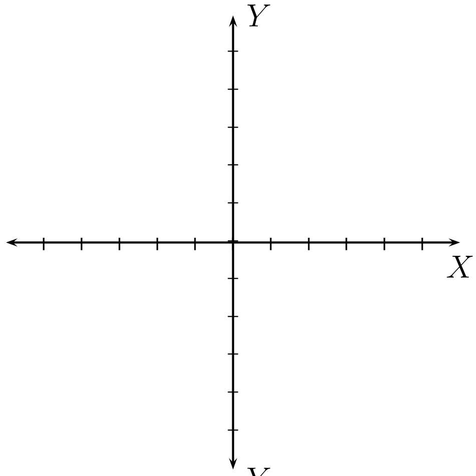


How are the graphs (e3) and (e4) related to (e)?

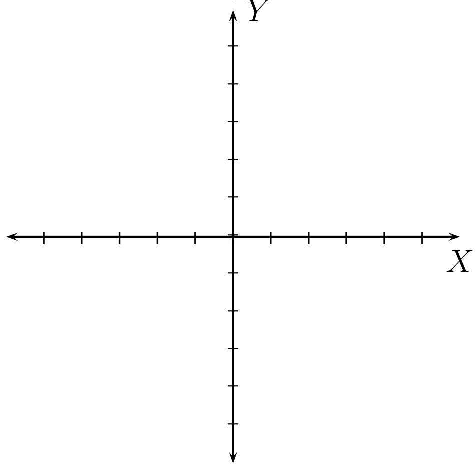
(p) $f(x) = \sqrt[3]{x}$



(p1) $g(x) = \sqrt[3]{x} - 3$

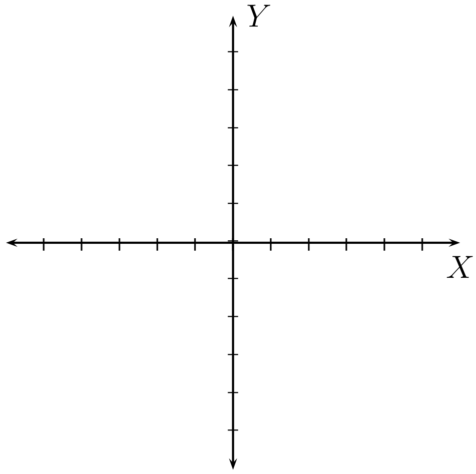


(p2) $h(x) = \sqrt[3]{x} + 4$

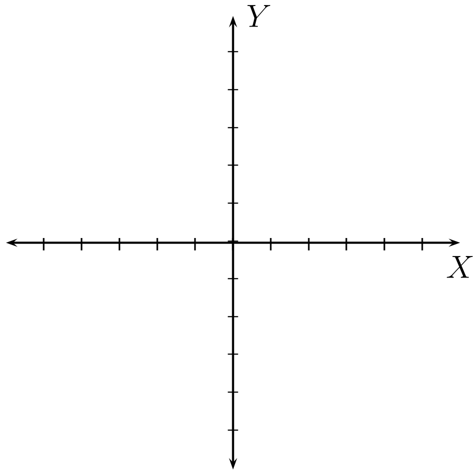


How are the graphs (p1) and (p2) related to (p)?

(p3) $k(x) = \sqrt[3]{x-3}$

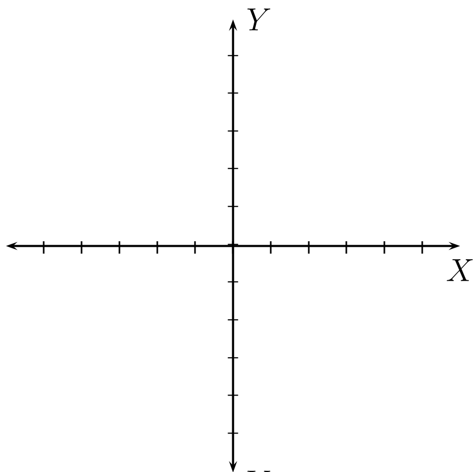


(p4) $l(x) = \sqrt[3]{x+4}$

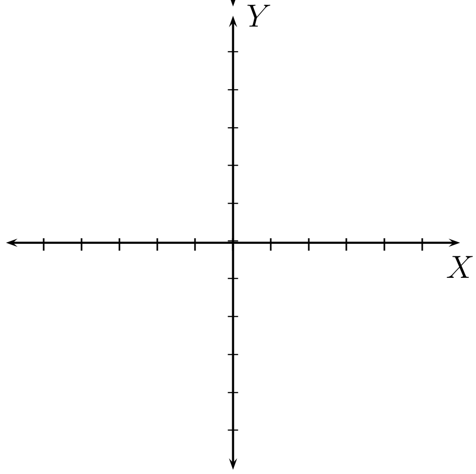


How are the graphs (p3) and (p4) related to (p)?

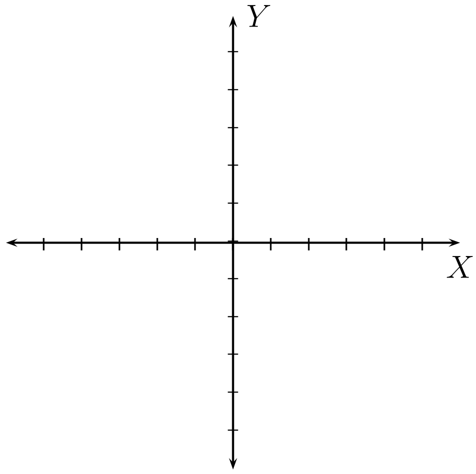
(q) $f(x) = \dfrac{1}{x}$



(q1) $g(x) = \dfrac{1}{x} - 3$

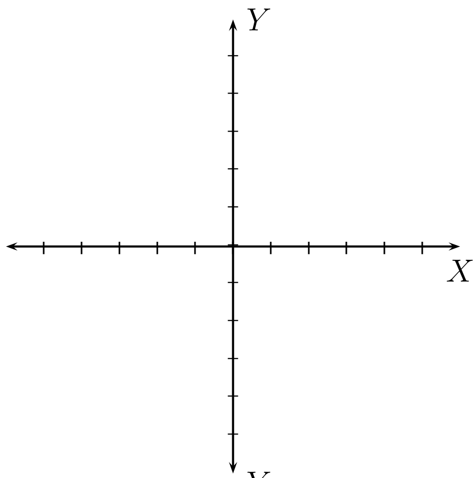


(q2) $h(x) = \dfrac{1}{x} + 4$

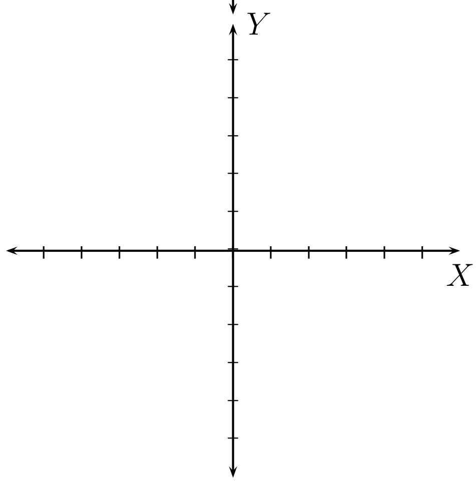


How are the graphs (q1) and (q2) related to (q)?

(q3) $k(x) = \dfrac{1}{x-3}$

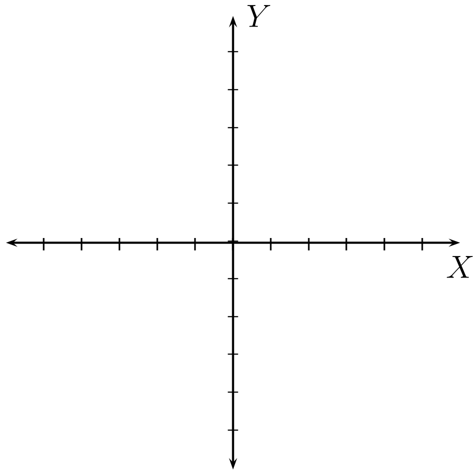


(q4) $l(x) = \dfrac{1}{x+4}$



How are the graphs (q1) and (q2) related to (q)?

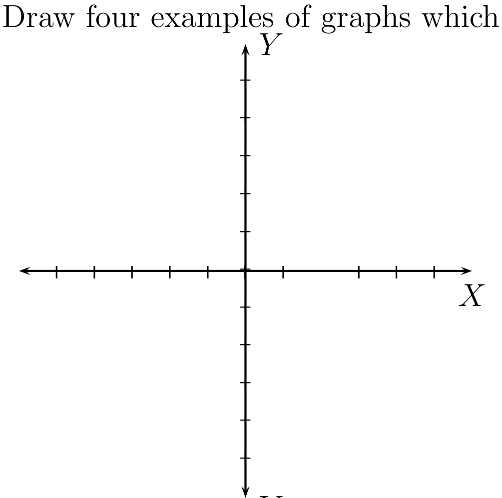
(s) $f(x) = 1$

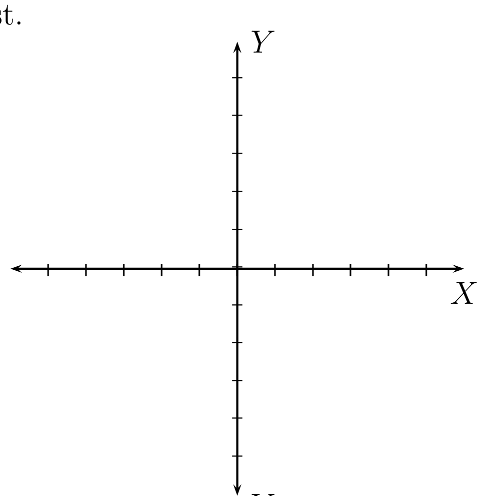


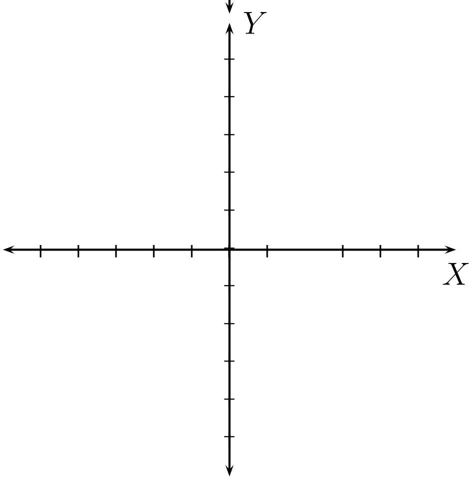
The graphs of $y = 1$, $y = x$, $y = x^2$, $y = x^3$, $y = |x|$, $y = \sqrt{x}$, $y = \sqrt[3]{x}$, and $y = \dfrac{1}{x}$ are important.

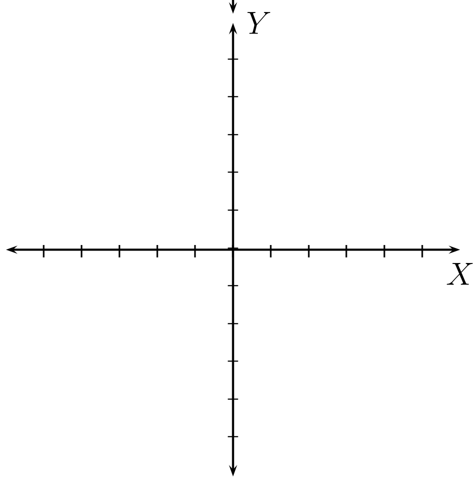
(23) What is the vertical line test for a graph? Explain in your words why the vertical line test works.

(24) Draw four examples of graphs which fail the vertical test.









(25) Draw a graph of the function f with the given properties.

(a) The domain of f is $[-3, 5]$

The range of f is $[-2, 4]$

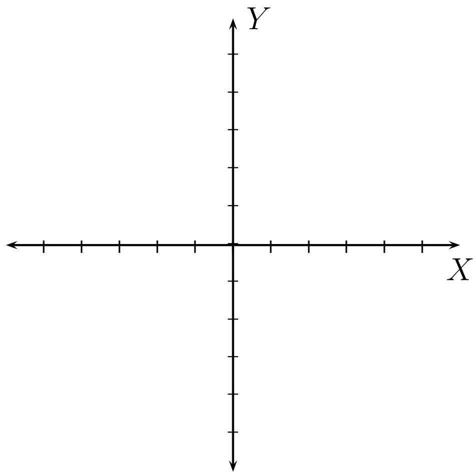
$f(-3) = 1$

$f(-1) = -1$

$f(4) = 3$

The x-intercepts are at -2 and 1

The y-intercept is at -2



(b) The domain of f is $[-5, 5]$

The range of f is $[-4, 4]$

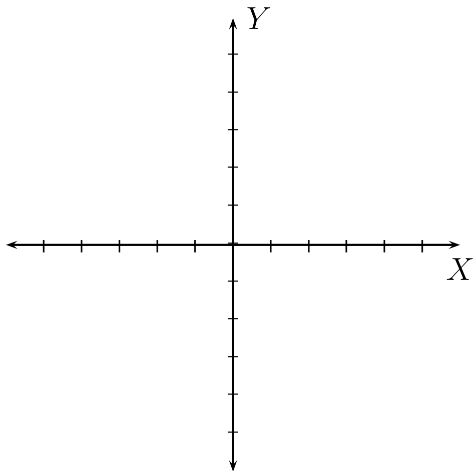
$f(-2) = 3$

$f(-1) = 3$

$f(5) = 3$

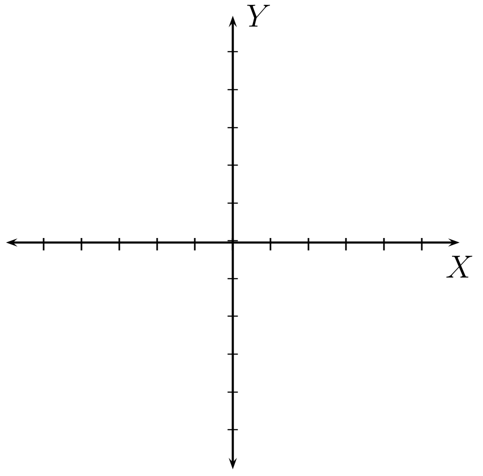
The x-intercepts are at 2 and 4

The y-intercept is at -2

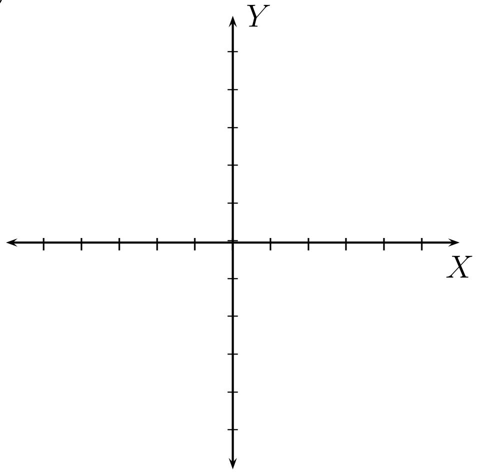


2. More on Functions and their Graphs

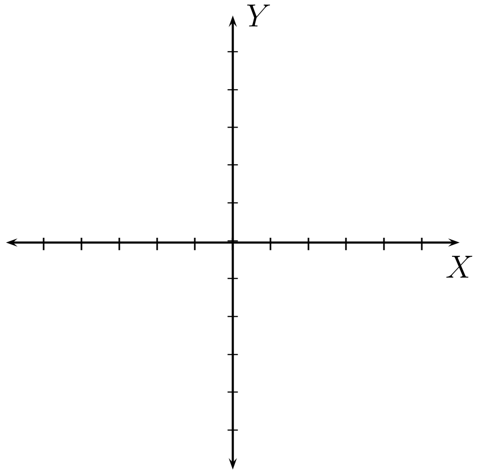
(1) When is a function said to be **increasing** on an open interval I? Draw the graph of a function which is increasing on the open interval $(-2, 5)$



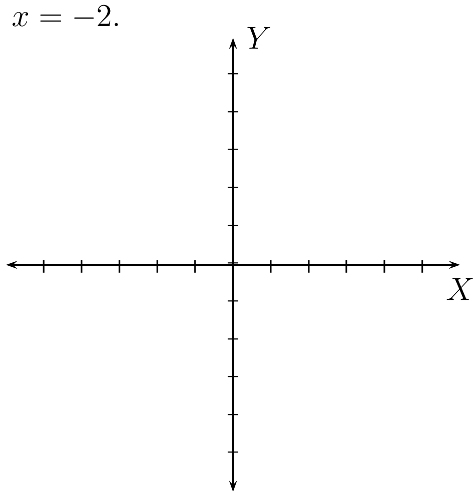
(2) When is a function said to be **decreasing** on an open interval I? Draw the graph of a function which is decreasing on the open interval $(-2, 5)$



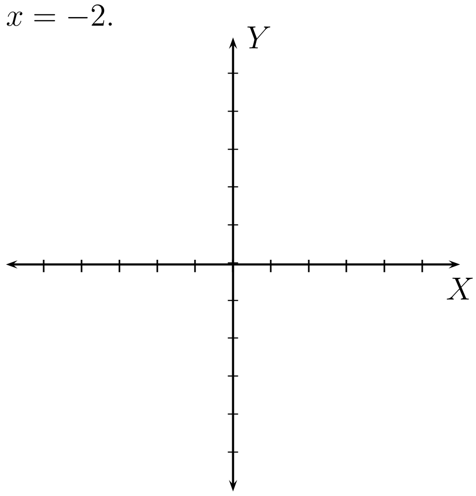
(3) When is a function said to be **constant** on an open interval I? Draw the graph of a function which on the open interval $(-2, 5)$



(4) When is a function value $f(a)$ said to be a **relative maximum**? Draw the graph of a function with a relative maximum function value of 1 at $x = -2$.

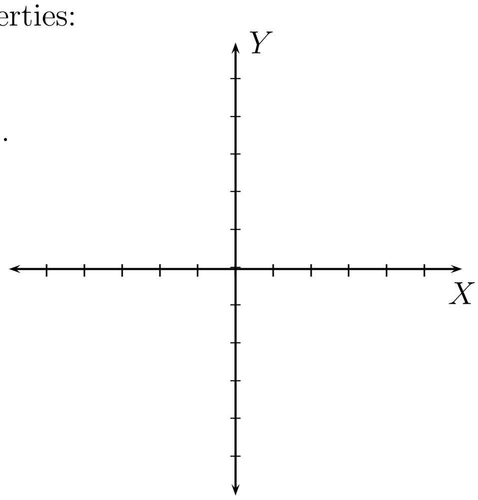


(5) When is a function value $f(a)$ said to be a **relative minimum**? Draw the graph of a function with a relative minimum function value of 1 at $x = -2$.



(6) Draw the graph of a function f with the following properties:

The domain of f is $(-\infty, \infty)$.
The range of f is $(-4, \infty)$.
f is decreasing on the intervals $(-\infty, -3)$ and $(-2, 1)$.
f is increasing on the intervals $(-3, -2)$ and $(2, \infty)$.
f is constant on the interval $(1, 2)$.
f has a relative minimum value of -2 at $x = -3$.
f has a relative maximum value of 2 at $x = -2$.
f has x-intercepts at $-2.5, 0$, and 4.



(7) When is a function said to be **even**? Give five examples of even functions and explain why they are even.

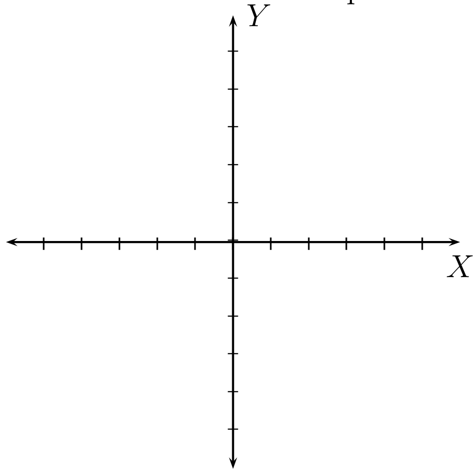
(8) When is a function said to be **odd**? Give five examples of even functions and explain why they are even.

(9) Recall the list of important graphs we saw in lesson (1).

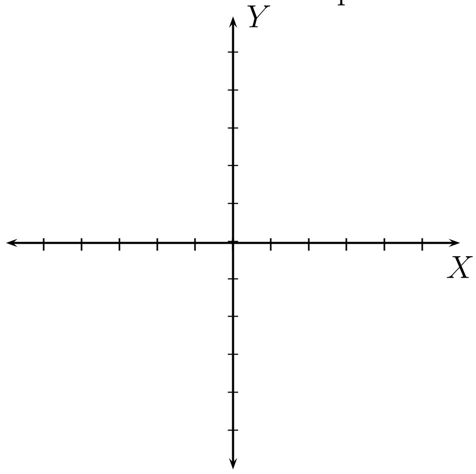
$$y = 1, y = x, y = x^2, y = x^3, y = |x|, y = \sqrt{x}, y = \sqrt[3]{x}, y = \frac{1}{x}.$$

Classify these functions as even, odd, and neither even nor odd?

(10) What kind of symmetry does the graph of an even function have? Draw an example.

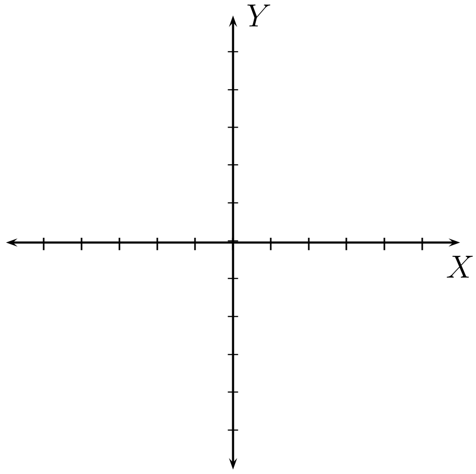


(11) What kind of symmetry does the graph of an odd function have? Draw an example.

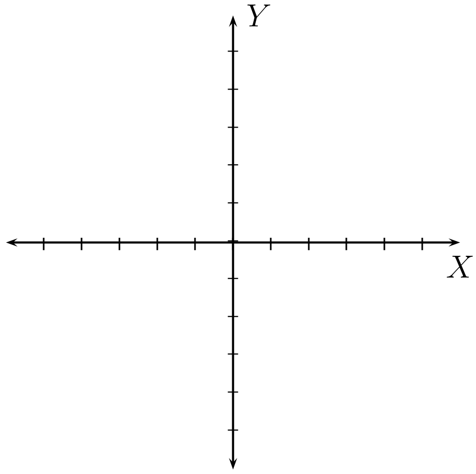


(12) Draw graphs of following piecewise defined functions:

$$f(x) = \begin{cases} x-3 & \textit{if} \quad x \leq 2 \\ 4 & \textit{if} \quad 2 < x \leq 3 \\ x-4 & \textit{if} \quad x > 3 \end{cases}$$



$$g(x) = \begin{cases} x^2 & \text{if} \quad x \le -1 \\ x & \text{if} \quad -1 < x \le 2 \\ x+2 & \text{if} \quad x > 2 \end{cases}$$



(13) Find and simplify the difference quotient

$$\frac{f(x+h)-f(x)}{h}; \quad h \neq 0.$$

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = \dfrac{1}{x}$

(e) $f(x) = 3x^2 + 4x + 5$

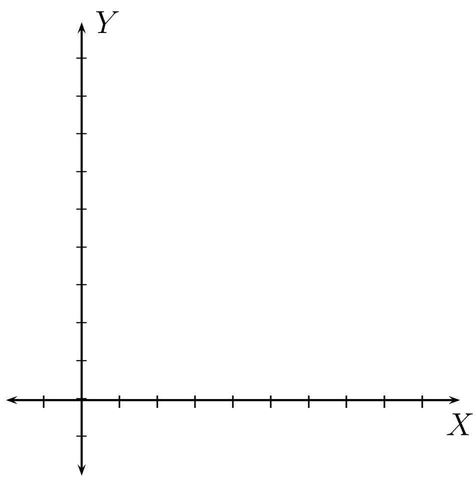
(f) $f(x) = -3x^2 - 4x + 5$

(g) $f(x) = \sqrt{x}$

(h) $f(x) = \sqrt{x+5}$

(i) $f(x) = 2$

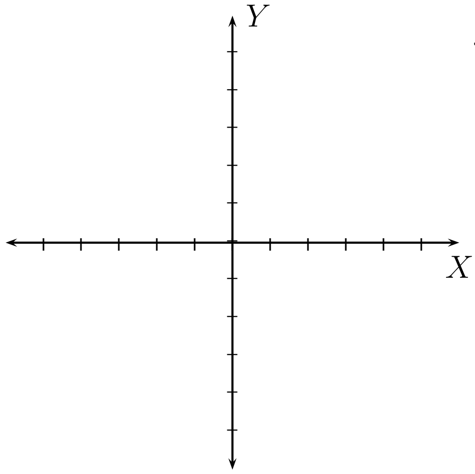
(14) Write a piecewise defined function that models the cellular phone billing plan, where you pay \$ 32 per month for 300 minutes and \$ 0.20 for every additional minute. Graph this piecewise defined function.



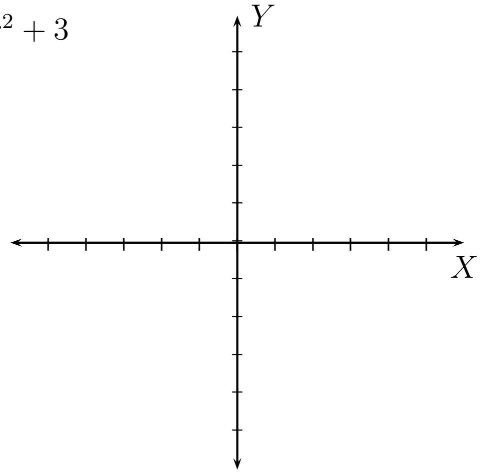
3. Transformations of functions

(1) Graph the following (plot three points at the same corresponding location on the graph):

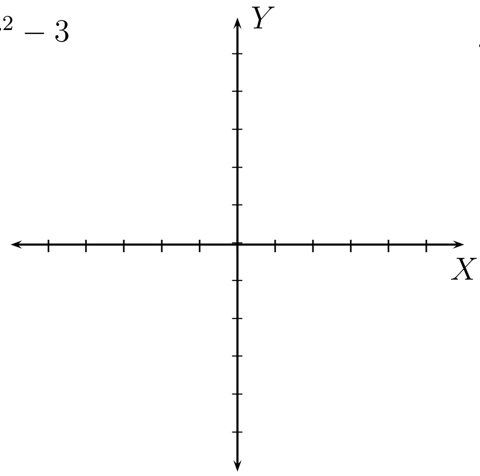
$f(x) = x^2$



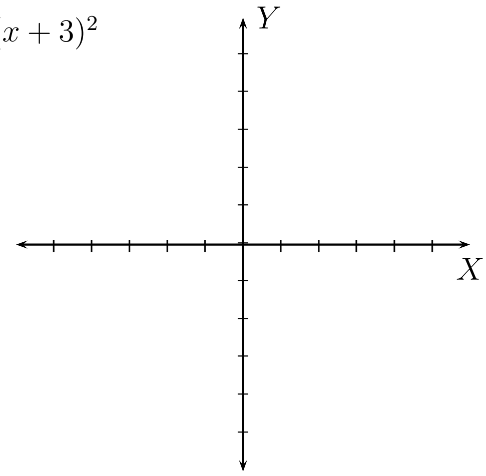
$f_1(x) = x^2 + 3$



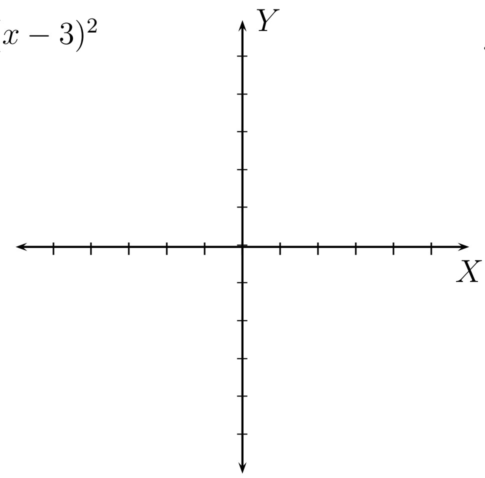
$f_2(x) = x^2 - 3$



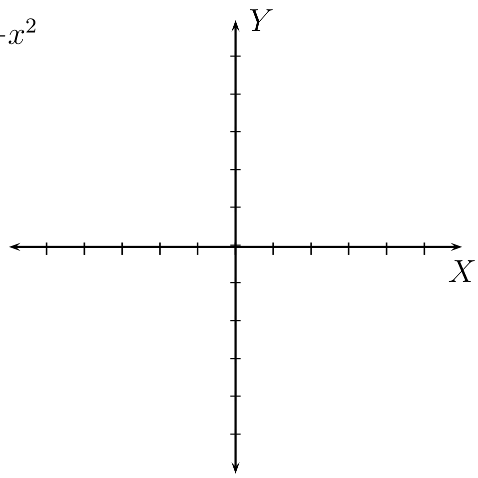
$f_3(x) = (x + 3)^2$



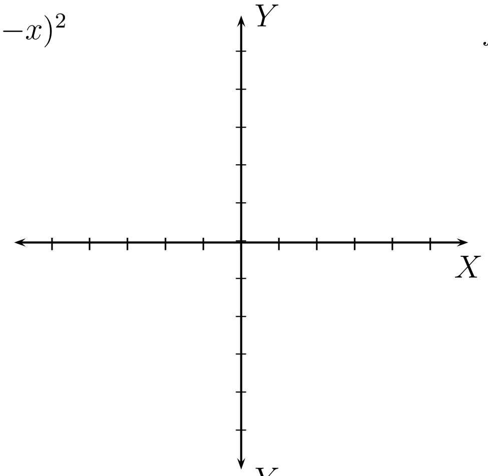
$f_4(x) = (x - 3)^2$



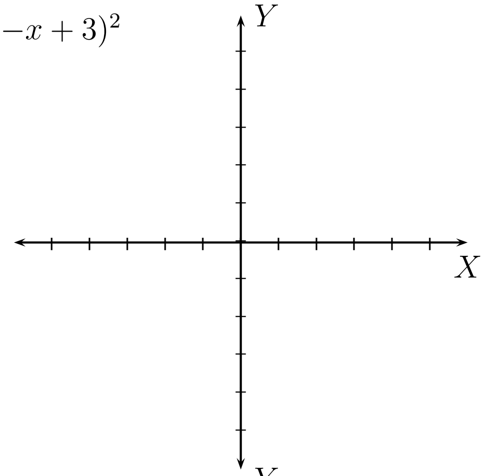
$f_5(x) = -x^2$



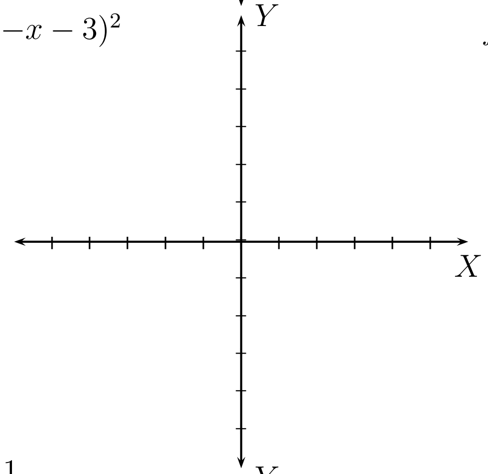
$f_6(x) = (-x)^2$



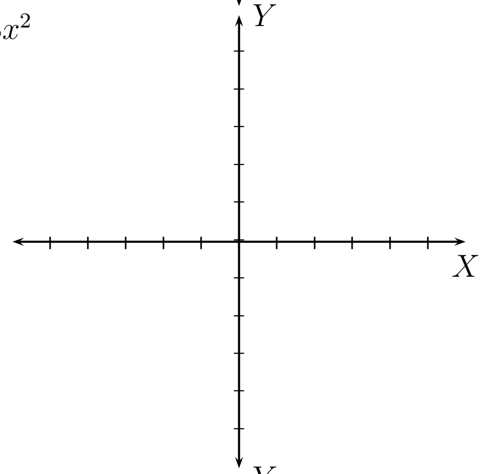
$f_7(x) = (-x+3)^2$



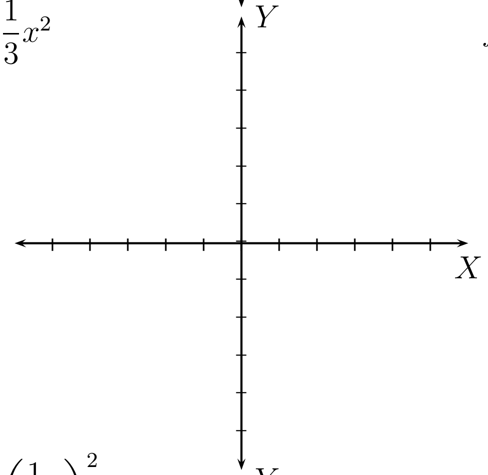
$f_8(x) = (-x-3)^2$



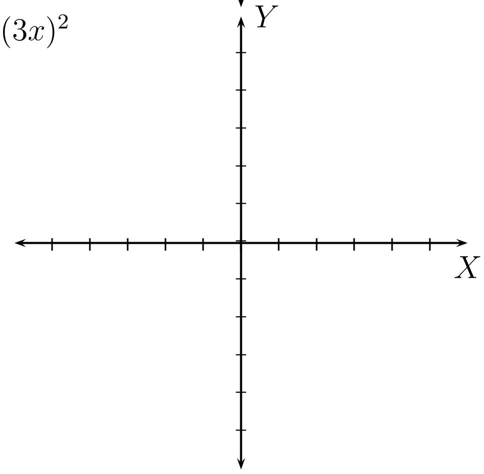
$f_9(x) = 3x^2$



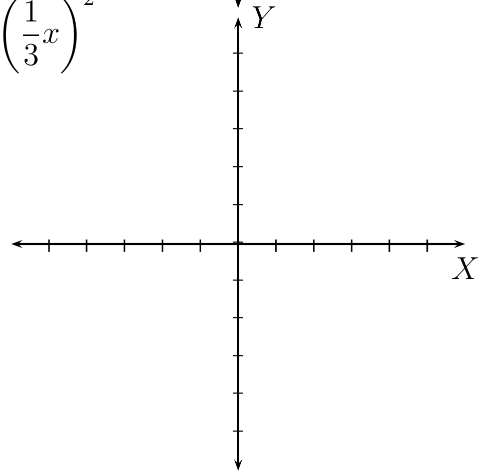
$f_{10}(x) = \dfrac{1}{3}x^2$



$f_{11}(x) = (3x)^2$

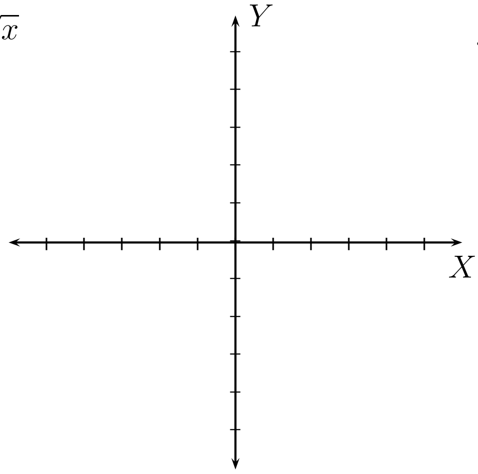


$f_{12}(x) = \left(\dfrac{1}{3}x\right)^2$

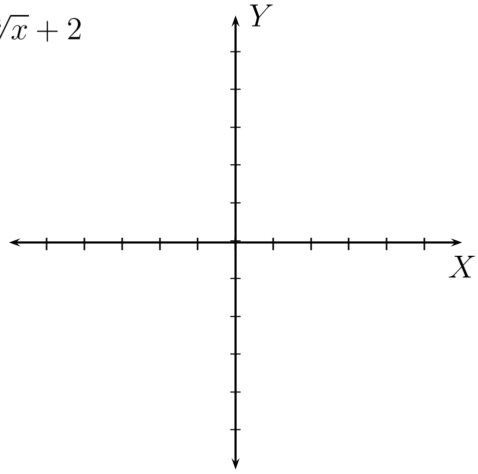


(2) Graph the following (plot three points at the same corresponding location on the graph):

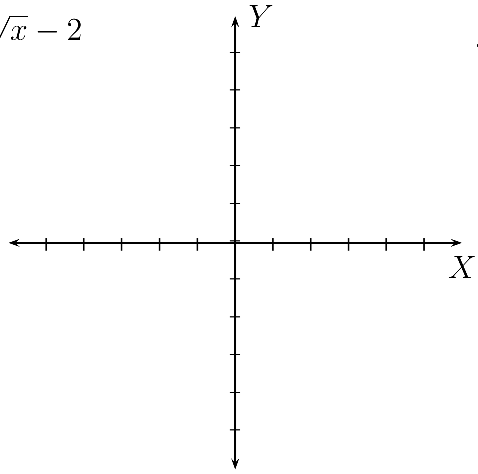
$f(x) = \sqrt[3]{x}$



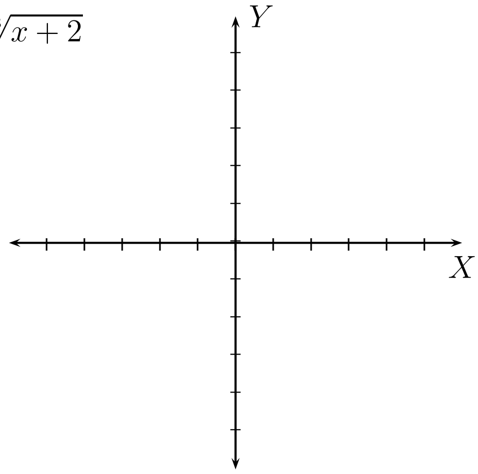
$f_1(x) = \sqrt[3]{x} + 2$



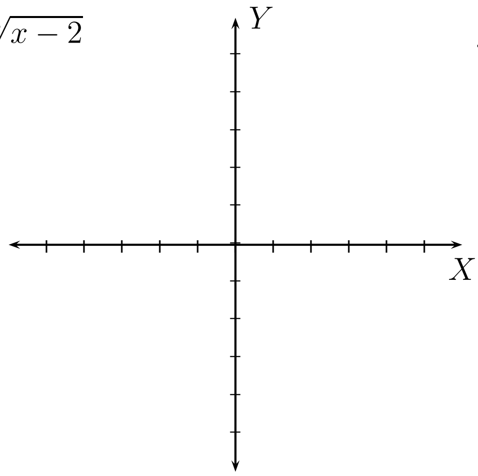
$f_2(x) = \sqrt[3]{x} - 2$



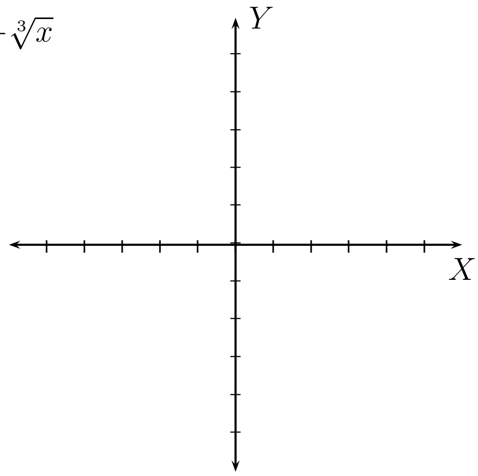
$f_3(x) = \sqrt[3]{x+2}$



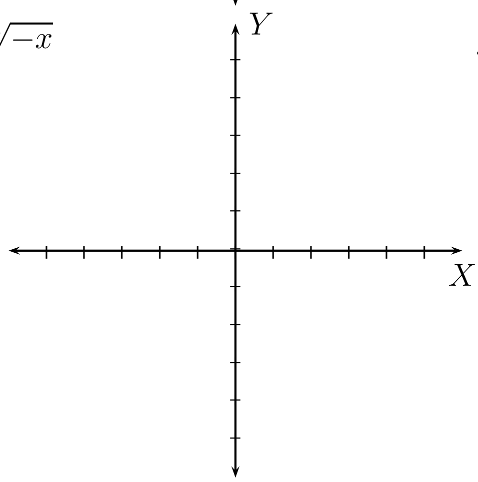
$f_4(x) = \sqrt[3]{x-2}$



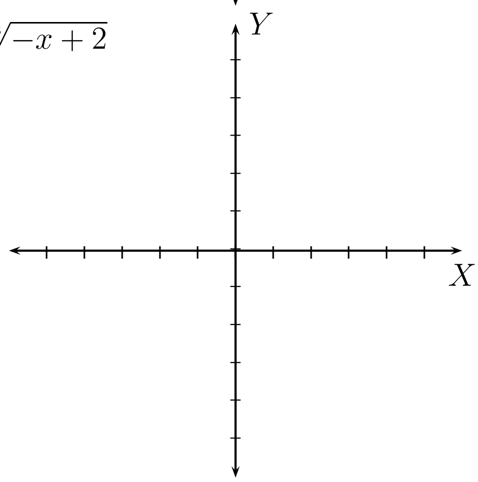
$f_5(x) = -\sqrt[3]{x}$



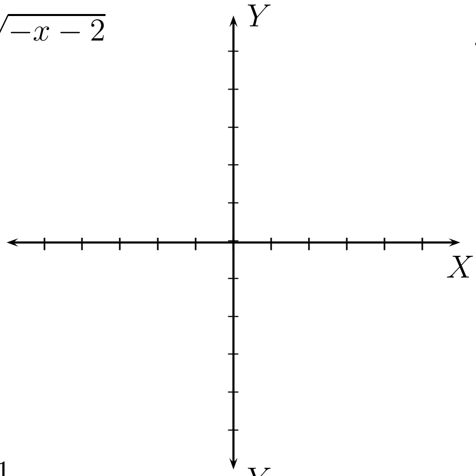
$f_6(x) = \sqrt[3]{-x}$



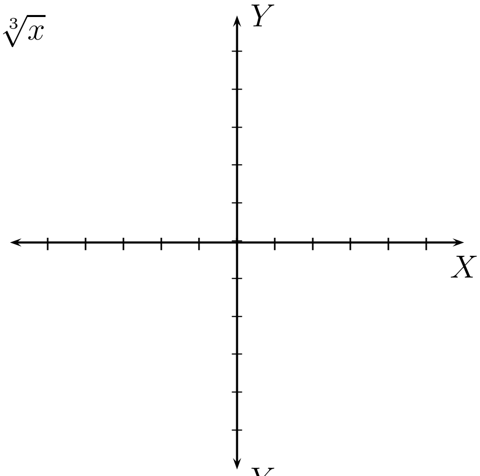
$f_7(x) = \sqrt[3]{-x+2}$



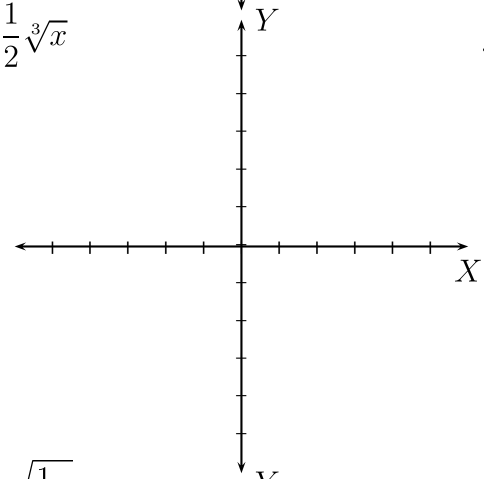
$f_8(x) = \sqrt[3]{-x-2}$



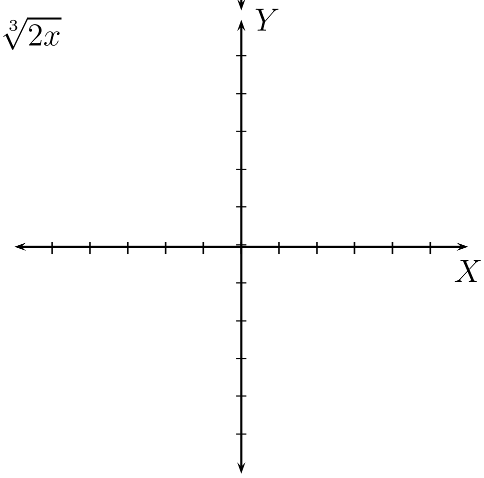
$f_9(x) = 2\sqrt[3]{x}$



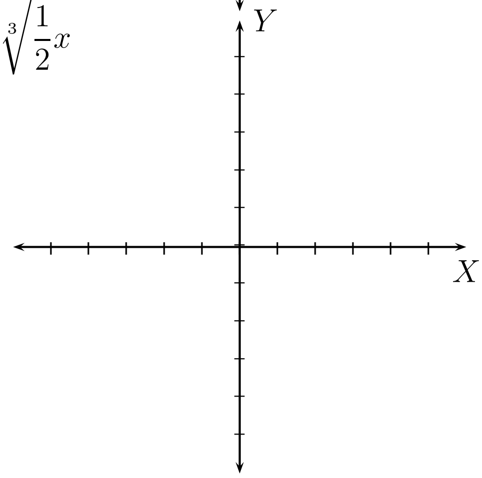
$f_{10}(x) = \frac{1}{2}\sqrt[3]{x}$



$f_{11}(x) = \sqrt[3]{2x}$



$f_{12}(x) = \sqrt[3]{\frac{1}{2}x}$

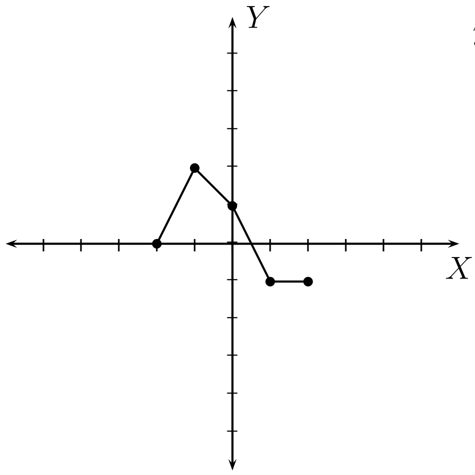


Summarize your conclusions: **For** $c > 0$

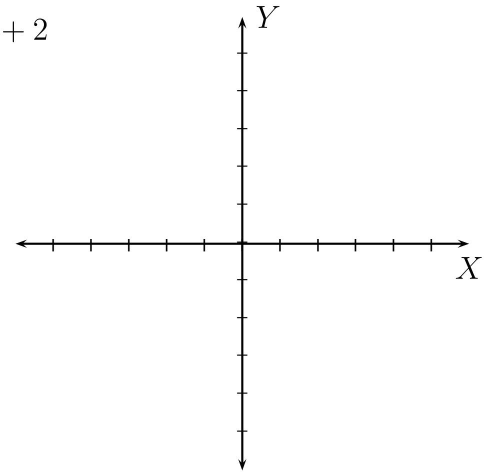
To graph	Draw the graph of f and	To graph	Draw the graph of f and
$y = f(x) + c$		$y = f(x) - c$	
$y = f(x + c)$		$y = f(x - c)$	
$y = -f(x)$		$y = f(-x)$	
$y = cf(x), c > 1$		$y = cf(x), c < 1$	
$y = f(cx), c > 1$		$y = f(cx), c < 1$	

Compare your table with the Table 1.4 on page 213 of your textbook.

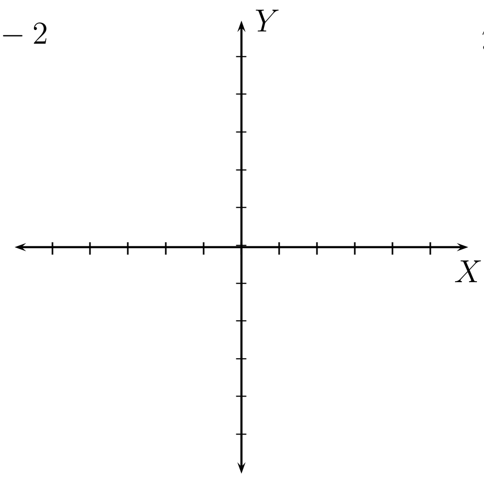
(1) The graph of g is given. Graph the given transformed functions (plot the **five** highlighted points at the same corresponding location on the graphs):



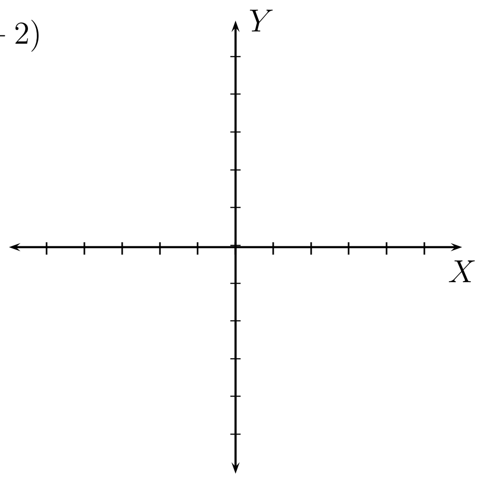
$y = g(x) + 2$



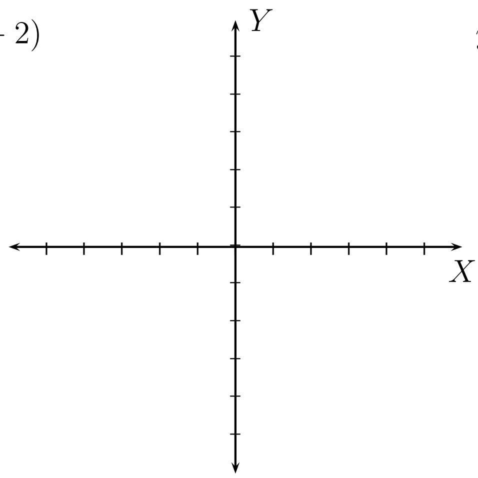
$y = g(x) - 2$



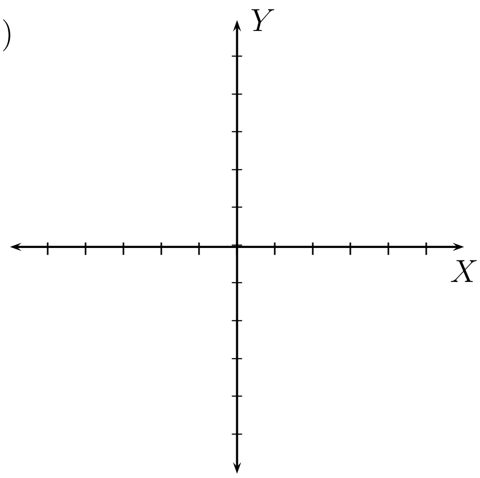
$y = g(x + 2)$



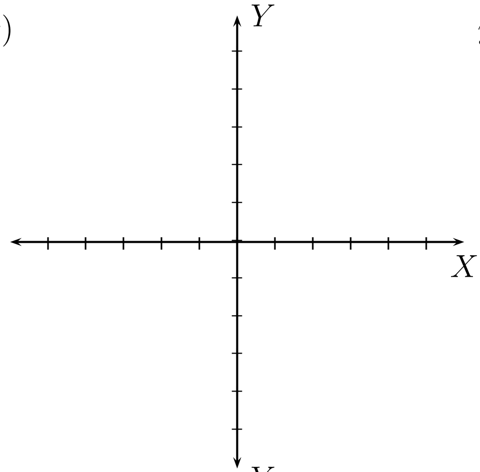
$y = g(x - 2)$



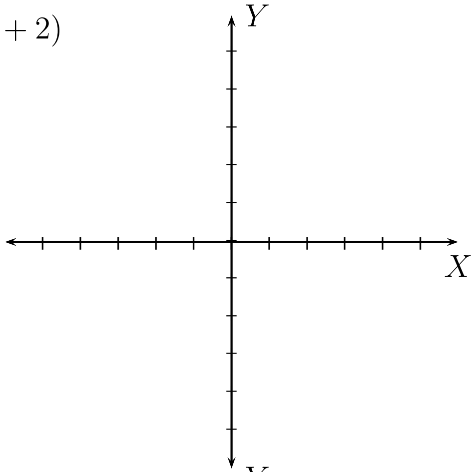
$y = -g(x)$



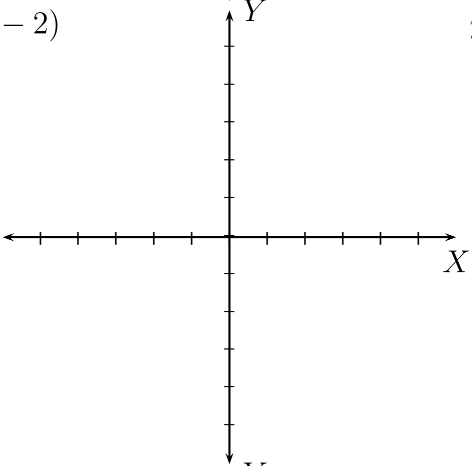
$y = g(-x)$



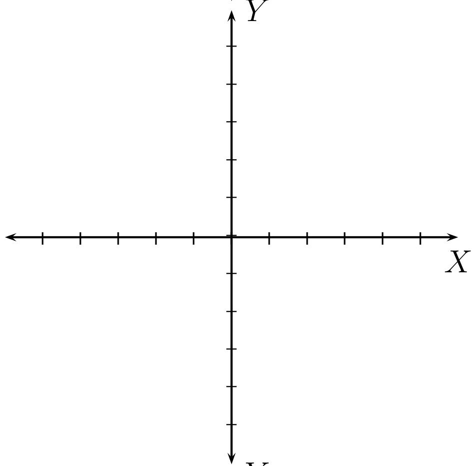
$y = g(-x+2)$



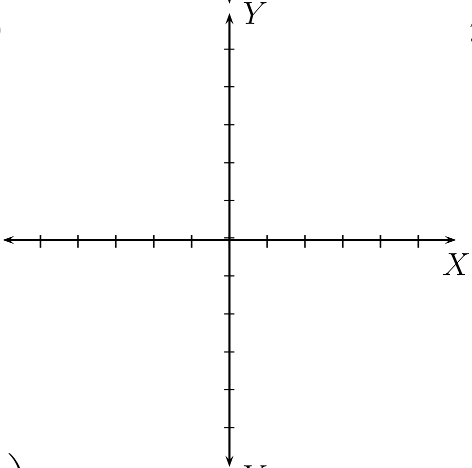
$y = g(-x-2)$



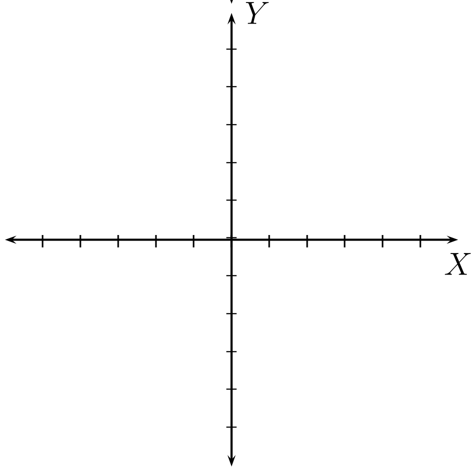
$y = 2g(x)$



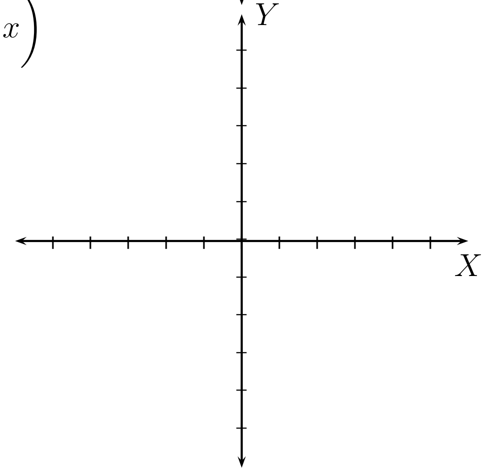
$y = \frac{1}{2}g(x)$



$y = g(2x)$

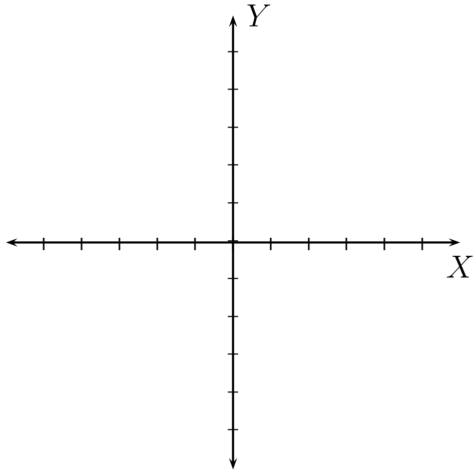


$y = g\left(\frac{1}{2}x\right)$

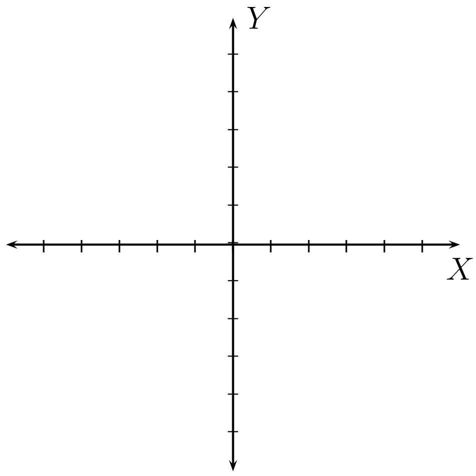


(2) Using transformations draw the graphs (plot at least three points):

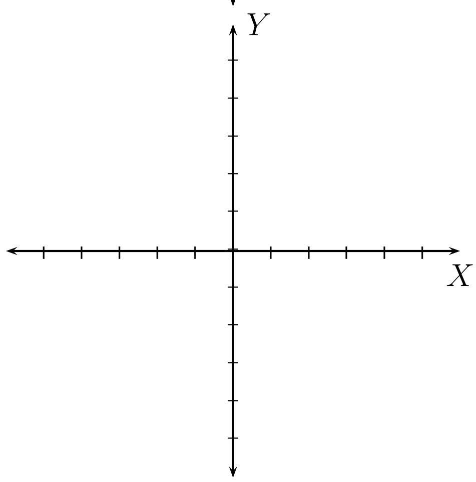
$f(x) = -2x + 3$



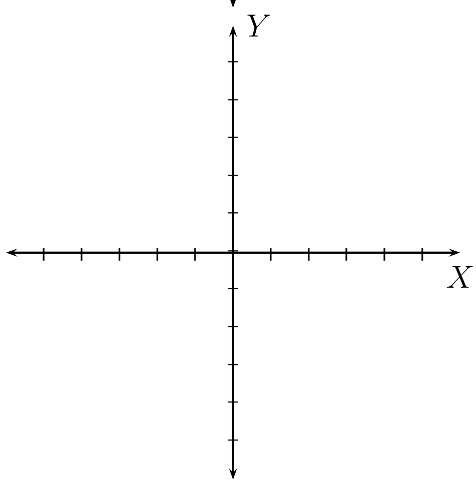
$f(x) = (-x + 3)^2 + 1$



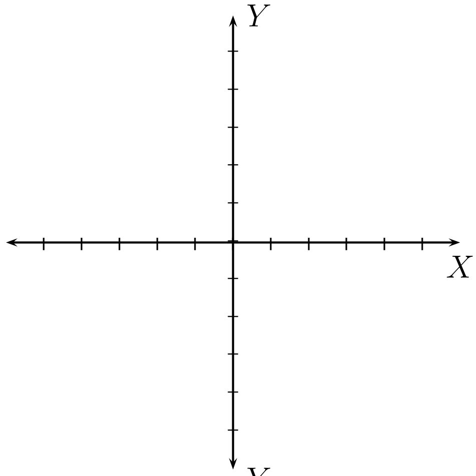
$f(x) = (-x - 3)^3 - 1$



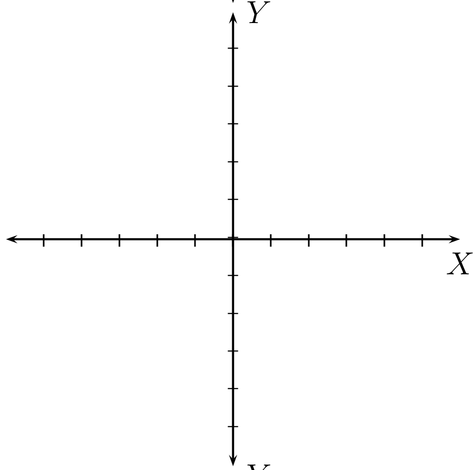
$f(x) = |-x + 4| - 3$



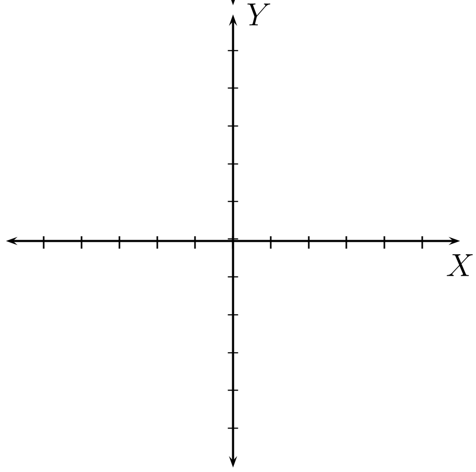
$f(x) = \sqrt{2x} - 1$



$f(x) = \sqrt[3]{\frac{1}{2}x} + 1$



$f(x) = \frac{1}{-x-4} + 2$



4. Combinations of functions; composite functions

(1) What is the domain of a function?

(2) Find the domain of the following functions:

- $f(x) = 3x^4 + 5x^2 - 3$
- $f(x) = 7x^{100} - 5x$
- $f(x) = 20$
- $f(x) = \dfrac{1}{x+2}$
- $f(x) = -\dfrac{x+5}{x-3}$
- $f(x) = \dfrac{4x+5}{(x+2)(x-3)}$

- $f(x) = \dfrac{4x+5}{x^2-x-6}$

- $f(x) = \dfrac{x+5}{x+5}$

- $f(x) = \dfrac{x+5}{(x+5)(x-2)}$

- $f(x) = \dfrac{x-5}{x^2-2x-15}$

- $f(x) = \dfrac{2x+5}{2x^2+11x+5}$

- $f(x) = \dfrac{3x-1}{6x^2+5x-6}$

- $f(x) = \dfrac{x+5}{x^2-25}$

- $f(x) = \dfrac{2x+3}{6x^2-7x-20}$

- $f(x) = \sqrt{x}$

- $f(x) = \sqrt{x+5}$

- $f(x) = \sqrt[3]{x}$

- $f(x) = \sqrt[3]{x+5}$

- $f(x) = \sqrt{x+5} + \sqrt{x+2}$

- $f(x) = \sqrt{x+5} + \sqrt{x-2}$

- $f(x) = \sqrt{x-5} + \sqrt{x+2}$

- $f(x) = \sqrt{x-5} + \sqrt{x-2}$

- $f(x) = \sqrt{x+5} + \sqrt{2-x}$

- $f(x) = \sqrt{x-5} + \sqrt{2-x}$

- $f(x) = \sqrt{5-x} + \sqrt{x+2}$

- $f(x) = \dfrac{x+2}{\sqrt{x+5}}$

- $f(x) = \dfrac{\sqrt{x+5}}{x+2}$

(3) Find $f+g$, $f-g$, fg, and $\dfrac{f}{g}$, and find their respective domains.

- $f(x) = 4x-1$, $g(x) = x^2 - 13x - 30$

- $f(x) = 4x - 1,\ g(x) = x^2$

- $f(x) = 2,\ g(x) = 7x$

- $f(x) = 4x,\ g(x) = 3x$

- $f(x) = 4,\ g(x) = \sqrt{x - 4}$

(4) Find $f \circ f$, $f \circ g$, $g \circ f$, $g \circ g$, $f \circ g(3)$, and $f \circ g(-3)$.

- $f(x) = 3x$, $g(x) = x + 8$.

- $f(x) = 4x - 1$, $g(x) = x^2$

- $f(x) = 2$, $g(x) = 7x$

- $f(x) = 4x$, $g(x) = 3x$

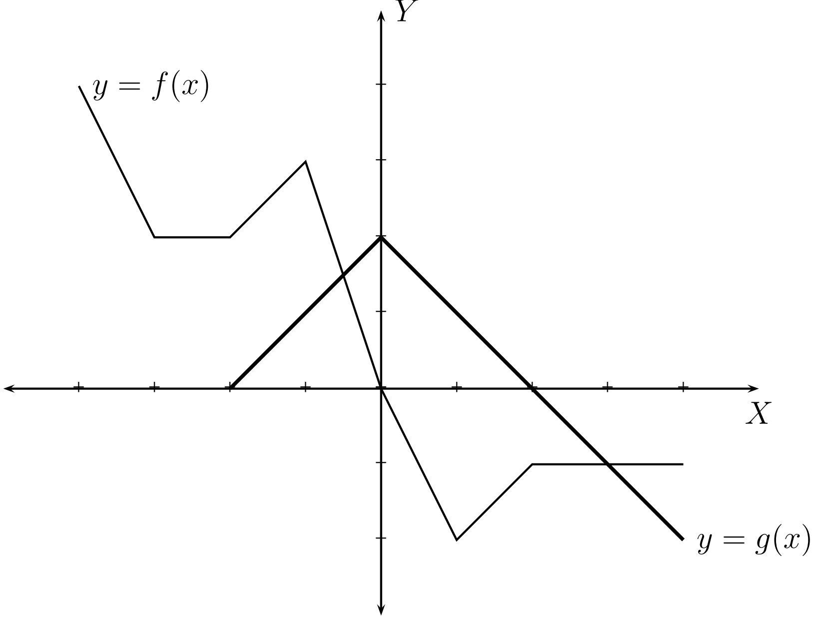
- $f(x) = 4,\ g(x) = \sqrt{x-4}$

- $f(x) = 3x - 1,\ g(x) = 5x^2 + 6x - 8$

- $f(x) = 3 - x,\ g(x) = 5x^2 + 6x - 8$

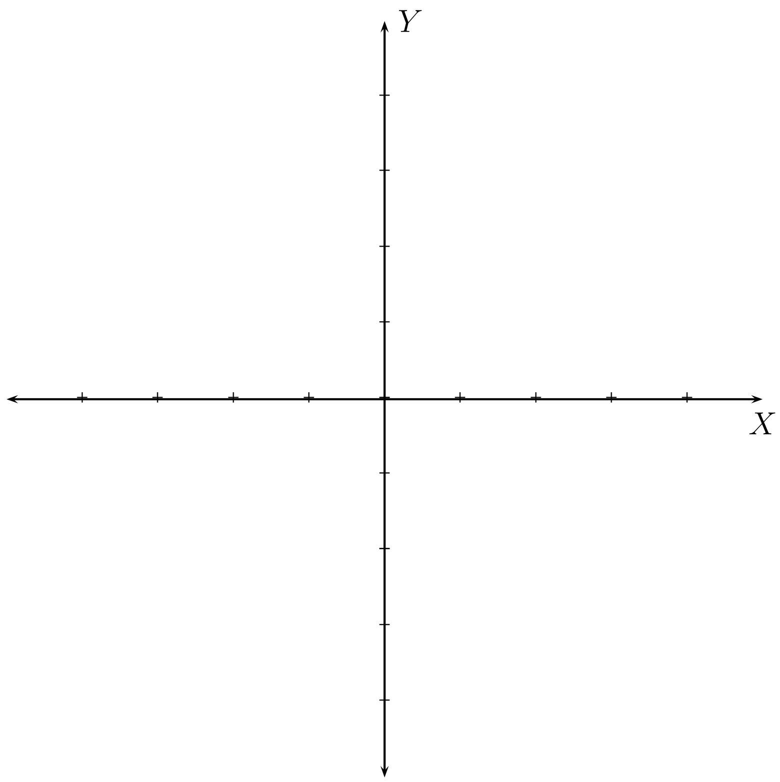
- $f(x) = x - 9,\ g(x) = 5x^2 + 6x - 8$

(5) Use the graphs of f and g and answer the following:



- Find $(f + g)(2)$
- Find $(f - g)(3)$
- Find $(g - f)(-1)$
- Find $(gf)(-1)$
- Find $(gf)(3)$
- Find $\dfrac{g}{f}(-1)$
- Find $\dfrac{g}{f}(3)$
- Find $(g - f)(-3)$

- Find $(g \circ f)(-3)$
- Find $(f \circ g)(-3)$
- Find the domain of $(g + f)$
- Find the domain of $\dfrac{g}{f}$
- Graph $f + g$ and $f - g$ on the same coordinate plane.



5. Inverse functions

(1) When is a functions g said to be the inverse function of f?

The inverse function of f is denoted by f^{-1}. This is not to be confused with $\frac{1}{f}$.

(2) Let $f(x) = x + 3$. What is $f^{-1}(x)$, and what is $\frac{1}{f}(x)$?

(3) For the functions given below, build the inverse **relation**. Then check whether the inverse **relation** is a function. In each of the cases f is a function from the set $\{a, b, c, d\}$ to itself.

f	f^{-1}	f	f^{-1}	f	f^{-1}
$a \mapsto b$		$a \mapsto b$		$a \mapsto a$	
$b \mapsto c$		$b \mapsto c$		$b \mapsto b$	
$c \mapsto d$		$c \mapsto d$		$c \mapsto c$	
$d \mapsto a$		$d \mapsto b$		$d \mapsto d$	

(4) Explain in your own words when does a function have an inverse **function**?

(5) What is a one-to-one function?

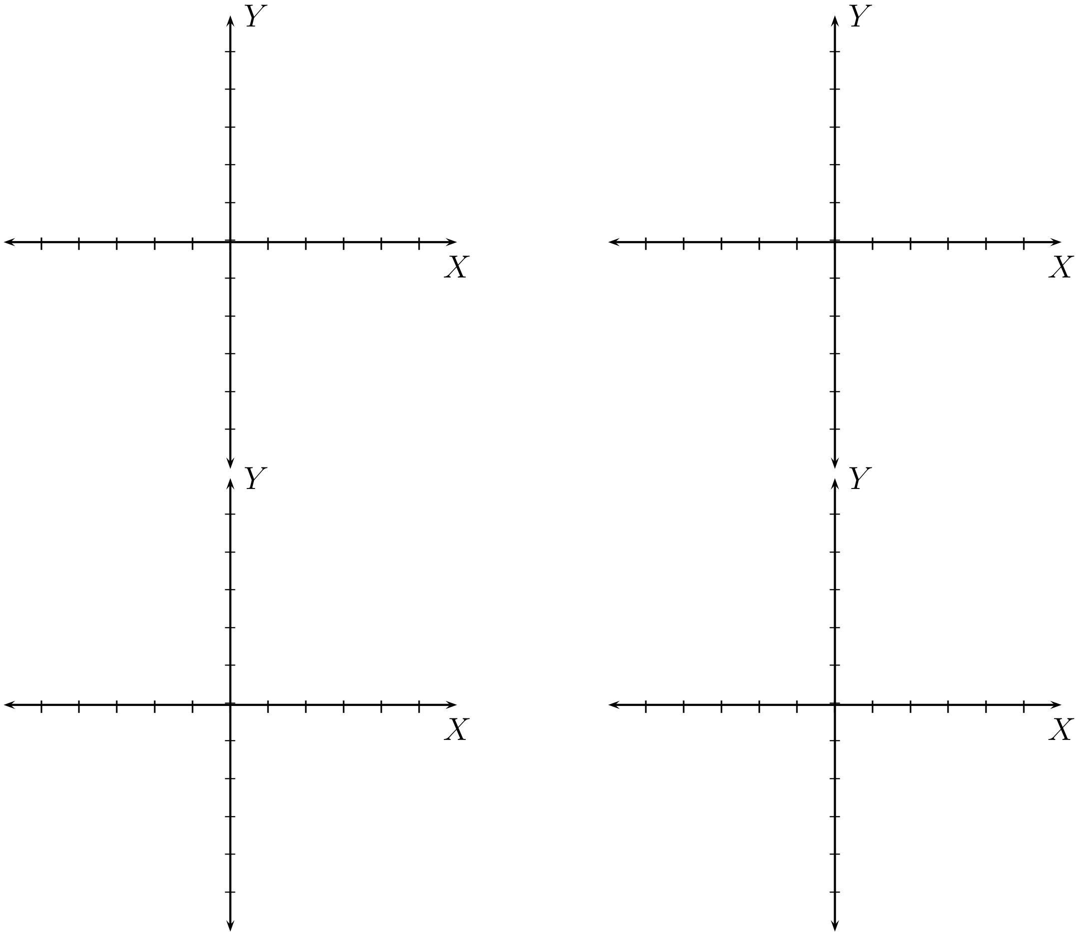
(6) Recall the functions $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, and $f(x) = \dfrac{1}{x}$. Which of these functions are one-to-one, and which are not? Explain.

(7) Recall the graphs of the functions $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, and $f(x) = \dfrac{1}{x}$. What property do the one-to-one functions share?

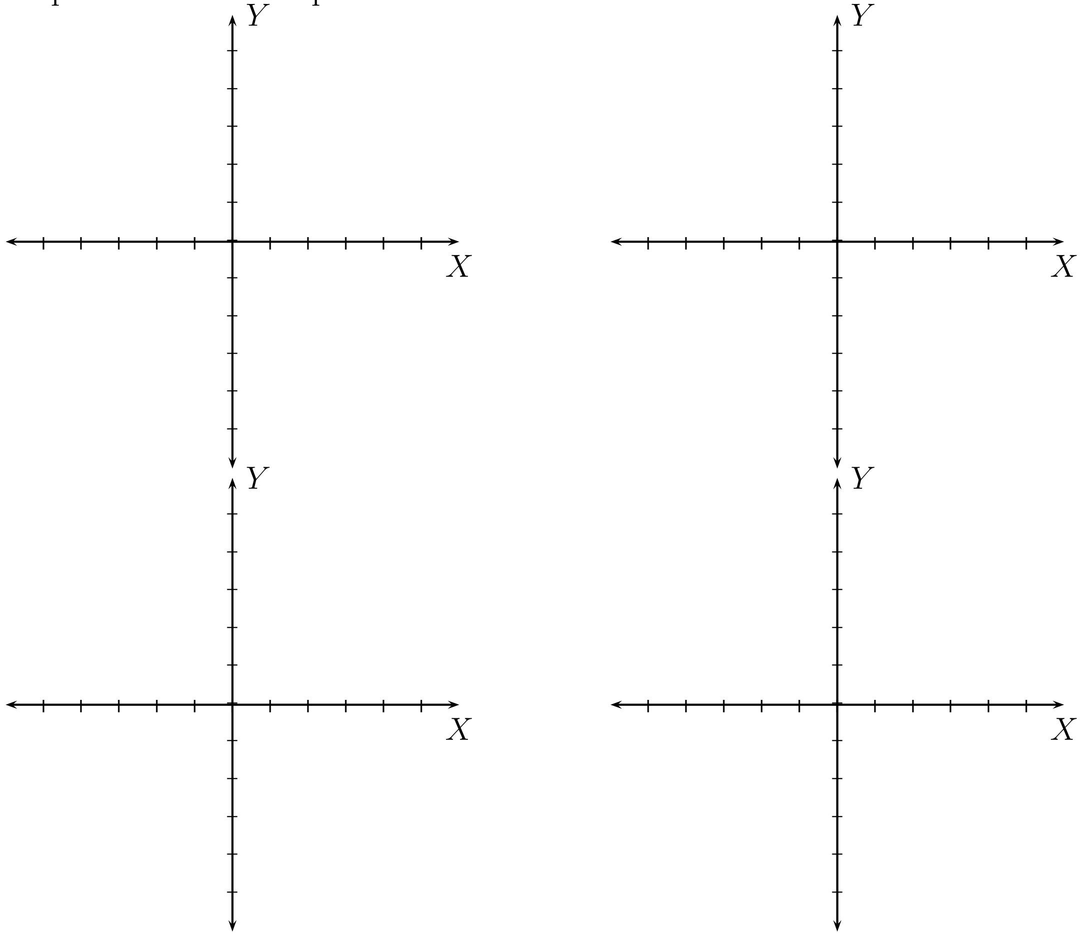
(8) State the Vertical line test for a graph.

(9) What does the Horizontal line test for a graph determine?

(10) Draw graphs of four functions which are not one-to-one, and hence do not have inverse functions.



(11) Draw graphs of four functions which are one-to-one. Draw the graphs of the inverse functions on the same respective coordinate planes.



(12) State the symmetry that you observe between the graph of a function and that of its inverse function.

(13) Determine whether functions f and g are inverses of each other.

- $f(x) = 5x,\ g(x) = \dfrac{x}{5}$

- $f(x) = 5x,\ g(x) = \dfrac{x}{6}$

- $f(x) = x + 2,\ g(x) = x - 2$

- $f(x) = 5x + 4,\ g(x) = \dfrac{x-5}{4}$

- $f(x) = 5x + 4,\ g(x) = \dfrac{x-4}{5}$

- $f(x) = x^3 + 2,\ g(x) = \dfrac{x}{3} - 2$

- $f(x) = x^3 + 2$, $g(x) = \sqrt[3]{x} - 2$

- $f(x) = x^3 + 2$, $g(x) = \sqrt[3]{x - 2}$

(14) Find the inverses of the following functions. Verify that the functions you obtained are indeed the inverses.

- $f(x) = x + 10$

- $f(x) = x - 5$

- $f(x) = 3x$

- $f(x) = \dfrac{x}{5}$

- $f(x) = 3x + 5$

- $f(x) = 4x - 5$

- $f(x) = \dfrac{x}{5} + 3$

- $f(x) = \dfrac{x}{5} - 3$

- $f(x) = x^2$ (There should be two distinct inverses.)

- $f(x) = (x - 3)^2$ (There should be two distinct inverses.)

- $f(x) = x^2 - 3$ (There should be two distinct inverses.)

- $f(x) = x^2 + 3$ (There should be two distinct inverses.)

- $f(x) = x^3$

- $f(x) = (x - 2)^3$

- $f(x) = x^3 + 2$

- $f(x) = \dfrac{1}{x}$ for $x \neq 0$.

- $f(x) = \dfrac{1}{x} + 5$ for $x \neq 0$.

- $f(x) = \dfrac{1}{x+5}$ for $x \neq ?$.

- $f(x) = \sqrt{x}$ for $x \geq 0$.

- $f(x) = \sqrt{x} + 5$ for $x \geq 0$.

- $f(x) = \sqrt{x-5}$ for $x \geq ?$.

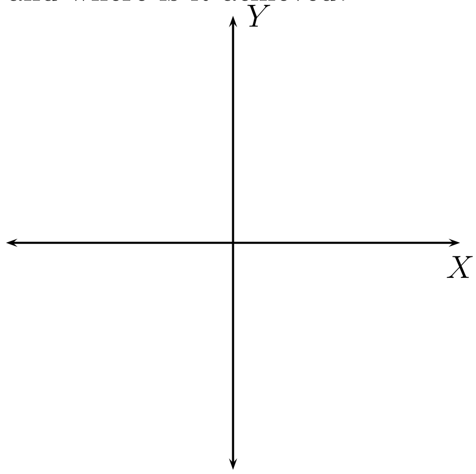
- $f(x) = \sqrt[3]{x}$

- $f(x) = \sqrt[3]{x} - 2$

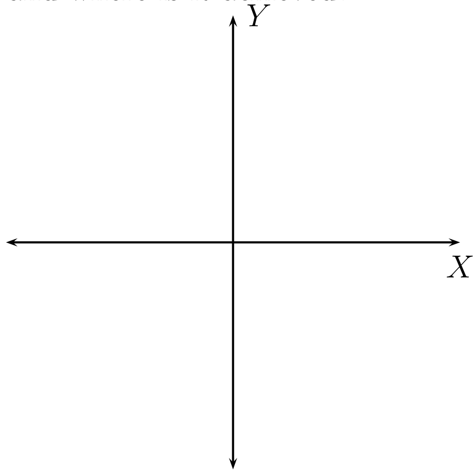
- $f(x) = \sqrt[3]{x - 2}$

6. Quadratic functions

(1) Draw the graph of the quadratic function $f(x) = 2(x-1)^2 + 3$. What is its vertex, axis of symmetry, domain, range? Does the parabola open up or down? Do you get a maximum value or a minimum value? What is the extremal value, and where is it achieved?



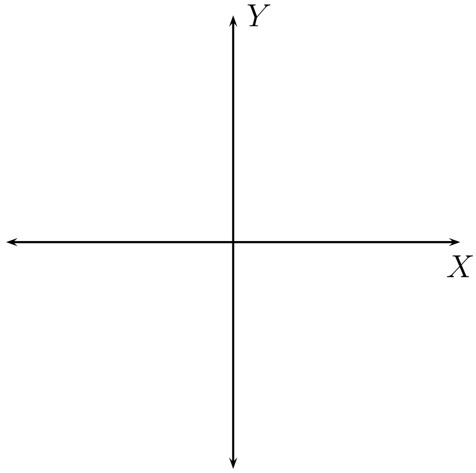
(2) Draw the graph of the quadratic function $f(x) = -2(x+1)^2 + 3$. What is its vertex, axis of symmetry, domain, range? Does the parabola open up or down? Do you get a maximum value or a minimum value? What is the extremal value, and where is it achieved?



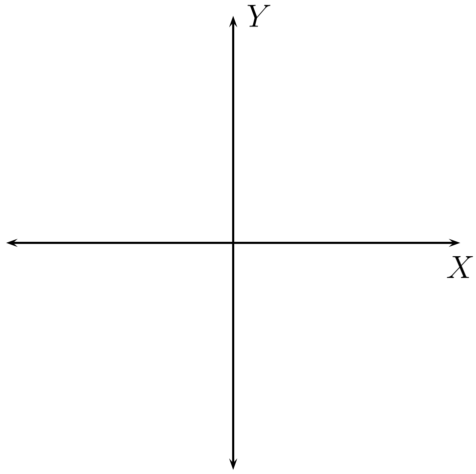
(3) Write down the standard form of a quadratic function. What is its vertex and its axis of symmetry? When does the parabola open up or down? When do you get a maximum value or a minimum value? What is the extremal value, and where is it achieved? What is its domain and range?

(4) Find the coordinates of the vertex, the x-intercepts (if any), the y-intercept, and determine whether the parabola opens up or down. Use this information, and at least two symmetric points on the parabola to sketch the graph. What is the axis of symmetry, domain, and range of the function. Find the minimum or maximum value and determine where it occurs.

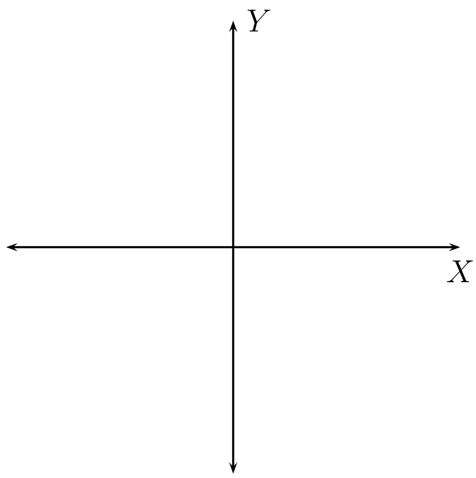
- $f(x) = 3(x-2)^2 + 4$



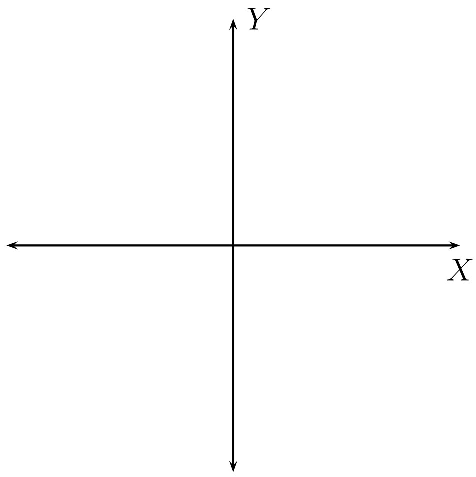
- $f(x) = -3(x-2)^2 - 4$



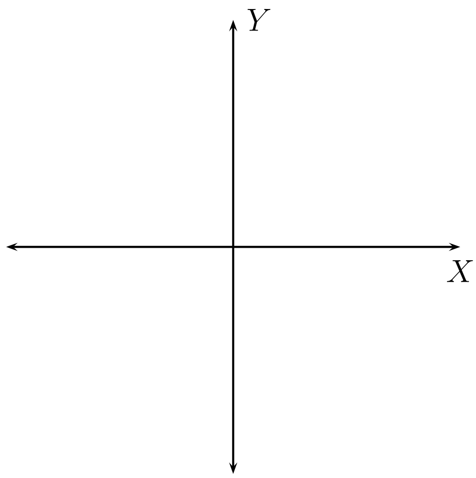
- $f(x) = \frac{1}{2}\left(x - \frac{2}{3}\right)^2 + \frac{4}{5}$



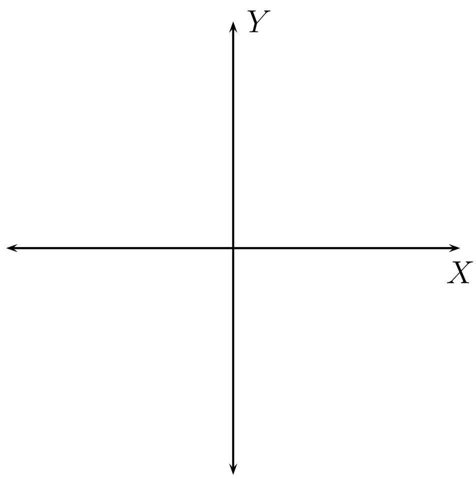
- $f(x) = 2x^2 + 12x + 19$



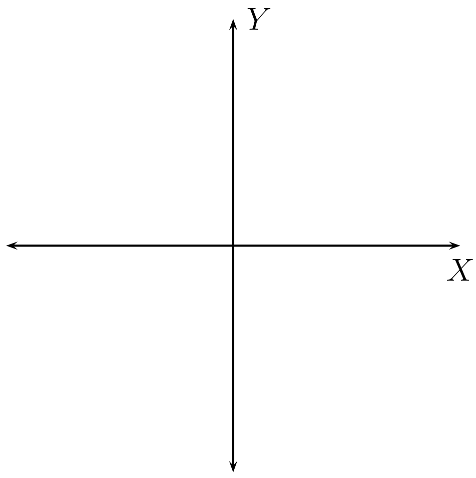
- $f(x) = x^2 + 6x - 10$



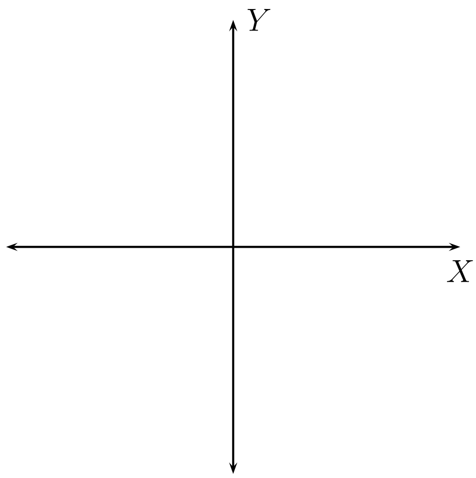
- $f(x) = -x^2 + 6x + 10$



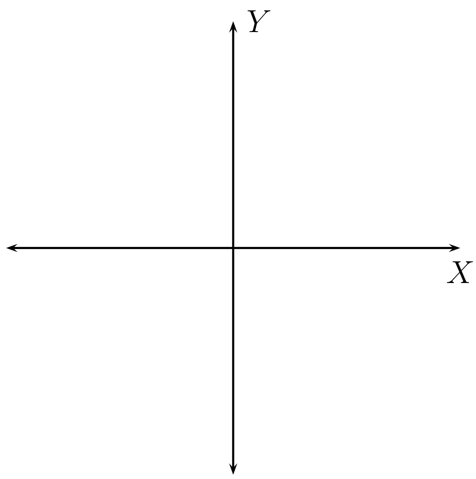
- $f(x) = -3x^2 + 30x - 77$



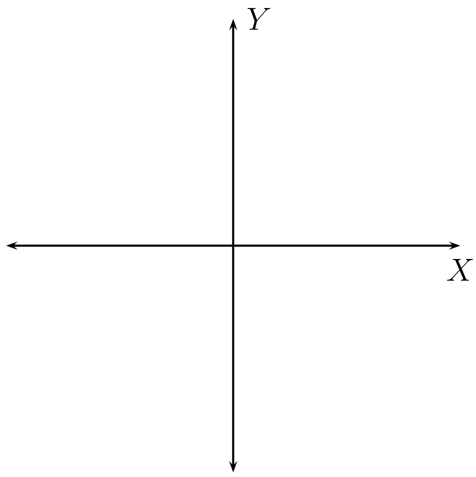
- $f(x) = 3 - (x-2)^2$



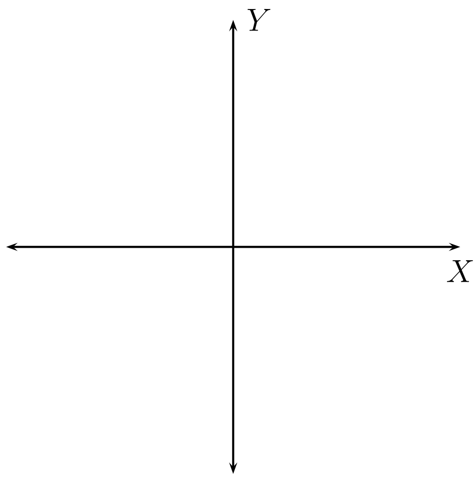
- $f(x) = \dfrac{1}{4} - \left(x + \dfrac{1}{2}\right)^2$



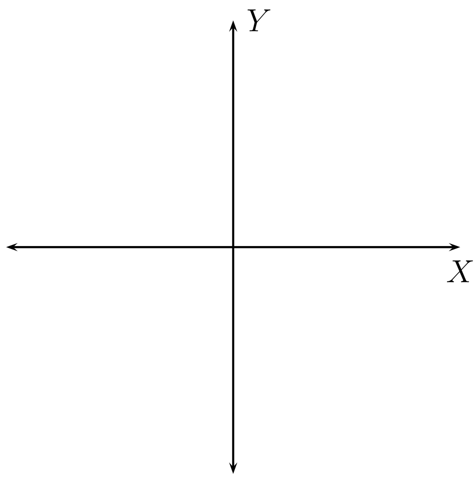
- $y - 2 = (x + 1)^2$



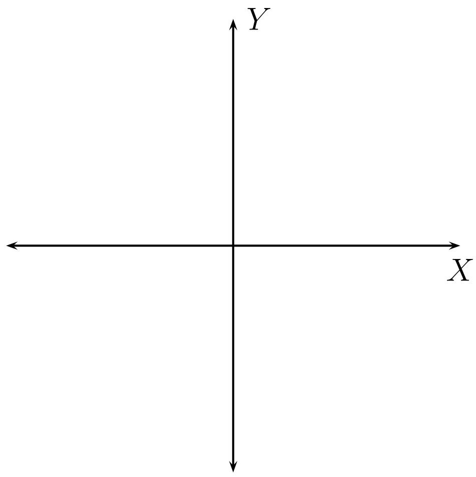
- $y + 1 = (x - 2)^2$



- $f(x) = 5 - 4x + 2x^2$



- $f(x) = -12x - 11 - 3x^2$



(5) Write an equation in standard form of each quadratic function whose parabola has the shape of $2x^2$ or $-3x^2$ with the property

- Vertex is $(1, 2)$, and the parabola opens up.

- Vertex is $\left(-\frac{1}{3}, \frac{2}{3}\right)$, and the parabola opens down.

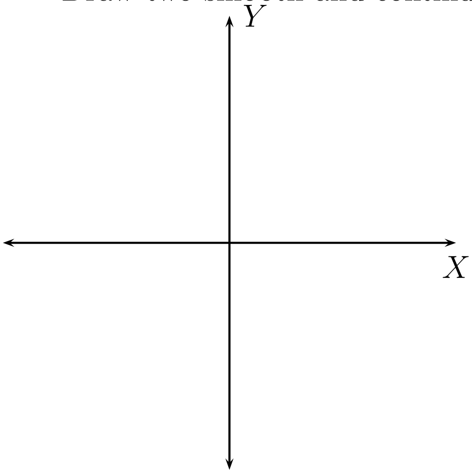
- Maximum value is 3 which occurs at $x = 1$.

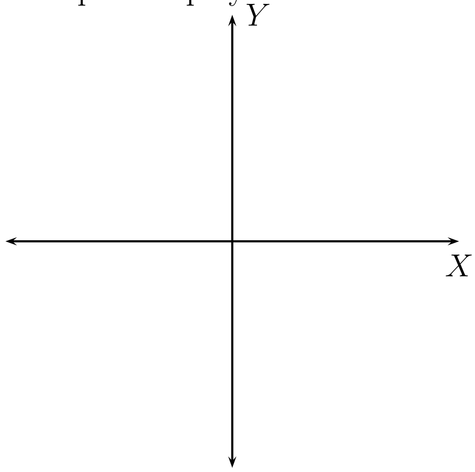
- Minimum value is 4 which occurs at $x = -2$.

7. Polynomial functions and their graphs

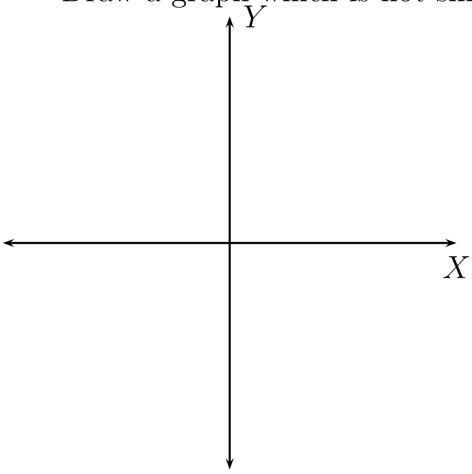
(1) What is a polynomial function? What is its degree, and its leading coefficient?

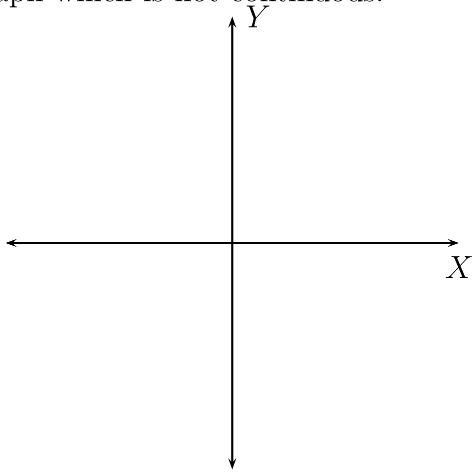
(2) Polynomial functions of degree 2 or higher have graphs that are smooth and continuous. Draw two smooth and continuous graphs which could represent polynomial functions.





Draw a graph which is not smooth, and draw a graph which is not continuous.





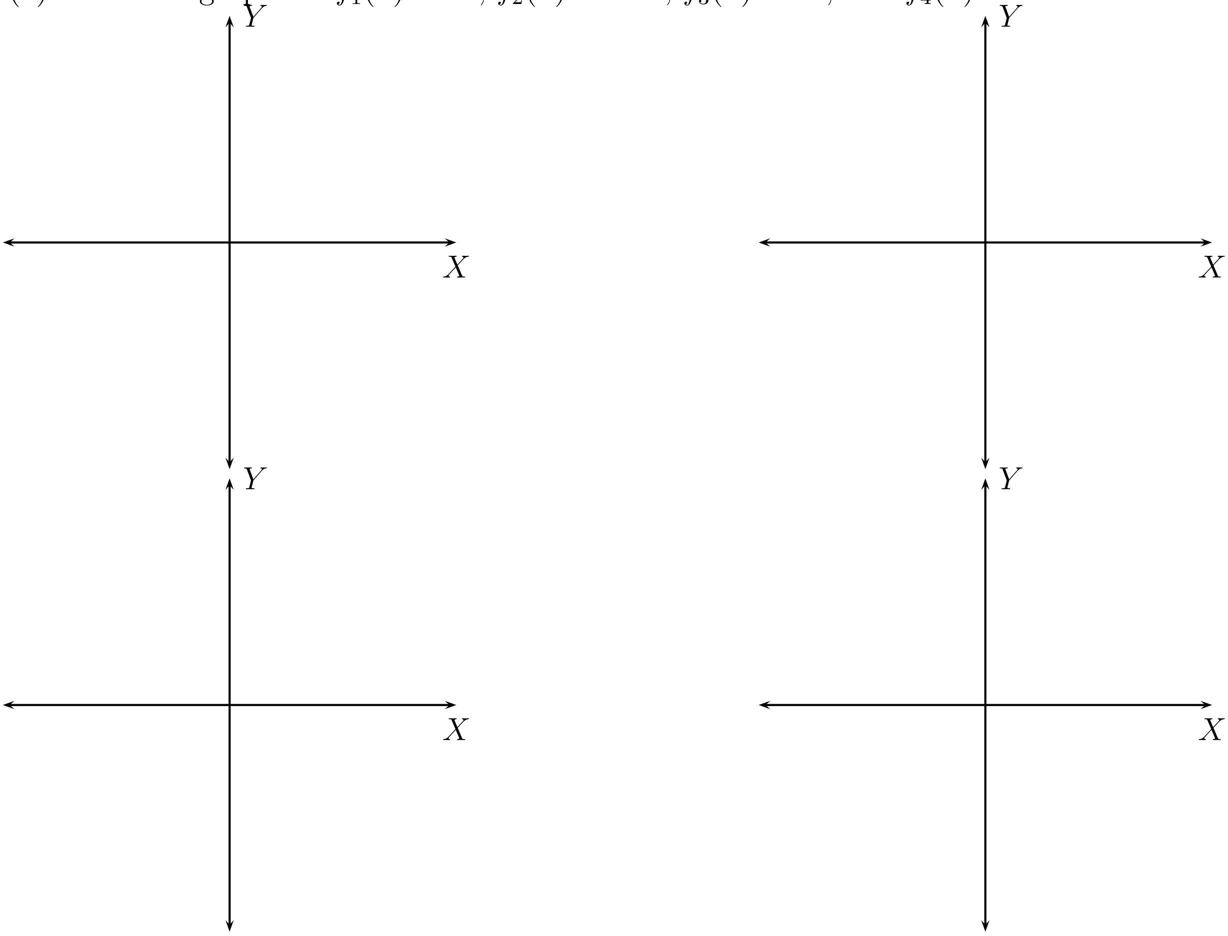
(3) Recall the important functions: $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sqrt{x}$, $f(x) = \sqrt[3]{x}$, $f(x) = |x|$, and $f(x) = \dfrac{1}{x}$. Which of these are polynomial functions? Are their graphs smooth and continuous?

Which of these have discontinuous graphs?

Which of these have non-smooth graphs?

Which of these are not polynomial and yet have continuous and smooth graphs?

(4) Draw the graphs of $f_1(x) = x^2$, $f_2(x) = -x^2$, $f_3(x) = x^3$, and $f_4(x) = -x^3$.

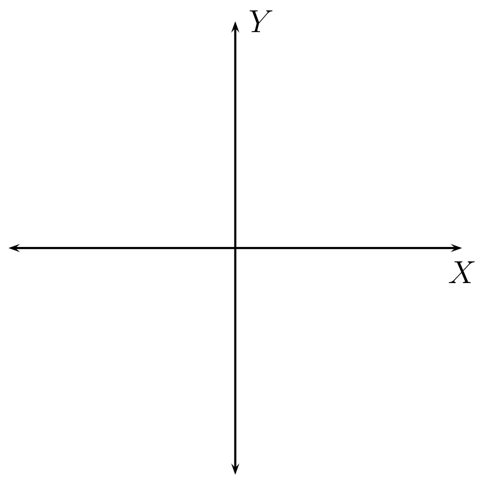


(5) State in your own words, the Leading Coefficient Test for the graph of a polynomial function.

(6) How do you find the y-intercept of the graph of a function? Explain with an example.

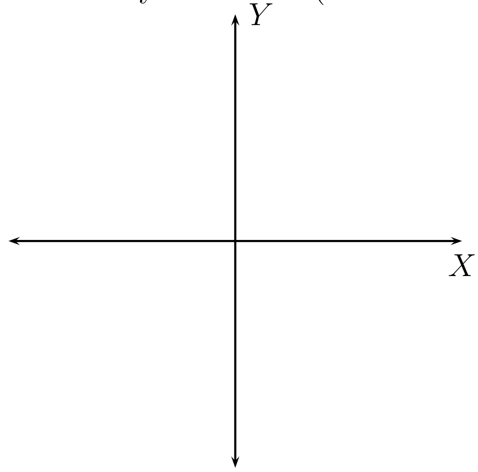
(7) When is a real number r said to be a **zero of a polynomial function?**

(8) What do the zeroes of a function tell us about the graph of the function? Illustrate your answer (use an example).

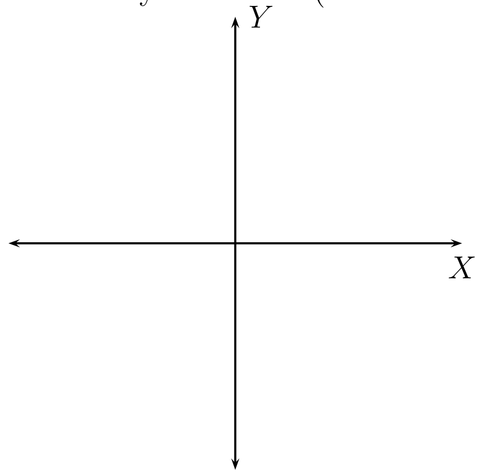


(9) What is the **multiplicity** of a zero of a polynomial function?

(10) Suppose a real number r is a zero of a polynomial function with odd multiplicy. What can you say about the graph of the function at $x = r$? Illustrate your answer (use an example).



(11) Suppose a real number r is a zero of a polynomial function with even multiplicy. What can you say about the graph of the function at $x = r$? Illustrate your answer (use an example).

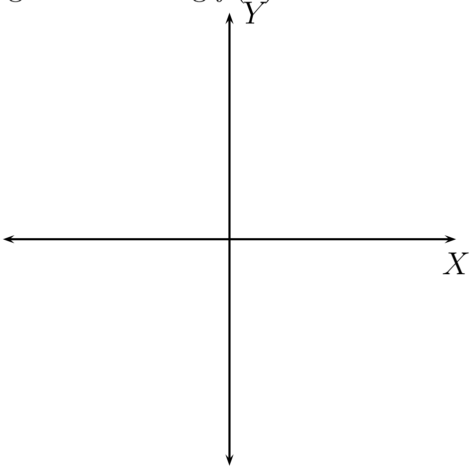


(12) State the Intermediate Value Theorem for polynomial functions.

(13) Let $f(x) = x^3 - 2x^2 - 5x + 6$. Show that $f(0) > 0$ and $f(2) < 0$. What can you conclude using intermediate value theorem?

(14) Write a paragraph on the turning points of polynomial functions.

(15) Using the information above, give a guideline of how you might draw the graph of a polynomial function. Illustrate your guideline using $f(x) = -2x^3 - 2x^2 + 10x - 6 = -2(x-1)^2(x+3)$.



Compare your guideline with that of the textbook (page 310).

(16) Determine which functions are polynomial functions. For those that are, identify the degree:

- $f(x) = x^3 - 4x + 5$
- $f(x) = x^{300} - 4$
- $f(x) = x^3 - 4x + \dfrac{5}{x}$
- $f(x) = x^3 - 4x + 5\sqrt{x}$
- $f(x) = 5$

(17) Use the Leading Coefficient Test to determine the end behaviour of the graph of the polynomial function.

- $f(x) = -4x^7 + 8x^2 - 32$
- $f(x) = -4x^8 + 8x^2 - 32$
- $f(x) = 4x^7 + 8x^2 - 32$
- $f(x) = 4x^8 + 8x^2 - 32$

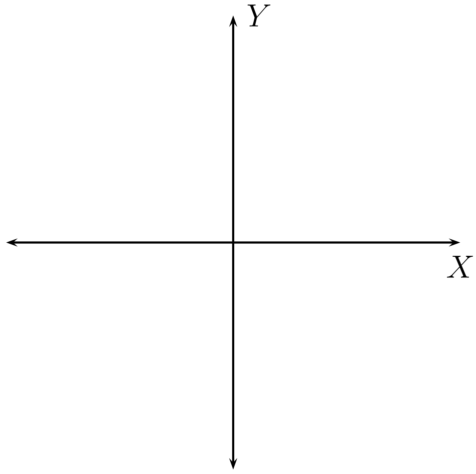
(18) Use the Intermediate Value Theorem to show that each polynomial has a real zero between the given integers.

- $f(x) = 2x^2 + 5x - 3$; between 0 and 1
- $f(x) = 2x^3 + x^2 - 6x$; between 1 and 2

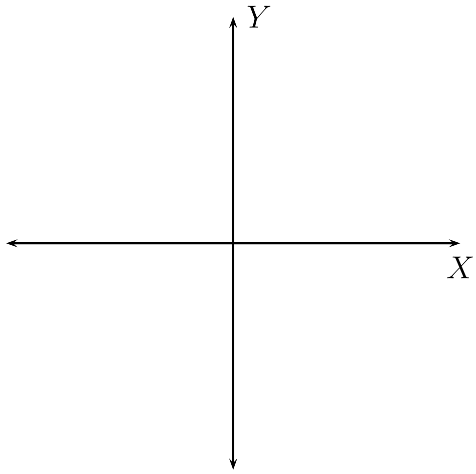
- $f(x) = 2x^3 - 3x^2 - 14x + 15$; between -2 and -3

(19) Graph the following polynomial functions:

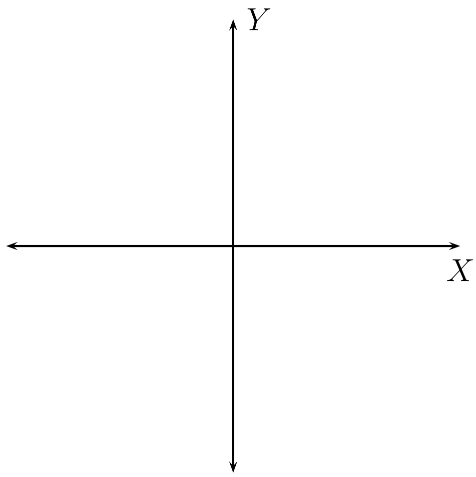
- $f(x) = 3x(x-2)^2(x+3)^3$



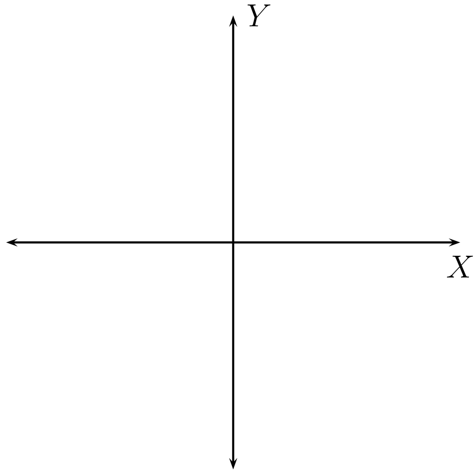
- $f(x) = -2x^2(x+2)^2(x-3)^2$



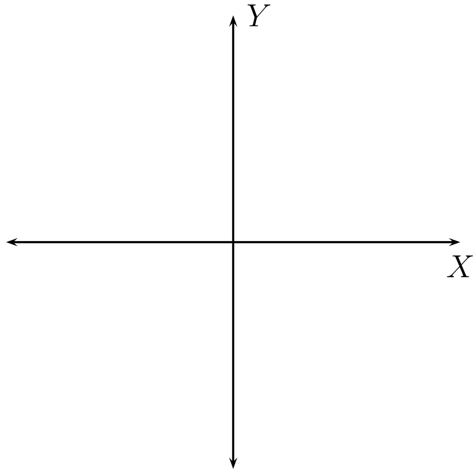
- $f(x) = -2x^2(x+2)^3(x-3)^2$



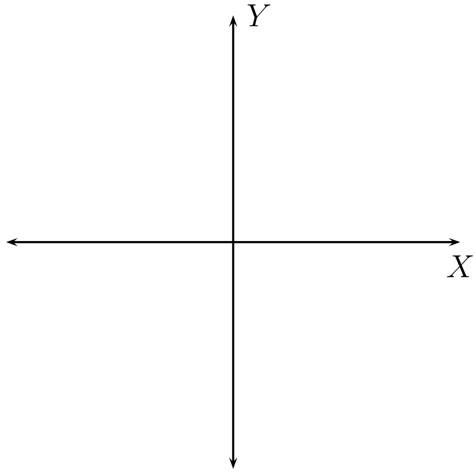
- $f(x) = 3x(x-2)^3(x+3)^3$



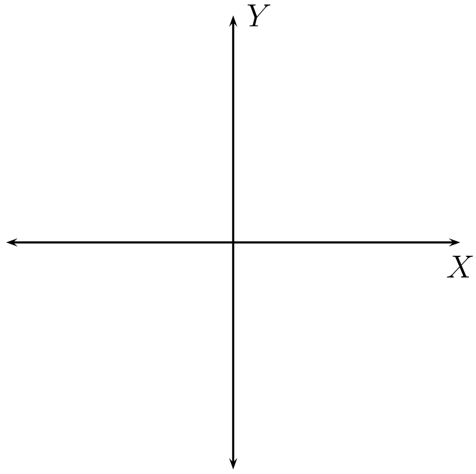
- $f(x) = x^4 - 25x^2$



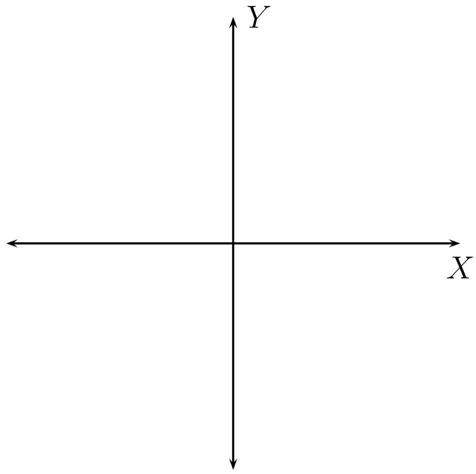
- $f(x) = x^3 + x^2 - 9x - 9$



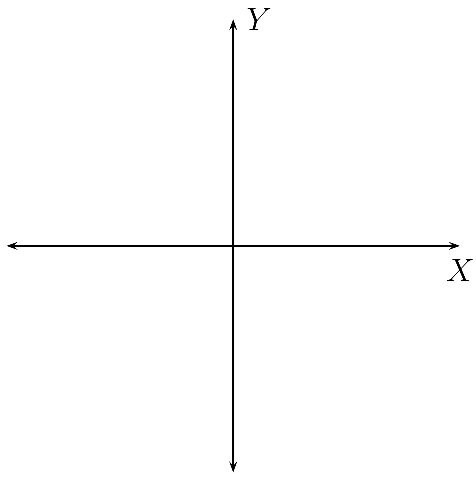
- $f(x) = x^3 - x^2 + 4x - 4$



- $f(x) = -x^3 + 7x^2 - 12x$



- $f(x) = 6x^3 - x^2 - 2x$



8. Dividing polynomials; Remainder and Factor Theorems

(1) Find the quotient and remainder when
- 182 is divided by 4
- 321 is divided by -4
- -223 is divided by 4
- -227 is divided by -4
- In all these cases how does the remainder compare with the divisor? How are the dividend, the remainder and the divisor related?

(2) State the Division Algorithm for polynomials.

(3) What can you say about the remainder when the divisor is $x - c$? Give five examples.

(4) What is the Factor Theorem? Give five examples.

(5) Find the quotient and the remainder, and verify the Division algorithm when

- $x^2 - 4x + 12$ is divided by $x + 4$

- $x^2 - 4x + 12$ is divided by $x - 8$

- $x^4 + x^3 + x^2 + 6x + 16$ is divided by $x^2 - 2x + 3$

- $8x - 6 - 10x^4 + 17x^3 + 6x^5 - 21x^2$ is divided by $3 + 2x^2$

- $-21x^2 - 10x^4 + 8x + 6x^5 + 17x^3 - 6$ is divided by $4x - 3 + 3x^3 - 5x^2$

(6) Use long division as well as synthetic division to find the quotient and remainder, when $f(x)$ is divided by $d(x)$. Use this information and the Remainder theorem to find the value of f at the given number.

- $f(x) = 3x^5 + x^4 - 9x^3 + 11x^2 - 13x + 15$, $d(x) = x - 1$. Find $f(1)$.

- $f(x) = 4x - 23 - 8x^2 + 4x^4 + x^3$, $d(x) = x + 2$. Find $f(-2)$.

- $f(x) = -18x^2 + 12x + 8x^4 + 2x^3 - 20$, $d(x) = 2x + 3$. Find $f\left(-\frac{3}{2}\right)$.

9. Zeroes of polynomial functions

(1) What is the Rational Zero Theorem? Explain using five examples.

(2) Complete the following sentence: If we count the multiple roots of a polynomial equation of degree n, $n \geq 1$, separately, then the equation has ____________ roots.
Explain using these examples:
- $x^2 + 12x + 36 = 0$
- $x^2 + 12x + 20 = 0$
- $x^4 + 10x^3 + 21x^2 - 40x - 100 = (x + 2)(x - 2)(x + 5)^2 = 0$

(3) Complete the following sentence: If $a + bi$ (for $b \neq 0$) is a root of a polynomial equation with real coefficients then ____________ is also a root.
Explain using these examples:
- $x^2 + 25 = 0$
- $x^2 + 4x + 13 = 0$
- $x^3 - 9x^2 + 31x - 39 = (x - 3)(x^2 - 6x + 13) = 0$

(4) What is Descarte's rule of signs? Explain using five examples.

(5) List all possible rational zeroes. Use synthetic division to test the possible rational zeroes to find one zero. Find all the remaining zeroes.

- $f(x) = x^3 - 3x^2 - 10x + 24$

- $f(x) = x^4 - 5x^2 + 4$

- $f(x) = x^4 - 2x^3 - 21x^2 + 22x + 40$

- $f(x) = 2x^3 + 9x^2 + 7x - 6$

(6) List all possible rational roots. Use synthetic division to test the possible rational roots to find one root. Find all the remaining roots.

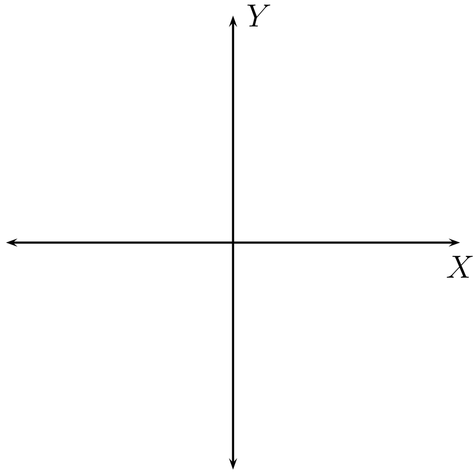
- $x^3 + 3x^2 - 10x - 24 = 0$

- $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$

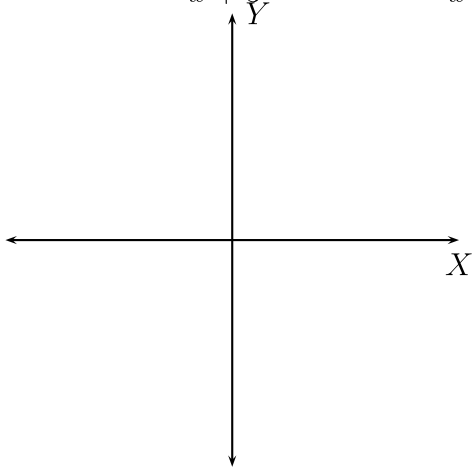
- $2x^3 + 9x^2 - 2x - 24 = 0$

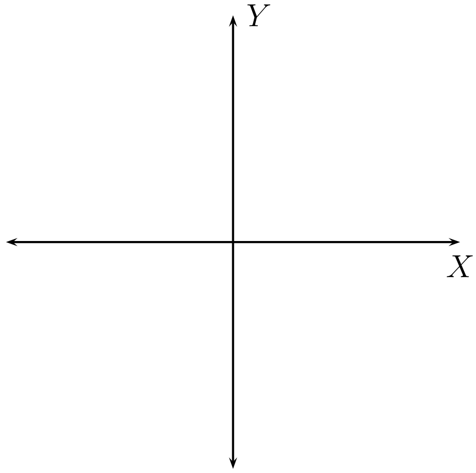
10. Rational functions and their graphs

(1) Recall the graph of $f(x) = \dfrac{1}{x}$

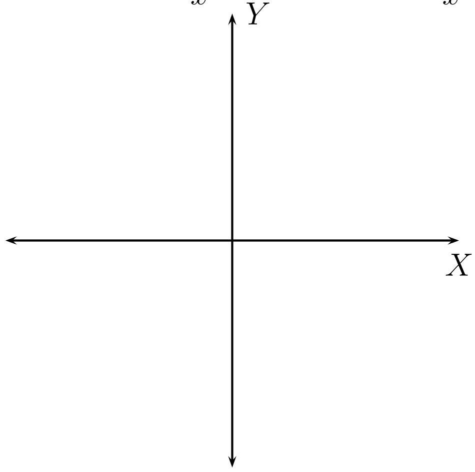


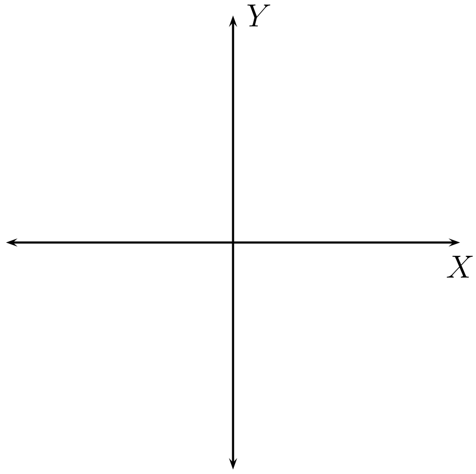
(2) Graph $g(x) = \dfrac{1}{x+3}$, and $h(x) = \dfrac{1}{x-3}$.



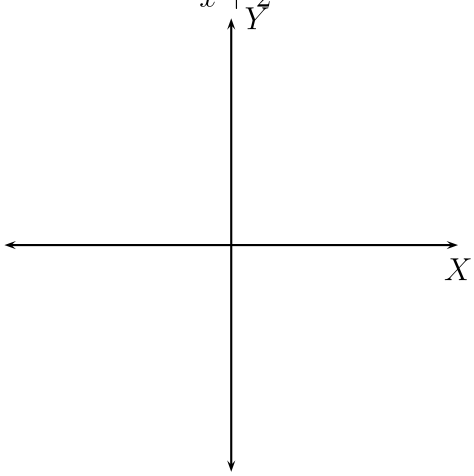


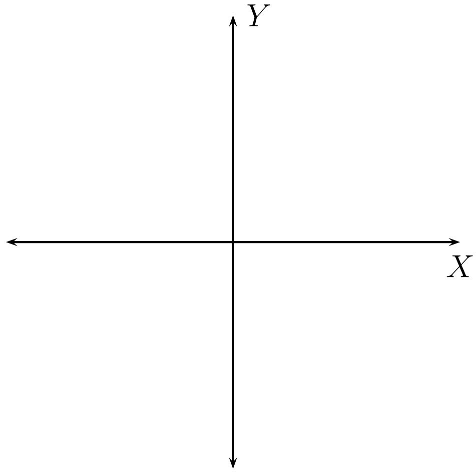
(3) Graph $k(x) = \dfrac{1}{x} + 3$, and $l(x) = \dfrac{1}{x} - 3$.



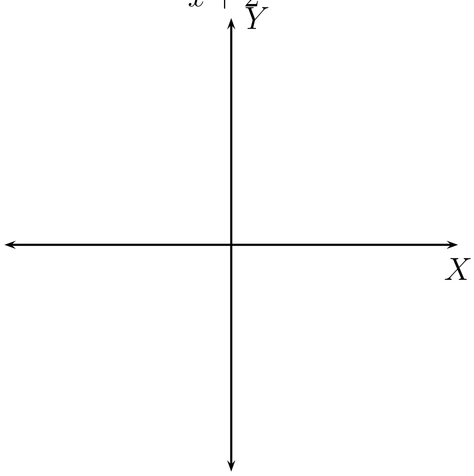


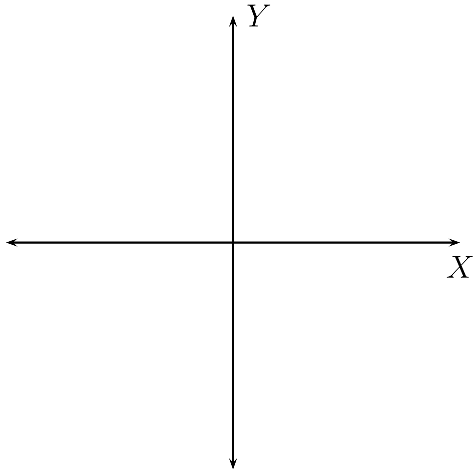
(4) Graph $m(x) = \dfrac{1}{x+2} + 3$, and $n(x) = \dfrac{1}{x-2} + 3$.



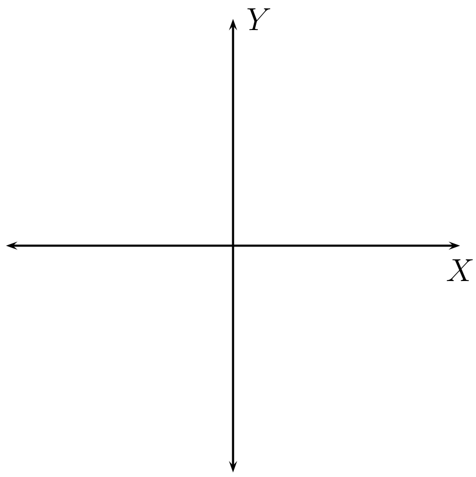


(5) Graph $p(x) = \dfrac{1}{x+2} - 3$, and $q(x) = \dfrac{1}{x-2} - 3$.

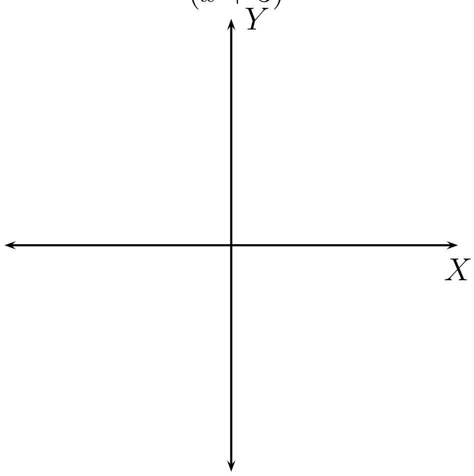


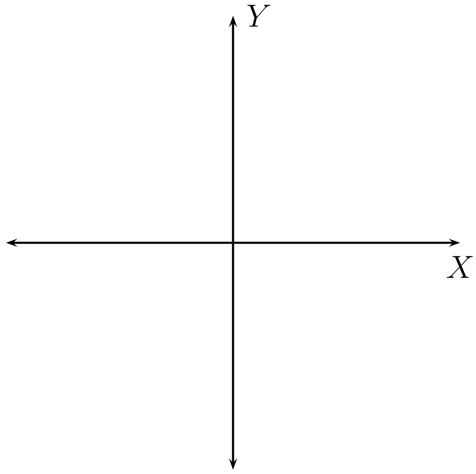


(6) Graph $f(x) = \dfrac{1}{x^2}$

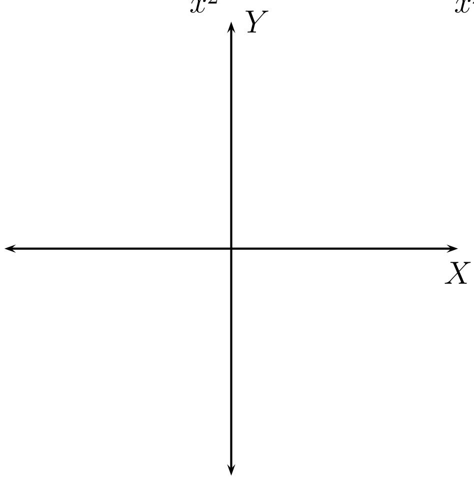


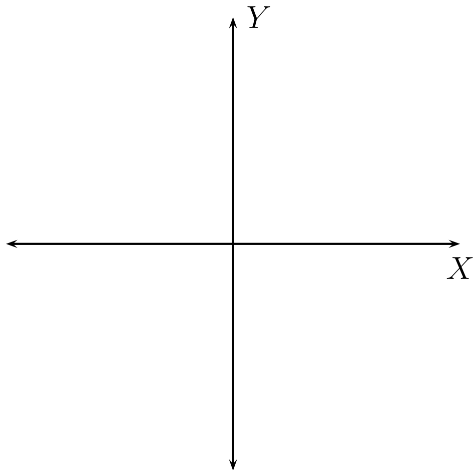
(7) Graph $g(x) = \dfrac{1}{(x+3)^2}$, and $h(x) = \dfrac{1}{(x-3)^2}$.



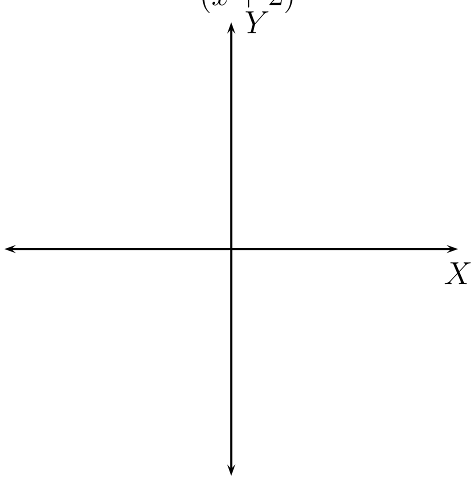


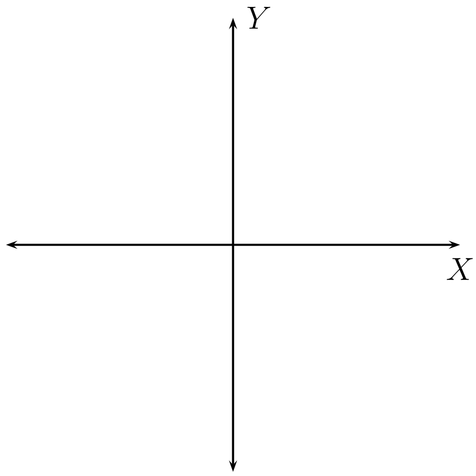
(8) Graph $k(x) = \dfrac{1}{x^2} + 3$, and $l(x) = \dfrac{1}{x^2} - 3$.



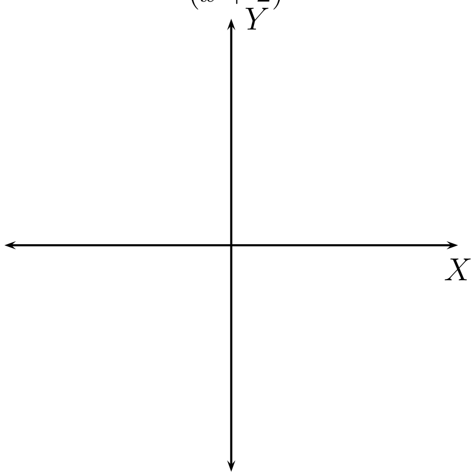


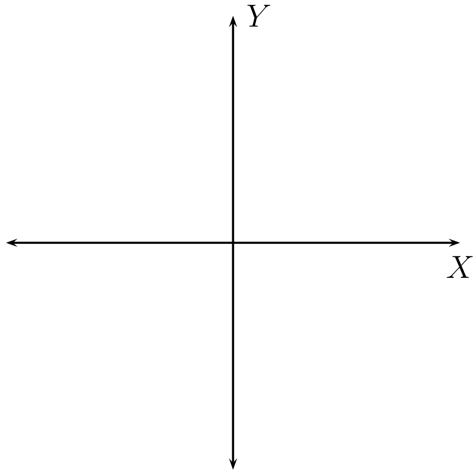
(9) Graph $m(x) = \dfrac{1}{(x+2)^2} + 3$, and $n(x) = \dfrac{1}{(x-2)^2} + 3$.





(10) Graph $p(x) = \dfrac{1}{(x+2)^2} - 3$, and $q(x) = \dfrac{1}{(x-2)^2} - 3$.





(11) What is a rational function? Give five examples.

(12) Using five examples, explain how you would find the domain of a rational function.

(13) What is the domain of

- $\dfrac{x-2}{x^2+5x+6}$

- $\dfrac{x+2}{x^2+5x+6}$

(14) How do you determine whether the graph of function f has symmetry about the y-axis? Give an example.

(15) How do you determine whether the graph of function f has symmetry around the origin? Give an example.

(16) Can the graph of function f have symmetry about the x-axis? Explain your answer.

(17) Using five examples, explain how you would find the x-intercepts of the graph of a rational function.

(18) Using five examples, explain how you would find the y-intercept of the graph of a rational function.

(19) Can a graph of a rational function have more than one x-intercepts? If yes, then give an example. If no, then explain why not.

(20) Can the graph of a rational function have more than one y-intercept? If yes, then give an example. If no, then explain why not.

(21) What is the equation of a vertical line passing through $(-2, 3)$?

(22) What is the equation of a horizontal line passing through $(-2, 3)$?

(23) Using five examples, explain what a vertical asymptote for the graph of a rational function is.

(24) Using five examples, explain what a horizontal asymptote for the graph of a rational function is.

(25) Give five examples of rational functions whose graphs have horizontal asymptote $y = 1$.

(26) Give five examples of rational functions whose graphs have horizontal asymptote $y = 0$.

(27) Give five examples of rational functions whose graphs have horizontal asymptote $y = -2$.

(28) Give five examples of rational functions whose graphs have horizontal asymptote $y = \frac{1}{2}$.

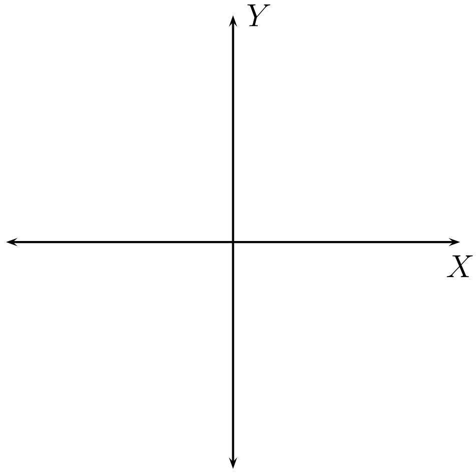
(29) Give five examples of rational functions whose graphs have no horizontal asymptotes.

(30) Give a guideline to draw the graph of a rational function.

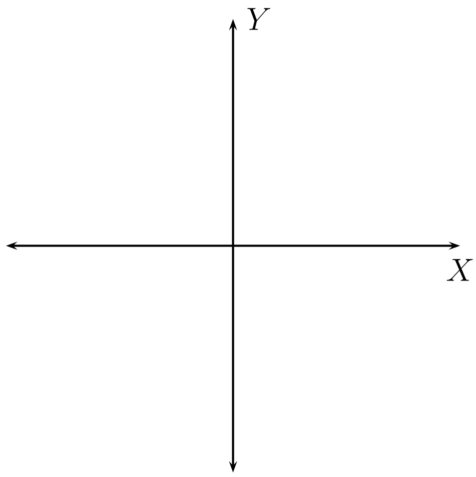
Compare your guideline with the one given in the textbook (page 347).

(31) Graph the following rational functions:

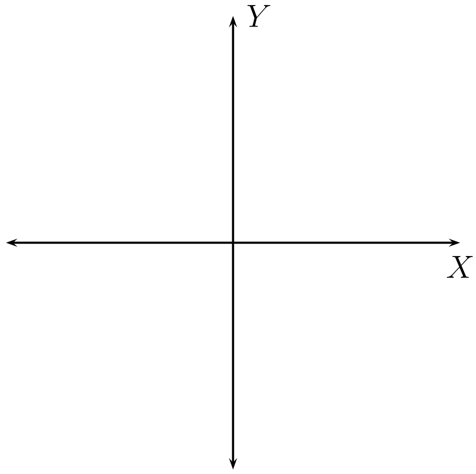
$$f(x) = \frac{x+2}{x^2+4x+3}$$



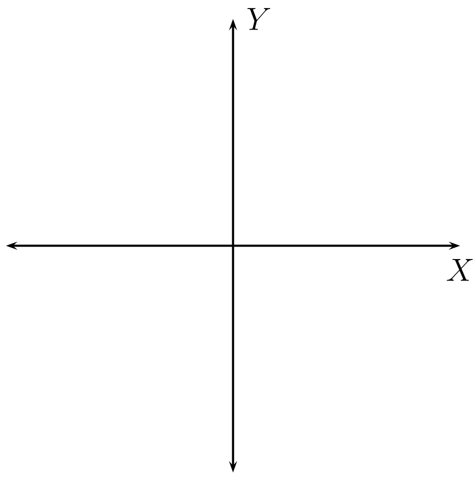
$$f(x) = \frac{2x^2}{x^2-4}$$



$$f(x) = \frac{-x+2}{x^2-3x-18}$$



$$f(x) = \frac{2}{x^2-1}$$



11. Polynomial and rational inequalities

Solve each inequality and graph the solution set on the real number line. Expres the solution using interval notation.

(1) $x + 5 \geq 0$

(2) $x - 3 < 2$

(3) $(x - 2)(x + 5) > 0$

(4) $(x + 5)(x - 3) \leq 0$

(5) $x^2 + 8x + 7 \geq 0$

(6) $x^2 - 6x - 40 < 0$

(7) $2x^2 < x + 6$

(8) $3x^2 > 5x$

(9) $(x + 1)(x - 3)(x + 2) \geq 0$

(10) $x^3 + x^2 - 4x - 4 < 0$

(11) $6x^2 - 5 > 13x$

(12) $\dfrac{x+2}{x-3} < 0$

(13) $\dfrac{x-4}{x+3} \geq 0$

(14) $\dfrac{-x+2}{x+5} < 0$

(15) $\dfrac{x-1}{x+4} < 3$

(16) $\dfrac{x}{x-3} > 2$

(17) $\dfrac{x+1}{3x+1} \leq 1$

(18) $x + 3 \geq x + 1$

(19) $x + 3 \leq x + 1$

(20) $\dfrac{1}{x^2 + 4} \geq 0$

(21) $\dfrac{1}{x^2 + 4} \leq 0$

12. Exponential functions

The Exponential function:

$$f(x) = b^x \textit{ for } b > 0 \textit{ and } b \neq 1.$$

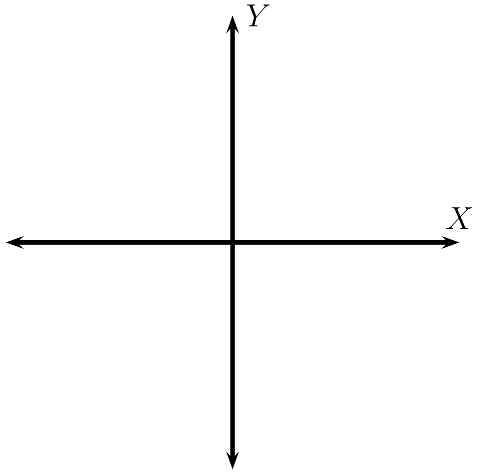
Here, b is the **base** of the exponential function.
What happens when $b = 1$? What is its graph?

What happens when $b < 0$?

What happens when $b = 0$?

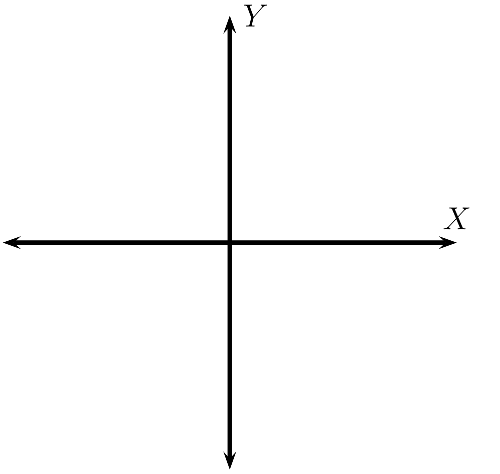
In what follows, whenever a problem asks you to graph any functions, provide at least 5 points. State its X or Y intercepts if any. Give its domain and range. Sketch the asymptotes as dotted lines, and give equations of the asymptotes.

(1) Graph $f(x) = 2^x$



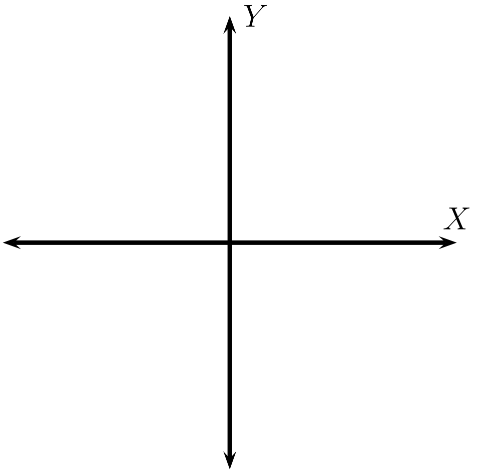
x	$y = 2^x$
-2	
-1	
0	
1	
2	

(2) Graph $f(x) = 3^x$



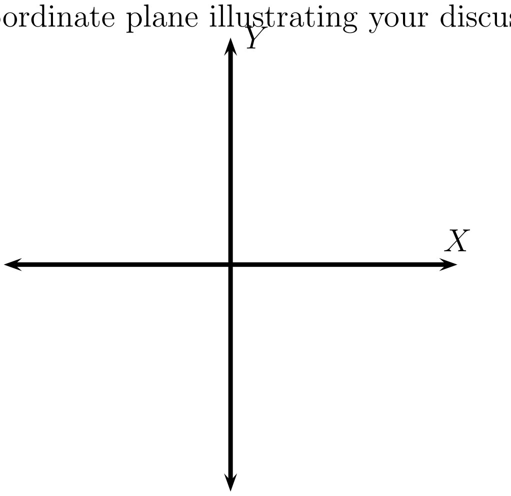
x	$y = 3^x$
-2	
-1	
0	
1	
2	

(3) Graph $f(x) = 5^x$

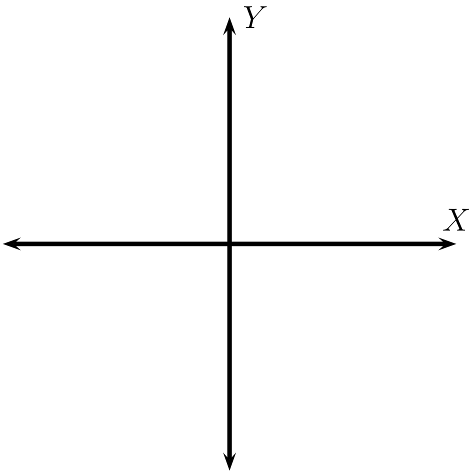


x	$y =$

(4) Discuss the graphs of $f(x) = b^x$ and $g(x) = c^x$ when $b > c > 1$. Your discussion should include concepts of *steepness*, and *horizonal asymptotes*. Draw two graphs on the same coordinate plane illustrating your discussion.



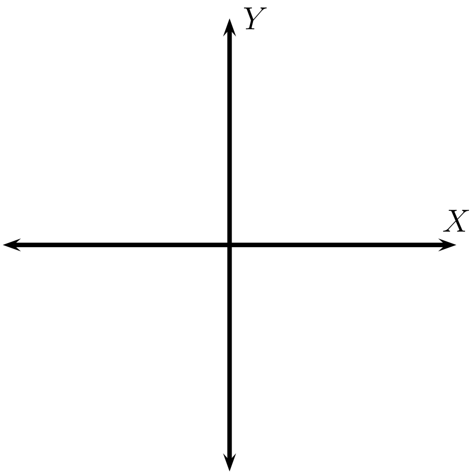
(5) Graph $f(x) = \left(\frac{1}{2}\right)^x$



x	$y =$

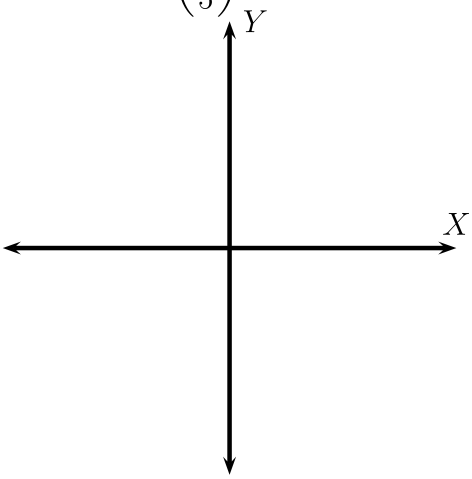
How does this graph compare with the graph of $y = 2^x$? Explain. What do you think is the graph of the function $g(x) = 2^{-x}$?

(6) Graph $f(x) = \left(\frac{1}{3}\right)^x$



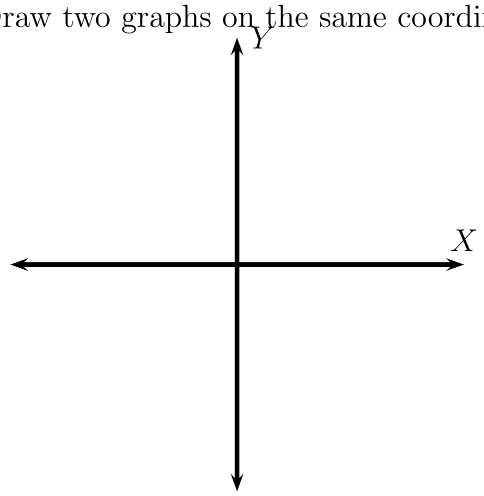
How does this graph compare with the graph of $y = 3^x$? Explain. What do you think is the graph of the function $g(x) = 3^{-x}$?

(7) Graph $f(x) = \left(\frac{1}{5}\right)^x$

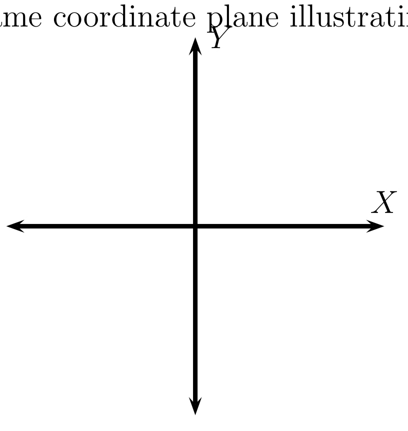


How does this graph compare with the graph of $y = 5^x$? Explain. What do you think is the graph of the function $g(x) = 5^{-x}$?

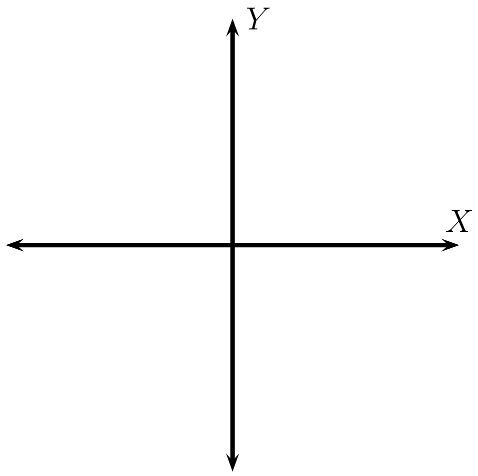
(8) Discuss the graphs of $f(x) = b^x$ and $g(x) = b^{-x}$ when $b > 1$. Your discussion should include concepts of *increasing/decreasing*, *horizontal asymptotes*, and *symmetry of the two graphs.* Draw two graphs on the same coordinate plane illustrating your discussion.



(9) Discuss the graphs of $f(x) = b^x$ and $g(x) = c^x$ when $0 < b < c < 1$. Your discussion should include concepts of *steepness*, and *horizonal asymptotes.* Draw two graphs on the same coordinate plane illustrating your discussion.

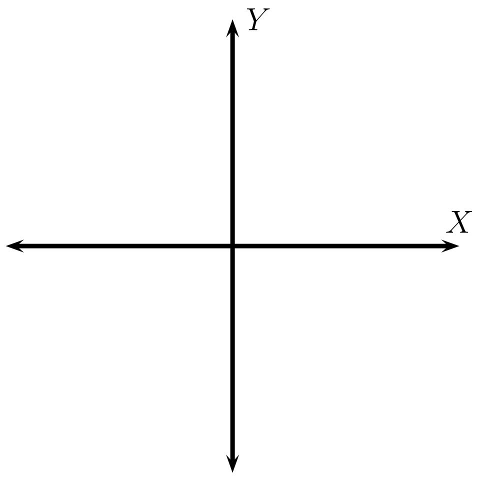


(10) Graph $f(x) = 2^x + 1$



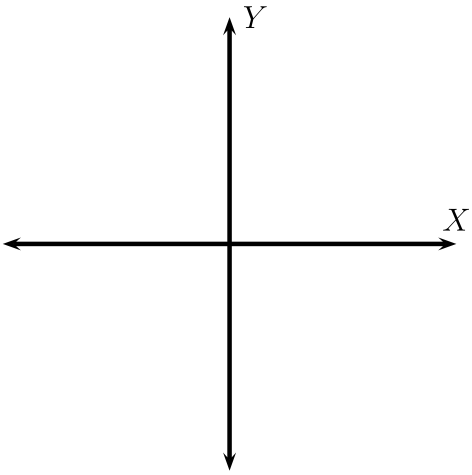
How does this graph compare with the graph of $y = 2^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(11) Graph $f(x) = 3^x - 2$



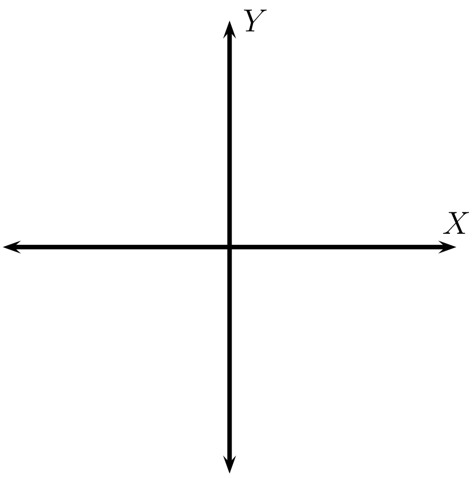
How does this graph compare with the graph of $y = 3^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(12) Graph $f(x) = \left(\frac{1}{3}\right)^x - 2$



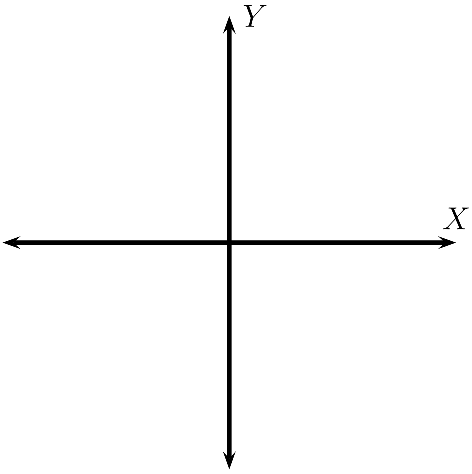
How does this graph compare with the graph of $y = \left(\frac{1}{3}\right)^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(13) Graph $f(x) = \left(\frac{1}{5}\right)^x + 3$



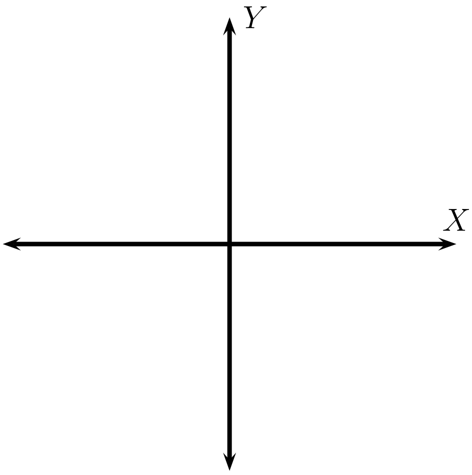
How does this graph compare with the graph of $y = \left(\frac{1}{5}\right)^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(14) Graph $f(x) = 2^{x+1}$



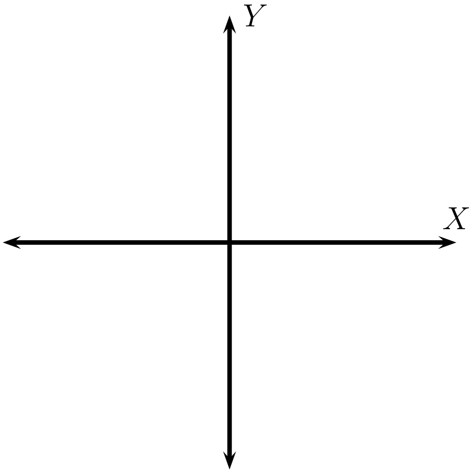
How does this graph compare with the graph of $y = 2^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(15) Graph $f(x) = 3^{x-2}$



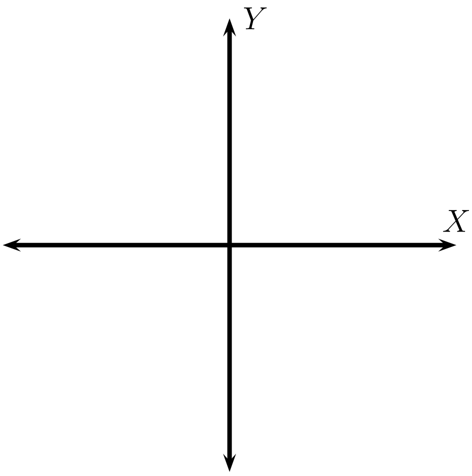
How does this graph compare with the graph of $y = 3^x$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(16) Graph $f(x) = \left(\frac{1}{3}\right)^{x+2}$



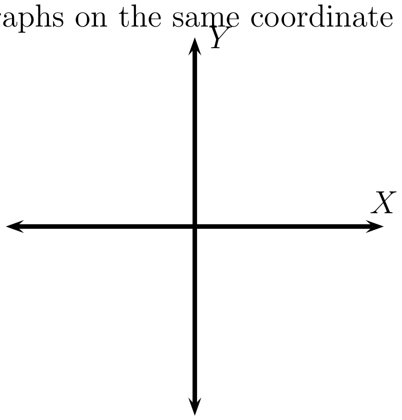
How does this graph compare with the graph of $y = \left(\frac{1}{3}\right)^{x}$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

(17) Graph $f(x) = \left(\frac{1}{5}\right)^{x-3}$

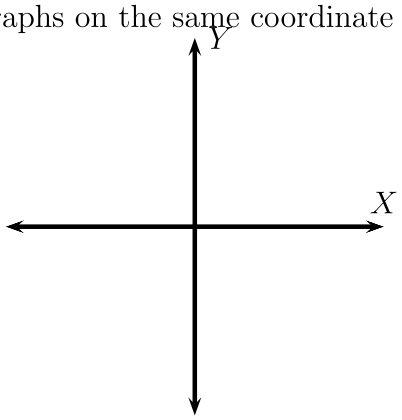


How does this graph compare with the graph of $y = \left(\frac{1}{5}\right)^{x}$? Explain. Your discussion should include the effect on the *horizontal asymptote*.

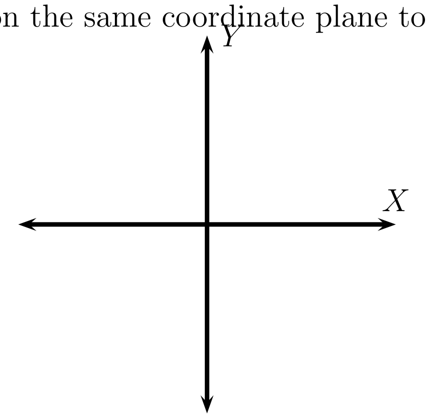
(18) Let $c > 0$. Explain how the graph $y = b^x - c$ is related to the graph $y = b^x$. Draw two graphs on the same coordinate plane to illustrate your answer.



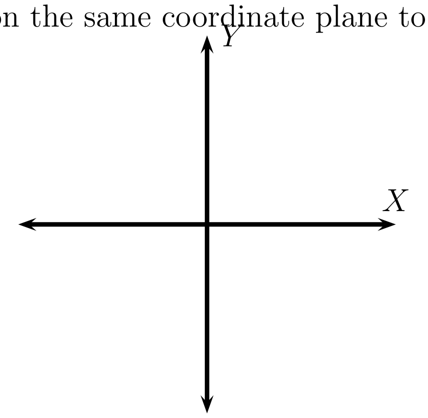
(19) Let $c > 0$. Explain how the graph $y = b^x + c$ is related to the graph $y = b^x$. Draw two graphs on the same coordinate plane to illustrate your answer.



(20) Let $c > 0$. Explain how the graph $y = b^{x+c}$ is related to the graph $y = b^x$. Draw two graphs on the same coordinate plane to illustrate your answer.

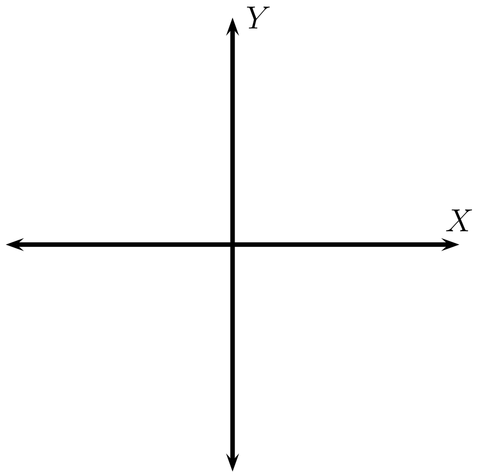


(21) Let $c > 0$. Explain how the graph $y = b^{x-c}$ is related to the graph $y = b^x$. Draw two graphs on the same coordinate plane to illustrate your answer.

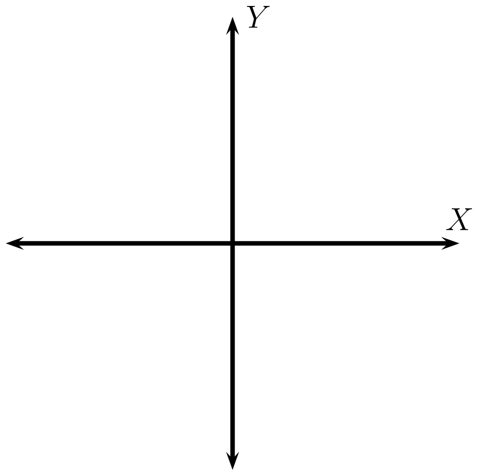


The **Euler number** e is defined as the limit of $\left(1+\dfrac{1}{n}\right)^n$ as n approaches ∞. We will see the significance of e in a course on Calculus. An approximate value of e is 2.718281827.

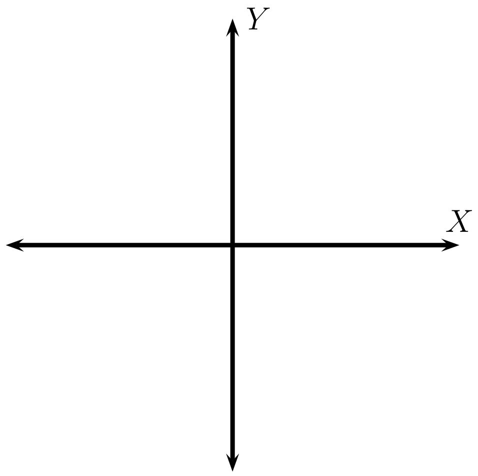
(22) Graph $f(x) = e^x$



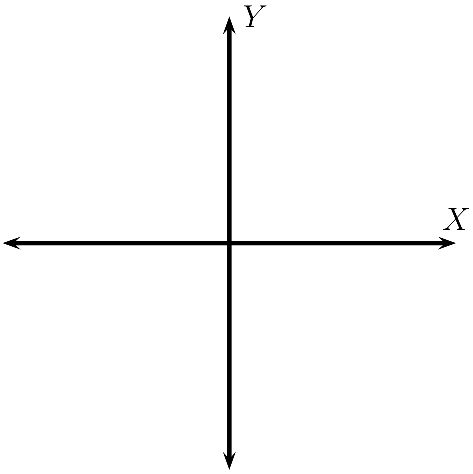
(23) Graph $f(x) = -e^x$



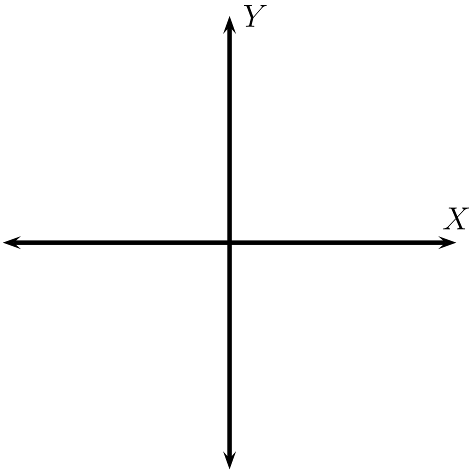
(24) Graph $f(x) = e^{-x}$



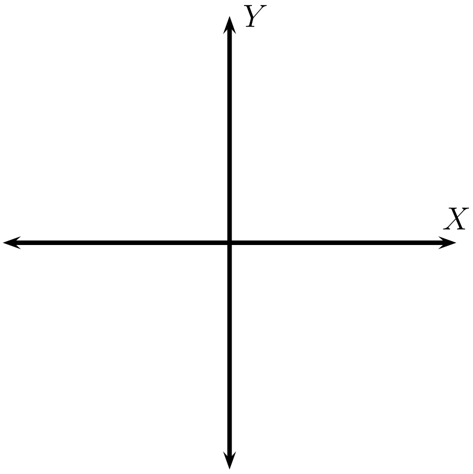
(25) Graph $f(x) = e^x + 2$



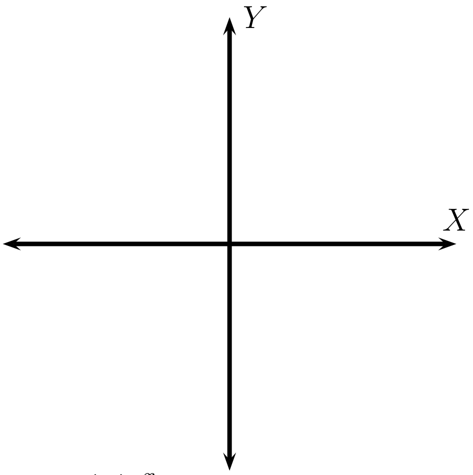
(26) Graph $f(x) = e^x - 2$



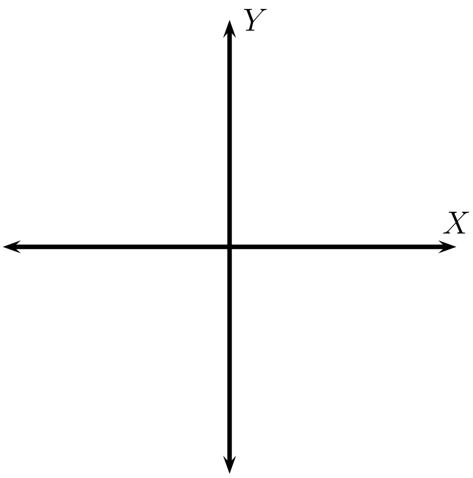
(27) Graph $f(x) = e^{x-3}$



(28) $f(x) = \left(\frac{5}{2}\right)^x$



(29) $f(x) = \left(\frac{2}{5}\right)^x$



Solve for x

(1) $2^x = 256$

(2) $3^x = 81$

(3) $4^{x-1} = 64$

(4) $5^{x+1} = \frac{1}{25}$

(5) $6^{x-3} = \frac{1}{36}$

13. Logarithmic functions

The Logarithmic function is the **inverse** function of the exponential function.
For $b > 0, b \neq 1$,

$$\log_b x = y \textit{ if and only if } b^y = x.$$

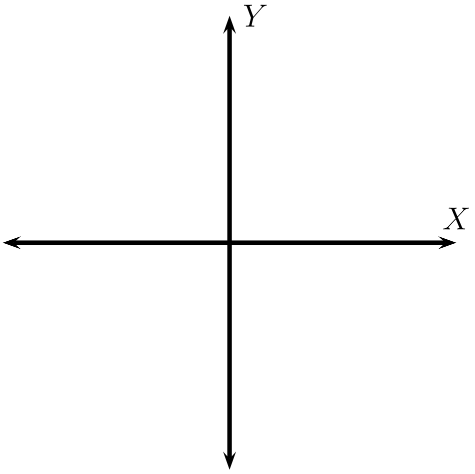
What values of x is $\log_b x$ undefined? When is $\log_b x = 0$?

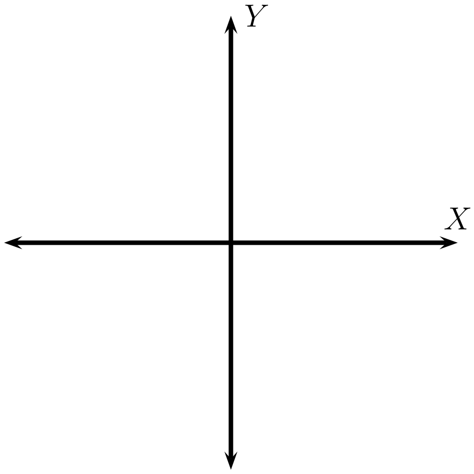
(1) Convert each statement to a radical equation.
  (a) $2^4 = 16$.
  (b) $3^5 = 243$.
  (c) $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$.
  (d) $\left(\frac{2}{3}\right)^{-4} =$ ____________.

(2) Convert each statement to a logarithmic equation. How is this form different from the radical form?.
  (a) $2^4 = 16$.
  (b) $3^5 = 243$.
  (c) $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$.
  (d) $\left(\frac{2}{3}\right)^{-4} =$ ____________.

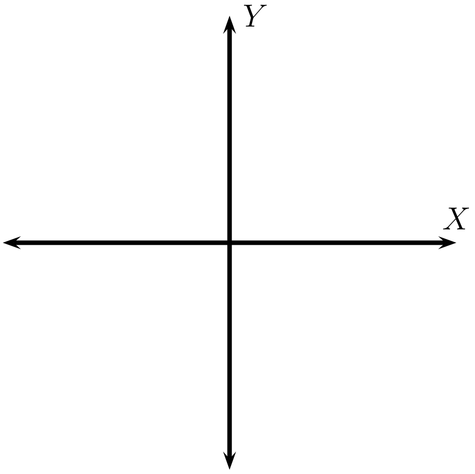
(3) Convert each statement to exponential form:
  (a) $\log_{10} 1000 = 3$.
  (b) $\log_{\frac{1}{3}} 9 = -2$.
  (c) $\log_{25} 5 = \frac{1}{2}$.
  (d) $\log_5 25 = 2$.

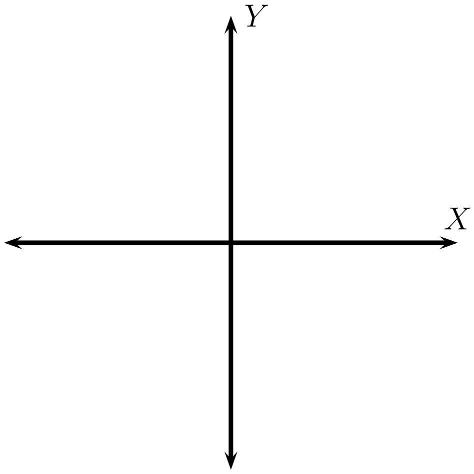
(4) Graph (plot at least five points). What are the X or Y intercepts if any. Give the range and domain of the graph. Sketch the relevant asymptote as a dotted line and give its equation.
$f(x) = 2^x$ and $g(x) = \log_2 x$



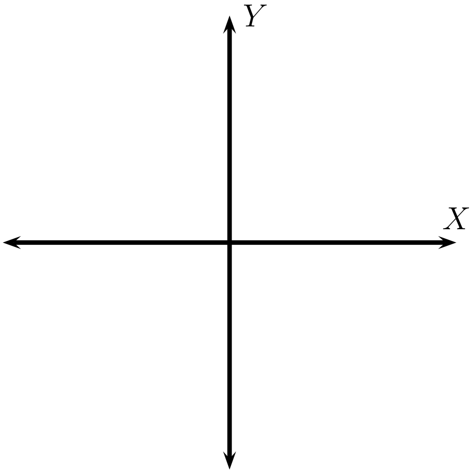


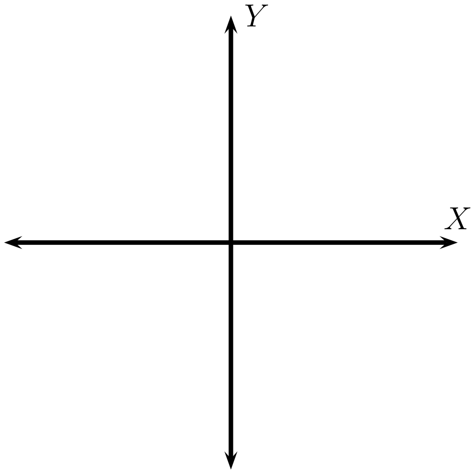
$f(x) = 3^x$ and $g(x) = \log_3 x$



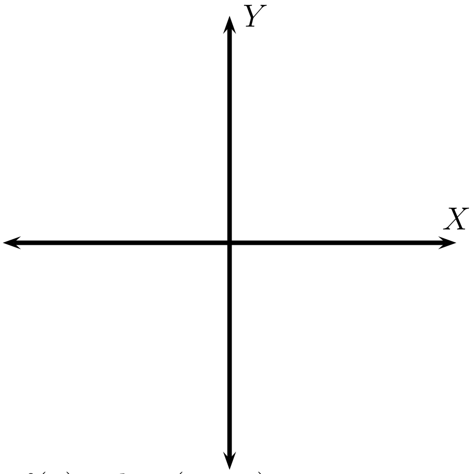


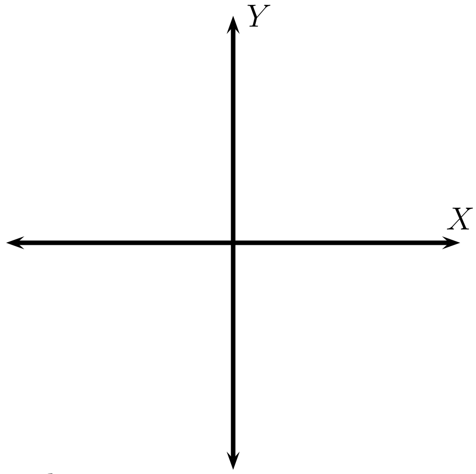
$f(x) = 10^x$ and $g(x) = \log_{10} x$





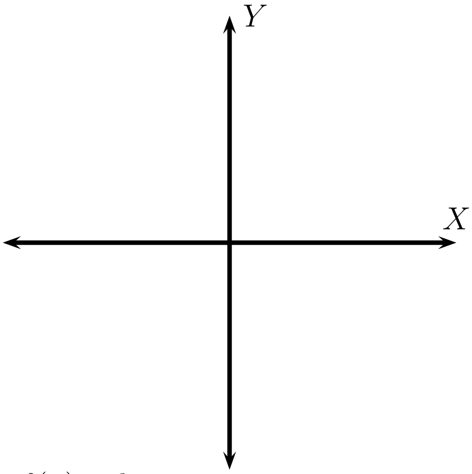
$f(x) = e^x$ and $g(x) = \ln x$ (The function $\log_e$ is denoted by ln and is called the natural logarithm)

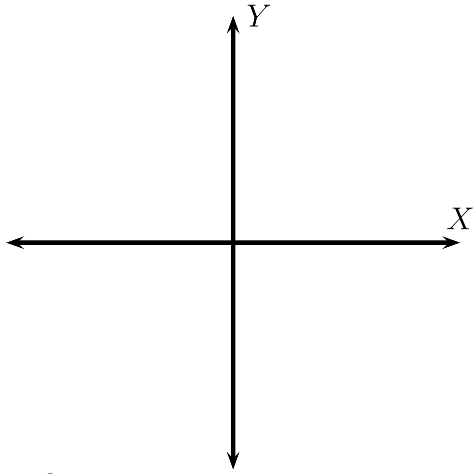




$f(x) = \log_3(x-1)$

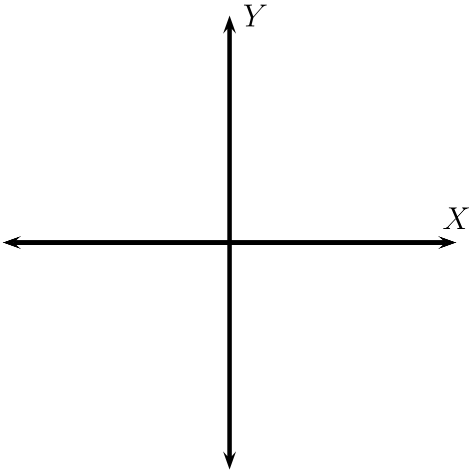
$f(x) = \log_3 x + 2$

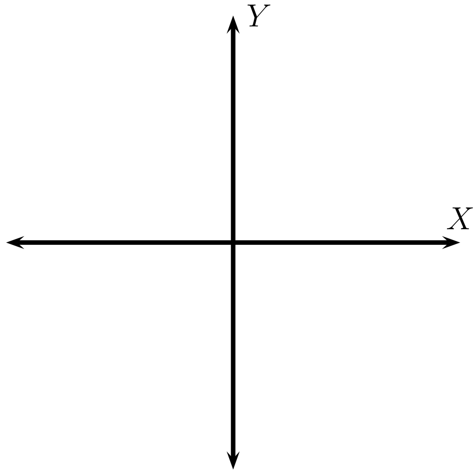




$f(x) = \log_2 x - 1$

$f(x) = \ln x - 1$





(5) Evaluate each expression without using a calculator

(a) $\log_3 27$

(b) $\log_5 \dfrac{1}{125}$

(c) $\log_{10} \sqrt{10}$

(d) $\log_7 \dfrac{1}{\sqrt[3]{7}}$

(e) $\log_3 3$

(f) $\log 1000,000$

(g) $\ln e^9$

(h) $\ln \dfrac{1}{e^{22}}$

(i) $3^{\log_3 1234}$

(j) $e^{\ln 27}$

(k) $10^{\log 43}$

(6) Fill in the blanks:

(a) For $b > 0, b \neq 1$, $\log_b 1 =$ ____________

(b) For $b > 0, b \neq 1$, $b^{\log_b x} =$ ____________

(c) For $b > 0, b \neq 1$, $\log_b b^x =$ ____________

(d) For $b > 0, b \neq 1$, the domain of the function $\log_b$ is ____________.

(e) For $b > 0, b \neq 1$, the range of the function $\log_b$ is ____________.

14. Properties of logarithms

Recall that for $b > 0, b \neq 1$, $\log_b x = y$ if and only if $b^y = x$. Give proofs for the following:

- For $b > 0, b \neq 1$, we have $\log_b b^x = x$
- For $b > 0, b \neq 1$, we have $b^{\log_b x} = x$
- For $b, M, N > 0, b \neq 1$, we have $\log_b(M \cdot N) = \log_b M + \log_b N$.
- For $b, M, N > 0, b \neq 1$, we have $\log_b \left(\dfrac{M}{N}\right) = \log_b M - \log_b N$.
- For $b, M > 0, b \neq 1$, we have $\log_b(M^p) = p \log_b M$.

- For $b, c, M > 0, b, c \neq 1$, we have $\dfrac{\log_c M}{\log_c b} = \log_b M$.

Now use these properties to expand the following logarithmic expressions:

(1) $\log_3(243x)$

(2) $\log_5\left(\dfrac{125}{y}\right)$

(3) $\log_7\left(\dfrac{343\sqrt{MN}}{\sqrt[3]{7}}\right)$

(4) $\log\left(\dfrac{x^5\sqrt[5]{a+b+100}}{1000(x-2)^2}\right)$

(5) $\ln\sqrt[7]{\dfrac{e^3}{4}}$

(6) $\ln(M+N)$

Use the properties of the logarithm function to condense the following expressions:

(1) $\log 25 + \log 4$

(2) $\log_3 x^4 + 2\log_3 y - 5\log_3 z$

(3) $\dfrac{1}{2}\log x + \dfrac{1}{3}\log y$

(4) $3\ln(x+y) - 4\ln(a+b) + 2\ln z - \dfrac{2}{3}\ln p$

(5) $\log_6 x + \log_6(x+3) - \log_6(x^2-9)$

(6) $\log M \cdot \log N \div \log P$

Convert the following to expressions in log-expressions (common logarithm) and then use your calculator to find their values:

(1) $\log_6 34$

(2) $\log_{12} 123$

Convert the following to expressions in ln-expressions (natural logarithm) and then use your calculator to find their values:

(1) $\log_4 19.34$

(2) $\log_{11} 49$

15. Exponential and Logarithmic equations

Solve for x:

(1) $3^x = 243$

(2) $2^{x-1} = \frac{1}{32}$

(3) $4^x = 128$

(4) $8^{2x-5} = 32^{x+2}$

(5) $10^{\frac{x+4}{3}} = \sqrt[3]{10}$

(6) $e^{\frac{x+4}{3}} = \frac{1}{e^2}$

Solve each equation. Express the solution in terms of natural logarithm and use a calculator to find an approximate solution.

(1) $5^x = 300$

(2) $7^{3x+2} = 6^{x-1}$

(3) $3^{2x} + 7 \cdot 3^x + 10 = 0$

(4) $3^{2x} - 7 \cdot 3^x + 10 = 0$

Solve for x:

(1) $\log_4 64 = x$

(2) $\log_5 x = 1$

(3) $\log_2 x = 0$

(4) $\log_x 4 = 16$

(5) $\log_3 81 = x$

(6) $\log_2 \frac{1}{32} = x$

(7) $\log_{36} \frac{1}{6} = x$

(8) $\log_x 12 = 2$

(9) $\log_x 12 = \frac{1}{2}$

(10) $\log_x 9 = 2$

(11) $\log_2(x - 4) = 4$

(12) $\log_3 243 = (2x + 3)$

(13) $\log_{125} x = \dfrac{1}{3}$

(14) $\log_5 x = \dfrac{1}{3}$

(15) $\log_{10} x = 10$

(16) $\log_5(x^2 - 5x + 1) = 2$

(17) $\log_3(6x^2 - 5x + 23) = 3$

(18) $\log(3x + 1) - \log 4 = \log(2x + 1)$

(19) $\log(x-3)+\log 8=\log 90$

(20) $\log(2x+5)+\log 3=\log(3x+5)$

(21) $\log(x-12)+\log(x+2)=\log 3+\log x$

(22) $\log(x+12)-\log(x+2)=\log 3+\log x$

(23) $2\ln(x+1)=\ln 5+\ln x-\ln 6$

16. Angles and Radian measure

(1) What is one radian? Explain your answer with an illustration.

(2) 1 radian = ____________^{o}. So π radians = ____________^{o}. Express each angle in radians.

0^o	30^o	45^o	60^o
90^o	120^o	135^o	150^o
180^o	210^o	225^o	240^o
270^o	300^o	315^o	330^o
360^o	390^o	405^o	420^o

(3) Express each angle in degrees:

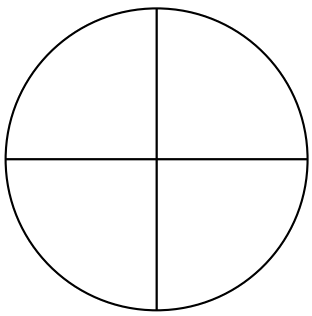
$\frac{\pi}{6}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$
$\frac{\pi}{4}$	$\frac{2\pi}{3}$	-5π
$\frac{\pi}{3}$	$\frac{3\pi}{4}$	-1.35

(4) Find a positive angle less than 360^o or 2π radians that is coterminal with the given angle.

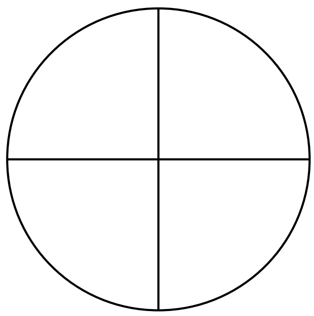
$\frac{17\pi}{6}$	$\frac{9\pi}{2}$	$-\frac{7\pi}{2}$
$\frac{9\pi}{4}$	$\frac{7\pi}{3}$	-5π
$\frac{13\pi}{3}$	$\frac{-11\pi}{4}$	$-\frac{15\pi}{2}$
430^o	-420^o	390^o
-530^o	-820^o	-1000^o

(5) Draw each angle in standard position:

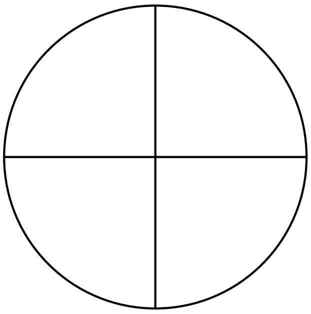
$\frac{17\pi}{6}$



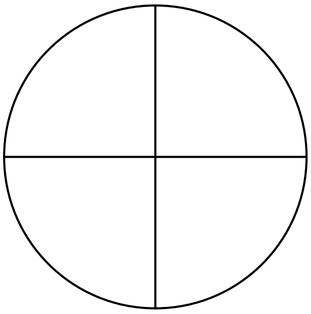
$\frac{9\pi}{2}$



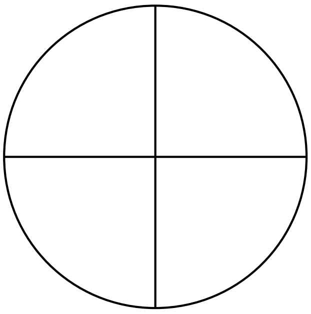
$-\dfrac{7\pi}{2}$



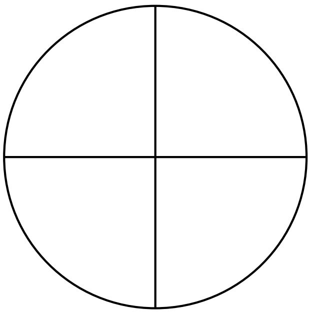
$\dfrac{7\pi}{3}$



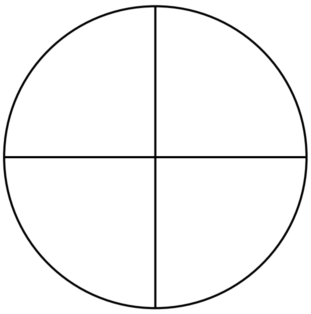
$\dfrac{13\pi}{3}$



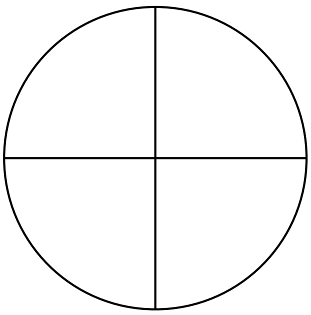
$-\dfrac{15\pi}{2}$



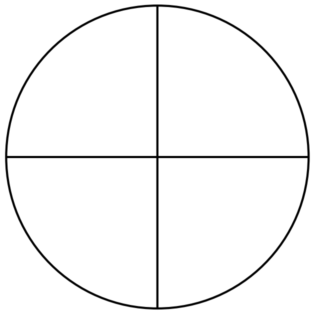
-420^o



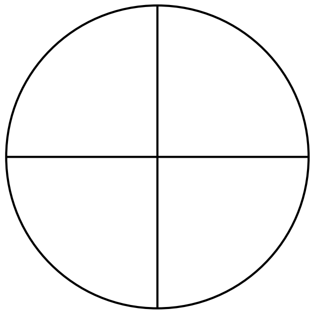
-530^o



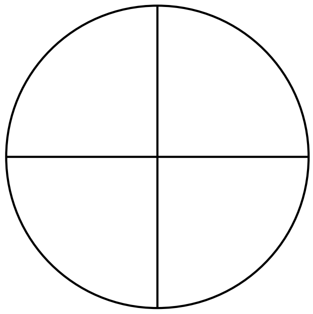
$\dfrac{9\pi}{4}$



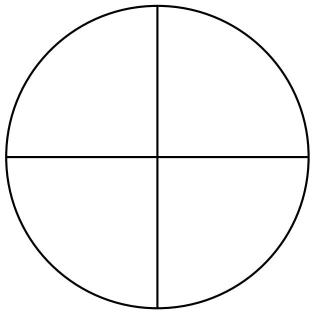
-5π



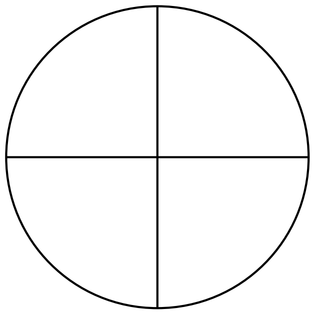
$-\dfrac{11\pi}{4}$



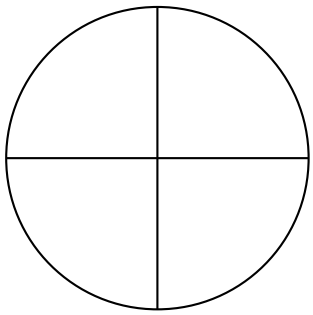
430^o



390^o



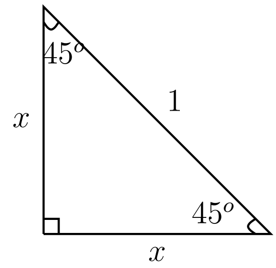
-820^o



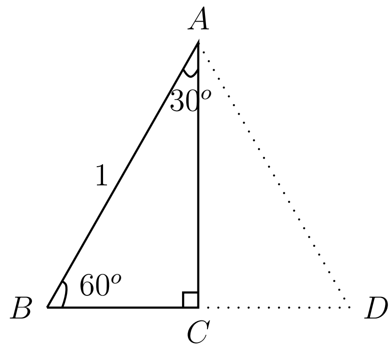
17. Right triangle trignometry

In this section we will start the study of **trignometric functions**. We first need the following two important calculations.

(1) Find the lengths of the legs of an isosceles right triangle with hypotenuse of length 1.



(2) Find the lengths of the legs of a $30^o - 60^o - 90^o$ triangle with hypotenuse of length 1.



$\triangle ABC$ is a $30^o - 60^o - 90^o$ triangle.
$\triangle ADC$ is a mirror image of $\triangle ABC$.
What kind of triangle is $\triangle ABD$? Explain.

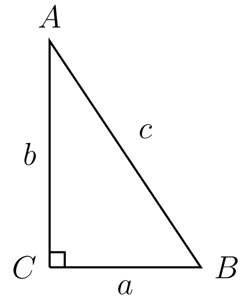
What is the length of segment BC?

What is the length of segment AC?

(3) The lengths of the legs of an isosceles right triangle with hypotenuse of length 1 are __________ and __________.

(4) The lengths of the legs of a $30^o - 60^o - 90^o$ triangle with hypotenuse of length 1 are __________ and __________.

Trignometric functions. Given is a right triangle $\triangle ABC$ with side lengths a, b, c. Note the naming scheme: The side opposite $\angle A$ has length a, the side opposite $\angle B$ has length b, and the side opposite $\angle C$ (the hypotenuse) has length c. For any acute angle,



$$\textbf{Cosine} \text{ of the angle} = \frac{\textit{length of its adjacent side}}{\textit{length of the hypotenuse}}$$

$$\textbf{Sine} \text{ of the angle} = \frac{\textit{length of its opposite side}}{\textit{length of the hypotenuse}}$$

$$\textbf{Tangent} \text{ of the angle} = \frac{\textit{length of its opposite side}}{\textit{length of its adjacent side}}$$

Notations and more trignometric functions: For an acute angle A,

$$\textit{Sine of angle } A = \sin(A)$$
$$\textit{Cosine of angle } A = \cos(A)$$
$$\textit{Tangent of angle } A = \tan(A) = \left(\frac{\sin(A)}{\cos(A)}\right)$$
$$\textit{Cosecant of angle } A = \csc(A) = \left(\frac{1}{\sin(A)}\right)$$
$$\textit{Secant of angle } A = \sec(A) = \left(\frac{1}{\cos(A)}\right)$$
$$\textit{Cotangent of angle } A = \cot(A) = \left(\frac{\cos(A)}{\sin(A)}\right) = \left(\frac{1}{\tan(A)}\right)$$

Use the figure above to fill in the following table:

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		

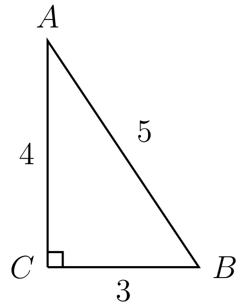
Use your results from the $45^o - 45^o - 90^o$ and $30^o - 60^o - 90^o$ triangles to fill in the following table:

Function	30^o	45^o	60^o
Sine			
Cosine			
Tangent			
Cosecant			
Secant			
Cotangent			

Fill in the table for the given right triangle:

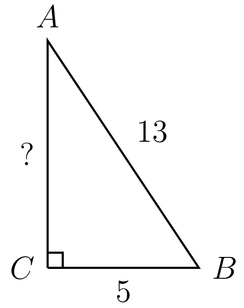
- Triangle 1.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



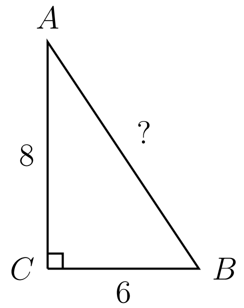
- Triangle 2.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



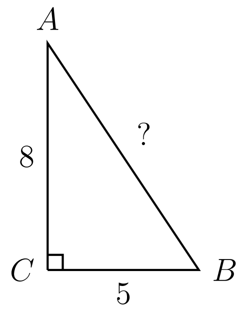
- Triangle 3.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



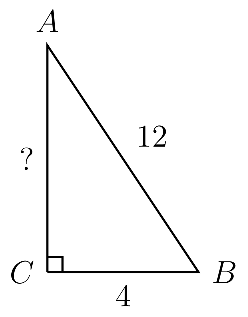
- Triangle 4.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



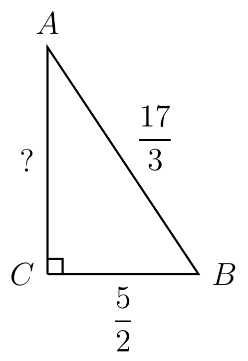
- Triangle 5.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



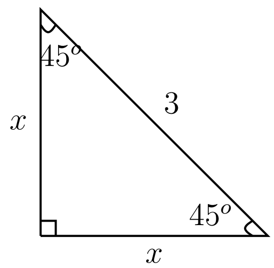
- Triangle 6.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



- Triangle 7.

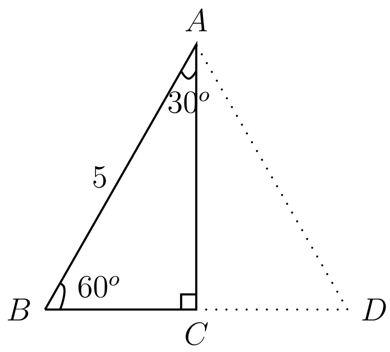
Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



Does this table look familiar?

- Triangle 8.

Function	A	B
Sine		
Cosine		
Tangent		
Cosecant		
Secant		
Cotangent		



Does this table look familiar?

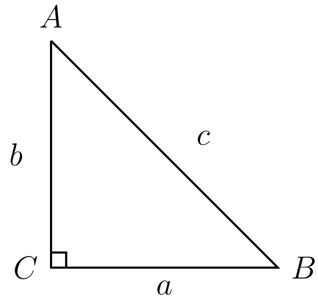
Complete the following tables (To find the measures of the angles you will need your calculators):

Function/Angle						
Sine	$\frac{3}{4}$					
Cosine		$\frac{2}{3}$				
Tangent			$\frac{3}{2}$			
Cosecant				$\frac{5}{4}$		
Secant					$\frac{6}{7}$	
Cotangent						$\frac{7}{8}$

Function/Angle						
Sine	$\frac{1}{2}$					
Cosine		$\frac{\sqrt{3}}{2}$				
Tangent			1			
Cosecant				$\frac{7}{4}$		
Secant					$\frac{9}{7}$	
Cotangent						$\frac{11}{8}$

Cofunctions :

$$\begin{aligned}
\sin(A) &= \cos(\underline{\hspace{1.5cm}}) = \cos(90 - A)\\
\cos(A) &= \sin(\underline{\hspace{1.5cm}}) = \sin(90 - A)\\
\tan(A) &= \cot(\underline{\hspace{1.5cm}}) = \cot(90 - A)\\
\cot(A) &= \tan(\underline{\hspace{1.5cm}}) = \tan(90 - A)\\
\csc(A) &= \sec(\underline{\hspace{1.5cm}}) = \sec(90 - A)\\
\sec(A) &= \csc(\underline{\hspace{1.5cm}}) = \csc(90 - A)
\end{aligned}$$



Recall that $90^o = \frac{\pi}{2}$ radians. Find a cofunction with the same value as the given expression:

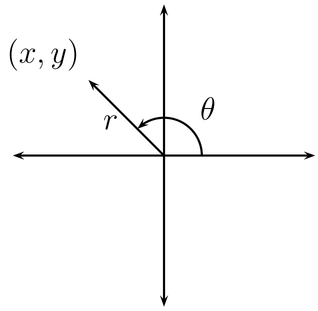
- $\sin 12^o$
- $\cos 15^o$
- $\tan 35^o$
- $\sin \frac{2\pi}{7}$
- $\cos \frac{3\pi}{7}$
- $\tan \frac{10\pi}{21}$

18. Trignometric functions: The unit circle

A point on the coordinate plane is determined by its x and y coordinates. These coordinates are called the **rectangular coordinates**.

Another way of describing a point on the coordinate plane is by using its **polar coordinates**, (r, θ) for $r > 0, 0 \leq \theta < 360^o$.

Here, r is the distance between the point (x, y) and the point $(0, 0)$; θ is the angle subtended by the ray joining $(0, 0)$ and (x, y) with the positive x-axis measured anticlockwise.
By convention, the point $(0, 0)$ in polar coordinates is also $(0, 0)$.



The **unit circle** is the circle centered at $(0, 0)$ and radius 1.

Equation for the unit circle is __________________________.

For a point $P = (x, y)$ with polar coordinates (r, θ),

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad \tan(\theta) = \frac{y}{x}$$

$$\csc(\theta) = \frac{r}{y} \qquad \sec(\theta) = \frac{r}{x} \qquad \cot(\theta) = \frac{x}{y}$$

When the point P is on the unit circle with polar coordinates (r, θ), we have $r =$ ______. So,

$$\sin(\theta) = \qquad \cos(\theta) = \qquad \tan(\theta) =$$

$$\csc(\theta) = \qquad \sec(\theta) = \qquad \cot(\theta) =$$

Find the rectangular coordinates of the point with polar coordinates

- $(1, 0^o)$

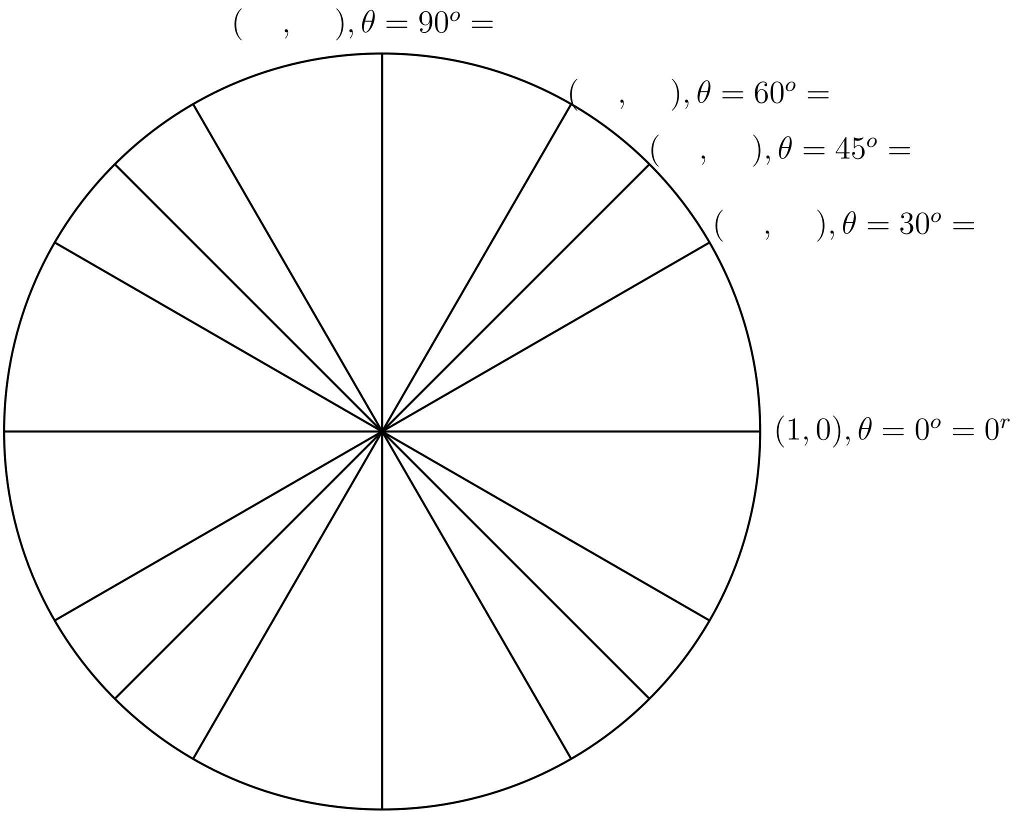
- $(1, 30^o)$

- $(1, 45^o)$

- $(1, 60^o)$

- $(1, 90^o)$

Find rectangular coordinates for all the end points of the radial segments shown on the unit circle below. Give the angles in both degree and radian form.



Here is a way of remembering the numbers you derived above:

$$0 < 1 < 2 < 3 < 4$$

Take square root throughout

Divide throughout by 2

How are these numbers to be used?

Recall the important cofunction properties:

- $\sin(A) = \cos(\underline{\hspace{3cm}})$
- $\cos(A) = \underline{\hspace{6cm}}$
- $\csc(A) = \underline{\hspace{6cm}}$
- $\sec(A) = \underline{\hspace{6cm}}$
- $\tan(A) = \underline{\hspace{6cm}}$
- $\cot(A) = \underline{\hspace{6cm}}$

Also recall the reciprocal (whenever defined) properties :

- $\csc(A) = \dfrac{1}{\sin(A)}$, and therefore $\sin(A) = \dfrac{1}{\csc(A)}$.
- $\sec(A) = \underline{\hspace{6cm}}$ and therefore $\underline{\hspace{7cm}}$
- $\cot(A) = \underline{\hspace{6cm}}$ and therefore $\underline{\hspace{7cm}}$

Using the circle, we get the following important trignometric identities (explain each one): **Even and Odd trignometric functions**

- $\cos(-A) = \underline{\hspace{6cm}}$ and therefore $\sec(-A) = \underline{\hspace{4.5cm}}$
- $\sin(-A) = \underline{\hspace{6cm}}$ and therefore $\csc(-A) = \underline{\hspace{4.5cm}}$.
- $\tan(-A) = \underline{\hspace{6cm}}$ and therefore $\cot(-A) = \underline{\hspace{4.5cm}}$.

Pythagorean identities

- $\sin^2(A) + \cos^2(A) = \underline{\hspace{6cm}}$

- $1 + \tan^2(A) =$ ________________________

- $1 + \cot^2(A) =$ ________________________

When is a function said to be periodic? What is the period of a periodic function?

State the **Periodic properties of the Sine and Cosine functions**

State the **Periodic properties of the Tangent and Cotangent functions**

Without using the calculator, find

- $\sin 3.2 \csc 3.2$
- $\sin^2 \dfrac{\pi}{9} + \cos^2 \dfrac{\pi}{9}$
- $\csc^2 30 - \cot^2 30$
- $\sin\left(-\dfrac{11\pi}{4}\right)$
- $\cos\left(-\dfrac{2\pi}{3} + 100\pi\right)$
- $\sin\left(-\dfrac{2\pi}{3} - 120\pi\right)$
- $\tan\left(-\dfrac{3\pi}{4} + 100\pi\right)$

19. Trignometric functions of any angle

(1) In the previous lesson we defined trignometric functions for acute angles. We are now ready to define trignometric functions for all angles, keeping in mind that division by 0 is undefined.
Let (x, y) be the terminal point of the ray given by an angle θ then
- $\sin(\theta) = \dfrac{y}{r}$
- $\cos(\theta) = \dfrac{x}{r}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{y}{x}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{r}{y}$
- $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{r}{x}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{x}{y}$

(2) Fill in the blanks:

(x, y)	$(5, 0)$	$(2, 5)$	$(0, 3)$	$(-3, 5)$	$(-3, 0)$	$(-4, -5)$	$(0, -5)$	$(3, -5)$
r								
$\sin(\theta)$								
$\cos(\theta)$								
$\tan(\theta)$								
$\csc(\theta)$								
$\sec(\theta)$								
$\cot(\theta)$								

(3) Fill in the blanks:

θ	0^o	30^o	45^o	60^o	90^o	120^o	135^o	150^o	180^o
$\sin(\theta)$									
$\cos(\theta)$									
$\tan(\theta)$									
$\csc(\theta)$									
$\sec(\theta)$									
$\cot(\theta)$									

θ	210^o	225^o	240^o	270^o	300^o	315^o	330^o	360^o
$\sin(\theta)$								
$\cos(\theta)$								
$\tan(\theta)$								
$\csc(\theta)$								
$\sec(\theta)$								
$\cot(\theta)$								

(4) Let point P be (x, y) in rectangular coordinates and (r, θ) in polar coordinates

(a) Given $\tan(\theta) = \dfrac{7}{12}$ and P is in the third quadrant. What is $\sin(\theta)$ and $\cos(\theta)$?

(b) Given $\cos(\theta) = -\dfrac{1}{\sqrt{12}}$ and P is in the second quadrant. What is $\sin(\theta)$ and $\tan(\theta)$?

(c) Given $\sin(\theta) = -\dfrac{2}{5}$ and P is in the fourth quadrant. What is $\cos(\theta)$ and $\tan(\theta)$?

(d) Given $\cot(\theta) = \dfrac{3}{4}$ and $\sin(\theta) > 0$, find $\sin(\theta)$ and $\cos(\theta)$.

The textbook uses reference angles to find exact values of trignometric functions. We will skip reference angles because we know to find exact values of trignometric functions already.

Fill in the blanks.

Function	0^o	-30^o	-45^o	-60^o	-90^o	-120^o	-135^o	-150^o	-180^o
$\sin(\theta)$									
$\cos(\theta)$									
$\tan(\theta)$									
$\csc(\theta)$									
$\sec(\theta)$									
$\cot(\theta)$									

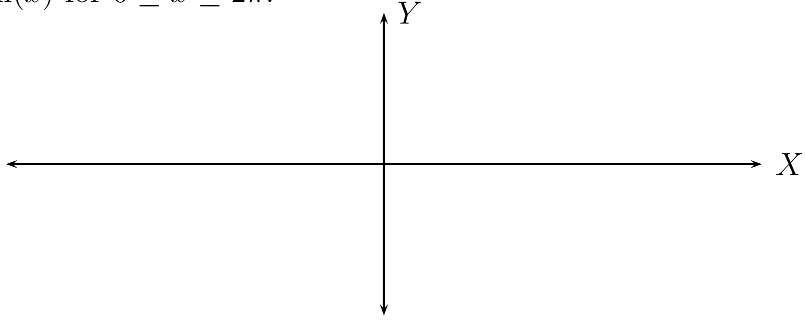
Function	-210^o	-225^o	-240^o	-270^o	-300^o	-315^o	-330^o	-360^o
$\sin(\theta)$								
$\cos(\theta)$								
$\tan(\theta)$								
$\csc(\theta)$								
$\sec(\theta)$								
$\cot(\theta)$								

- $\sin^2(x) + \cos^2(x) =$
- $\sin(-x) =$
- $\cos(-x) =$

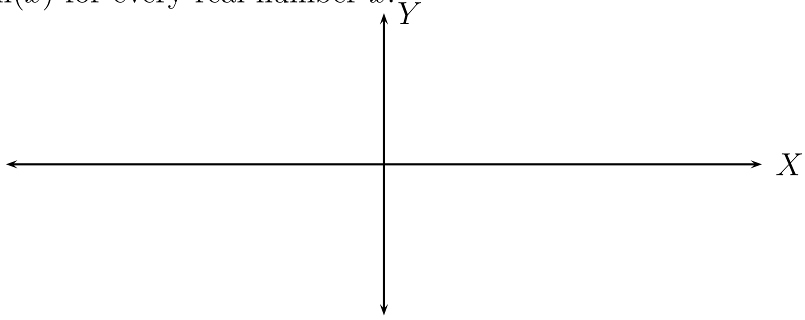
20. Graphs of Sine and Cosine functions

Graph (the variable x is measured in radians):

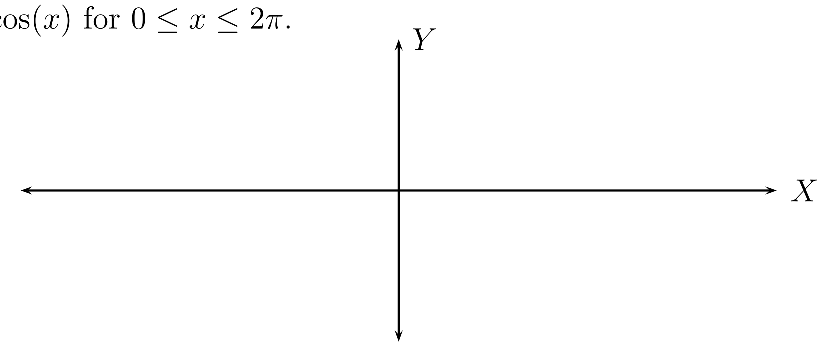
- $\sin(x)$ for $0 \le x \le 2\pi$.



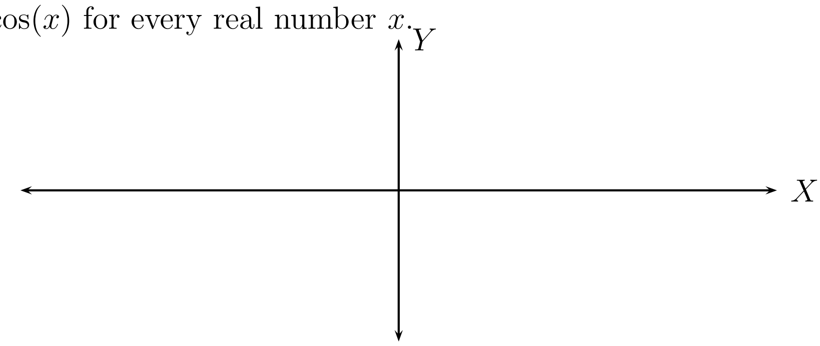
- $\sin(x)$ for every real number x.



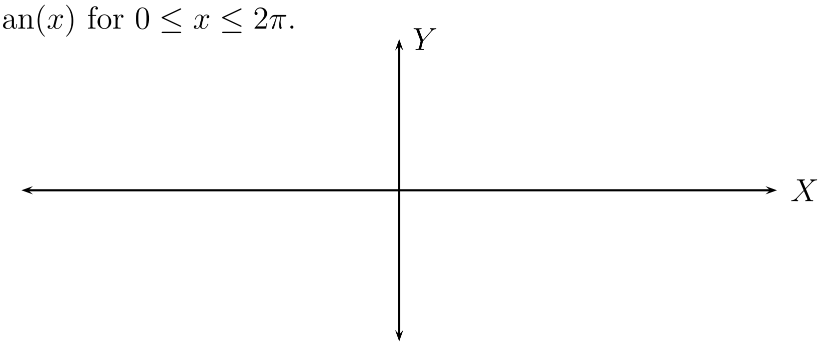
- $\cos(x)$ for $0 \le x \le 2\pi$.



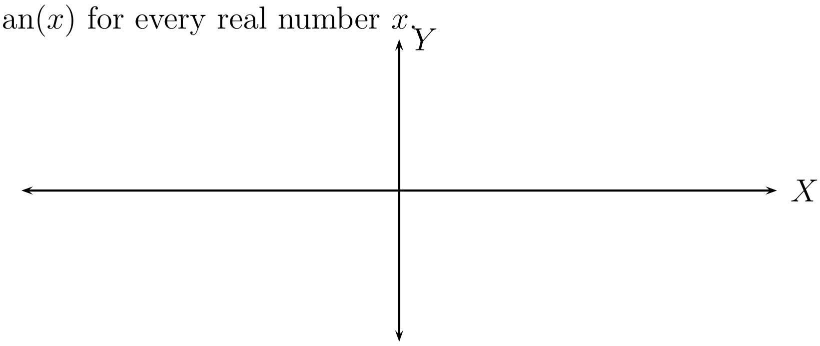
- $\cos(x)$ for every real number x.



- $\tan(x)$ for $0 \le x \le 2\pi$.



- $\tan(x)$ for every real number x.



For the graph of $y = \sin(x)$ or $y = \cos(x)$

- What is the fundamental cycle?

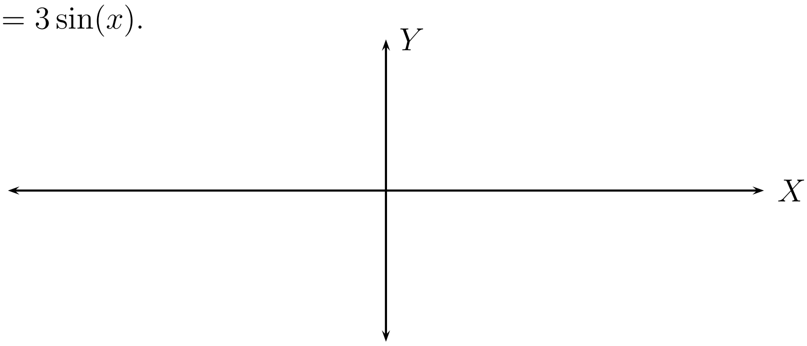
- The graph is periodic with period ____________.
- What is the sinusoidal axis of the graph?

- What is the amplitude?

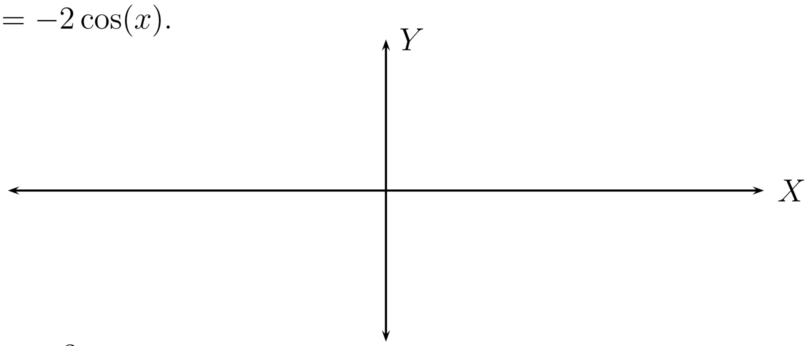
Given a trignometric function,

- Find its amplitude, period, and phase-shift.
- Graph the trignometric function.

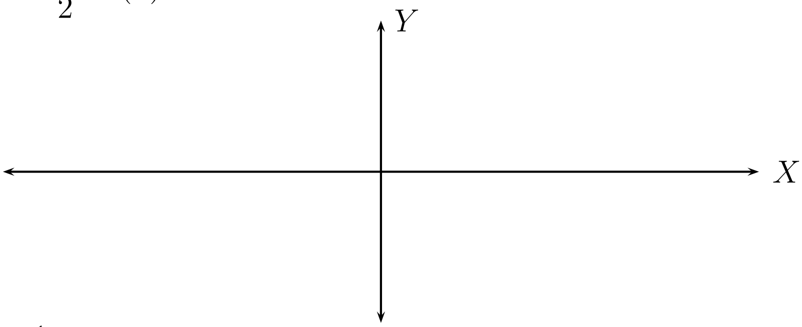
(1) $y = 3\sin(x)$.



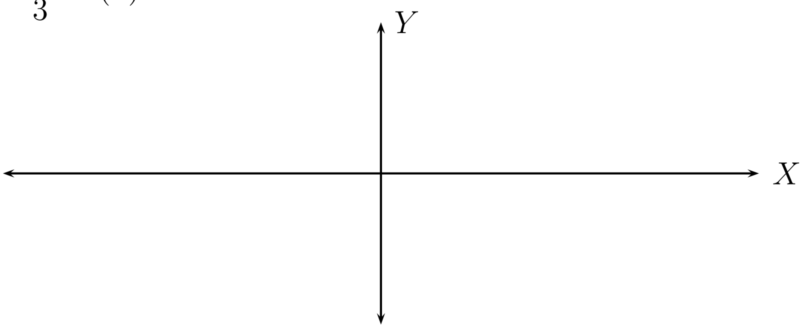
(2) $y = -2\cos(x)$.



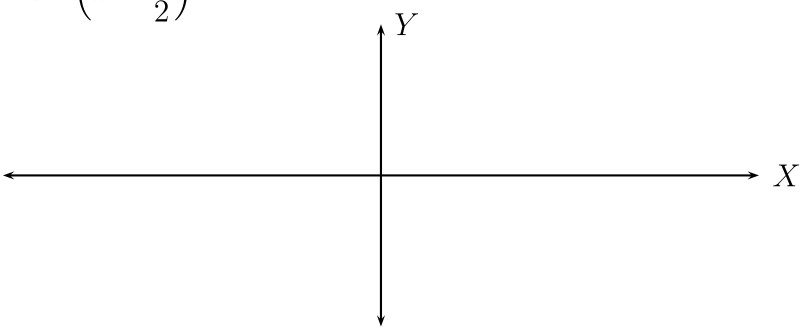
(3) $y = -\dfrac{3}{2}\sin(x)$.



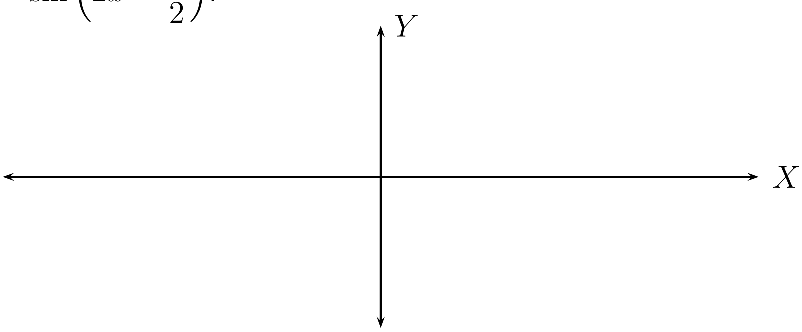
(4) $y = \dfrac{1}{3}\cos(x)$.



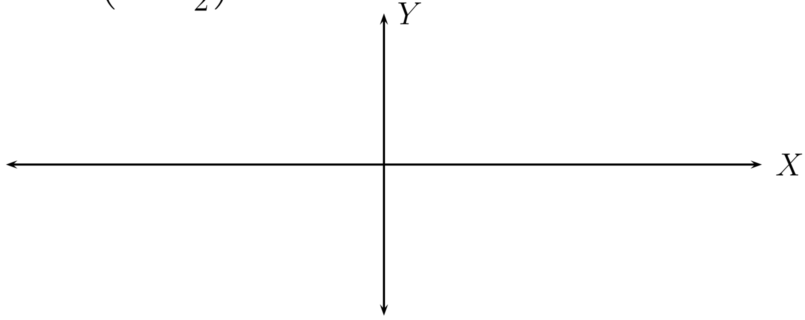
(5) $y = \sin\left(x - \dfrac{\pi}{2}\right)$.



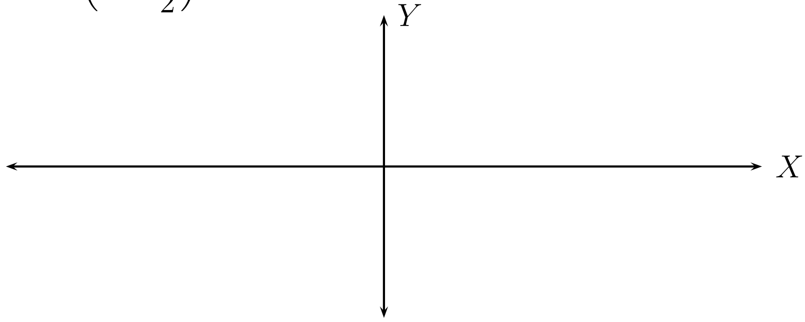
(6) $y = \sin\left(2x - \dfrac{\pi}{2}\right)$.



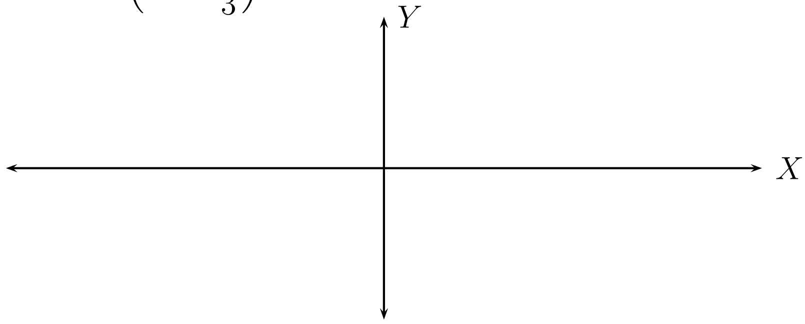
(7) $y = 3\sin\left(2x - \frac{\pi}{2}\right)$.



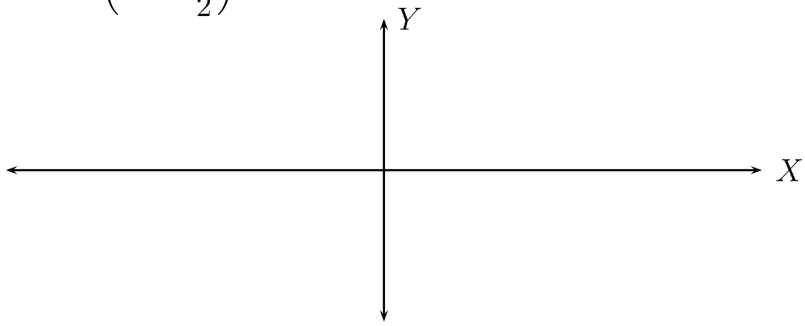
(8) $y = \cos\left(x + \frac{\pi}{2}\right)$.



(9) $y = -2\cos\left(2x + \frac{\pi}{3}\right)$.

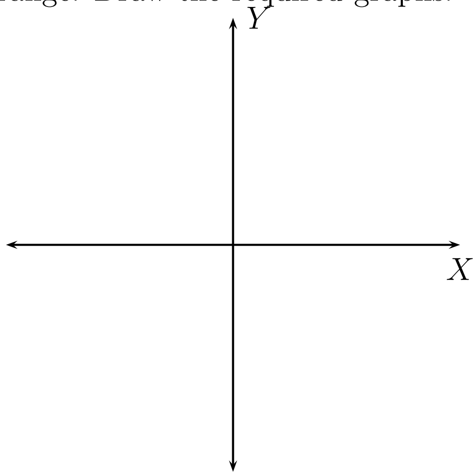


(10) $y = 3\cos\left(2x + \frac{\pi}{2}\right)$.

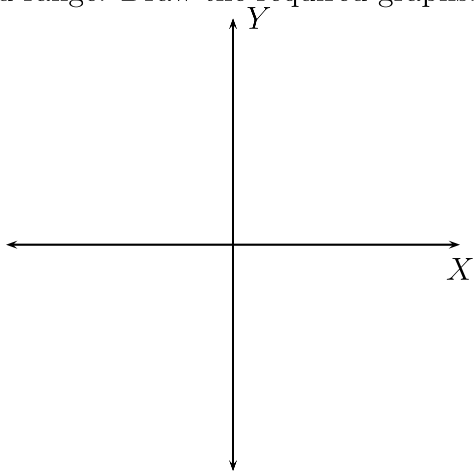


21. Inverse trignometric functions

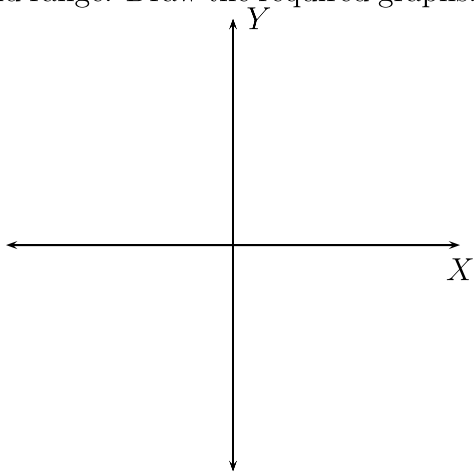
(1) Define the Inverse Sine Function. State its domain and range. Draw the required graphs.



(2) Define the Inverse Cosine Function. State its domain and range. Draw the required graphs.



(3) Define the Inverse Tangent Function. State its domain and range. Draw the required graphs.



(4) State the Inverse properties for the Sine function and its inverse.

(5) State the Inverse properties for the Cosine function and its inverse.

(6) State the Inverse properties for the Tangent function and its inverse.

(7) Without using the calculator, find the exact value in radians:

	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1}$									
$\cos^{-1}$									

	-1	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\tan^{-1}$							

(8) Without using calculators, find:

- $\sin\left(\sin^{-1}\dfrac{3}{5}\right)$

- $\sin^{-1}\left(\sin\dfrac{\pi}{5}\right)$

- $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$

- $\cos\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$

- $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right)$

- $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right)$

- $\cos\sin^{-1}\left(\dfrac{3}{4}\right)$

- $\sin\cos^{-1}\left(-\frac{4}{5}\right)$

- $\tan\sin^{-1}\left(-\frac{2}{3}\right)$

- $\cos\sin^{-1}\left(-\frac{1}{x}\right)$

- $\tan\cos^{-1}\left(\frac{2}{x}\right)$

22. Verifying trignometric identities

Recall the basic identities:

- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
- $\sec(x) = \dfrac{1}{\cos(x)}$
- $\csc(x) = \dfrac{1}{\sin(x)}$
- $\cot(x) = \dfrac{\cos(x)}{\sin(x)} = \dfrac{1}{\tan(x)}$
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(-x) = -\sin(x)$
- $\cos(-x) = \cos(x)$

Using these basic identities prove the following identities:

(1) $\tan(x)\csc(x)\cos(x) = 1$

(2) $\dfrac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)} = \cot(x) - \tan(x)$

(3) $\sec^2(x) - tan^2(x) = 1$

(4) $\sin(x) + \cos(x) = \dfrac{\tan(x) + 1}{\sec(x)}$

(5) $\ln(\csc(x)) = -\ln(\sin(x))$

(6) $\dfrac{\sec^4(x) - 1}{\tan^2 x} = 2 + \tan^2(x)$

(7) $\dfrac{\sin x}{\tan x} + \dfrac{\cos x}{\cot x} = \sin x + \cos x$

(8) $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x} = 0$

(9) $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}$

(10) $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

(11) $\dfrac{\sin x + \cos x}{\sin x} - \dfrac{\cos x - \sin x}{\cos x} = \sec x \csc x$

(12) $\ln e^{\tan^2 x - \sec^2 x} = -1$

In order to master this topic, please practise as many problems as you can from page 594 of the textbook.

23. Sum and Difference formulas

Here are two important **sum formulas**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Use these to give formulae for the following:

- $\sin(\alpha-\beta)$
- $\cos(\alpha-\beta)$
- $\tan(\alpha+\beta)$
- $\tan(\alpha-\beta)$

Without using calculators find the exact value of

- $\cos 15^o$
- $\sin 75^o$

- $\tan 105^o$

- $\cos\left(\dfrac{7\pi}{12}\right)$

- $\sin\left(\dfrac{\pi}{12}\right)$

- $\tan\left(\dfrac{4\pi}{3} - \dfrac{\pi}{4}\right)$

24. Trignometric Equations

(1) Determine whether the given x value is a solution to the given trignometric equation.

- $x = \frac{\pi}{4}$: $\sin x = 0$
- $x = 2\pi$: $\cos x = 1$
- $x = \frac{\pi}{6}$: $\tan x = \sqrt{3}$
- $x = \frac{7\pi}{6}$: $\sin x = -\frac{1}{2}$

(2) Find **all** the solutions to the given equation:

- $\sin x = \frac{1}{2}$
- $\cos x = \frac{1}{2}$
- $\tan x = 1$
- $\sin x = -\frac{\sqrt{3}}{2}$
- $\cos x = -\frac{\sqrt{3}}{2}$
- $\tan x = -\sqrt{3}$

(3) Solve each equation on the interval $[0, 2\pi)$

- $\sin \dfrac{3x}{2} = \dfrac{\sqrt{3}}{2}$
- $2 \cos 4x = -1$
- $\cot \dfrac{5x}{4} = -\sqrt{3}$
- $\sin \left(3x - \dfrac{\pi}{5}\right) = \dfrac{\sqrt{2}}{2}$
- $\sin^2 x - 1 = 0$

- $4\cos^2 x - 3 = 0$

- $3\sec^2 x - 4 = 0$

- $2\cos^2 x - 3\cos x + 1 = 0$

- $2\sin^2 x - \sin x - 1 = 0$