Chapter 7

Factoring

Vocabulary

- A factor (of an integer)
- A factor (of a polynomial)
- To factor
- Greatest common divisor
- To factor completely
- Difference of squares
- Quadratic trinomial
- Monic polynomial

7.1 Introduction to factoring

There are a number of circumstances when it is convenient to see a polynomial not as a sum of terms but as a product of factors. The process of writing a polynomial as a product of factors is called "factoring." The main use we will see for factoring will be in Chapter 9, in order to solve some quadratic equations. However, factoring is also a basic technique for working with rational expressions (in intermediate algebra) and in solving higher-degree polynomial equations (in precalculus).

In this chapter, we will outline some basic techniques for factoring. By the end, we will have developed a kind of "checklist" that we can apply to try to factor any polynomial expression.

The word "factor," in the context of mathematics, always implies the operation of multiplication. Just like we use the word "term" to mean a quantity being added, the word "factor" will be used to represent a quantity being multiplied.

Example 7.1.1.
- *In the expression* $4 \cdot 7$*, there are two factors,* 4 *and* 7*.*
- *In the expression* $(3.79)(-1.2)(5.9)$*, there are three factors,* 3.79*,* -1.2 *and* 5.9*.*
- *In the expression* $3x$*, there are two factors,* 3 *and* x*.*

One thing to keep in mind is that factors might be more complicated that those in the previous example.

Example 7.1.2.
- *In the expression* $4x(x^2+3x+1)$*, there are three factors:* 4*,* x *and* x^2+3x+1*. Notice that the entire expression in parentheses (which is one "group" having three terms) is* one factor*.*
- *In the expression* $(x+1)(x+4)$*, there are two factors,* $x+1$ *and* $x+4$*. Notice that each of the two factors has two terms.*

Unfortunately, the word "factor" is used in other ways as well. While the basic meaning of a factor as "an expression appearing in a product" is not lost, in other uses of the word this basic meaning is hidden in the background.

For example, the integer 12, considered as an arithmetic expression, does not appear as a factor at all, in the sense that there is no multiplication indicated. However, we can write $12 = 3 \cdot 4$. So we can say that 3 and 4 are *factors of* 12. Likewise, -2 is also a factor of 12, since 12 can also be written as $(-2)(-6)$. Notice that, from this point of view, factors of an integer come in pairs: -2 is a factor of 12, and so is -6.

In summary, a **factor of an integer** is an integer which can be multiplied by another integer to give the original number. (Sometimes it is said that a factor of an integer "divides the original number evenly," but we want to emphasize that the word factor implies the operation of multiplication.) While the word "factor" in this sense is sometimes meant to only refer to positive numbers, we will need to consider both positive and negative factors.

Here are a few things to remember about factors of integers:

- 1 is a factor of every integer. (And so is -1.)
- Every integer is a factor of itself. (And so is its opposite a factor of itself.)
- A *prime* number is a positive integer with exactly two positive factors: 1 and itself.[1]

Example 7.1.3. *List all the factors of each of the following integers:*

(a) 12*;*

(b) -140*;*

[1] For this reason, the number 1 is not a prime number, since it has only one positive factor.

(c) 25;

(d) 17.

Answer. *(a) The factors of* 12 *are* 1, 2, 3, 4, 6, 12, −1, −2, −3, −4, −6 *and* −12.

(b) The factors of −140 *are* 1, 2, 4, 5, 7, 10, 14, 20, 28, 35, 70, 140, −1, −2, −4, −5, −7, −10, −14, −20, −28, −35, −70 *and* −140.

(c) The factors of 25 *are* 1, 5, 25, −1, −5, *and* −25.

(d) The factors of 17 *are* 1, 17, −1 *and* −17. *(Notice* 17 *is a prime number.)*

In a similar way, **a factor of a polynomial is another polynomial which, when multiplied by a third polynomial, gives the original polynomial**. In the case that that the polynomial is already written as a product, some factors are easy to see. For example, the polynomial $(3x + 2)(x - 1)$ has two factors, being $3x + 2$ and $x - 1$. Whether it has any other factors will be investigated in the remainder of the chapter.

What happens if a polynomial is not written as a product?

So far, we have used the word "factor" as a noun. However, due to the importance in various contexts of seeing an expression written as a product, the word "factor" is also used as a verb.

Factoring

To *factor* means to write as a product of two (or more) factors.

For example, to factor the number 12, we could write $3 \cdot 4$ or $2 \cdot 6$ or $(-1)(-12)$. From this example, you can see that there is usually more than one way to factor an integer.[2]

Likewise, when we are asked to factor a polynomial, the answer should be a product of two (or more) polynomials. For example, to factor $x^2 + 3x + 2$, we would write $(x + 1)(x + 2)$. You can check that the answer is correct by multiplying the two polynomials $x + 1$ and $x + 2$—you should get $x^2 + 3x + 2$. By the end of this chapter, you will see how to obtain that answer, if it weren't given to you like it was here. But you should notice something right away: **Factoring is the opposite process as multiplying**. This will be our guide to presenting all the various methods of factoring below.

[2](The *Fundamental Theorem of Arithmetic* states, however, that there is only one way to factor a positive integer into factors which are powers of prime numbers, up to the order in which the factors are written. This is called the prime factorization of the integer. You might remember algorithms for producing the prime factorization of a number, like the so-called "factor tree.")

NOTICE: For the remainder of this chapter, all of our polynomials will have *integer* coefficients. In particular, when we are asked to factor a polynomial with integer coefficients, we will insist that the factors should also have integer coefficients.

7.2 "Factoring out" the greatest common factor

Let's start out with an example where we can "cheat."

Example 7.2.1. *Factor:* $6x^3 + 21x^2$.

Answer. *One answer is* $3x^2(2x + 7)$.

To see why, refer to Example 6.30 in the last chapter. We can "cheat" because we already multiplied two polynomials to obtain $6x^3 + 21x^2$*, so when we are now asked to factor the same polynomial—to write it as a product—we can just refer back to the original multiplication problem.*

The problem, of course, is that on many occasions we will not be able to refer back to a multiplication problem to find an answer. But let's look a little more carefully at the preceding example to try to find a strategy.

Our "answer" $3x^2(2x + 7)$ has two factors: $3x^2$ and $2x + 7$. How are the related to the original polynomial $6x^3 + 21x^2$?

Notice that the original polynomial had two terms: $6x^3$ and $21x^2$. The coefficients of these two terms, 6 and 21, have two positive common factors: 1 and 3. (When we look for common factors, we will keep in mind that 1 is always a common factor.) Of these, the *greatest* common factor is 3—which is the coefficient of the factor $3x^2$.

In addition to the common factor of 3, the terms $6x^3$ and $21x^2$ have a variable part in common—they both involve powers of x. How many factors of x are *common*? Both terms include a factor of x, since $x^3 = x \cdot x^2$ and $x^2 = x \cdot x$. Both terms also include a factor of x^2, since $x^3 = x^2 \cdot x$ and $x^2 = x^2 \cdot 1$. But only one of the terms includes a factor of x^3, since the second term includes only two factors of x. Summarizing, the *greatest* common factor of x^3 and x^2—the greatest number of factors of x that are in common to both—is x^2. Notice that x^2 is the variable part of our original factor $3x^2$.

To summarize the preceding two paragraphs: the factor $3x^2$ is the **greatest common factor** (often abbreviated as GCF) of the two original terms $6x^3$ and $21x^2$. We obtained it by separately considering the coefficients and the variable parts of the terms and multiplying the result.

What about the other factor $2x + 7$ from our answer? How is this factor related to the original polynomial?

Notice what happens when we divide our original polynomial $6x^3 + 21x^2$ by the common factor $3x^2$ that we just discussed:

$$\frac{6x^3 + 21x^2}{3x^2}$$
$$\frac{6x^3}{3x^2} + \frac{21x^2}{3x^2}$$
$$2x + 7.$$

In other words, the second factor $2x+7$ is the quotient of the original polynomial by the greatest common factor.

Let's summarize the method that we have taken out of the preceding discussion. The process of factoring by finding a greatest common factor is often referred to as "factoring out" the greatest common factor.

Factoring out the greatest common factor

To factor a polynomial whose terms have a common factor:

1. Find the greatest common factor of all the terms of the original polynomial, considering both the coefficients and the variable parts.
2. Divide the original polynomial by the GCF from Step 1 to obtain the second factor.

The answer is the product of the polynomials from Steps 1 and 2.

NOTICE: It is possible to factor polynomials using a common factor that is not the GCF. For example, we could have factored $6x^3 + 21x^2$ above as $x^2(6x + 21)$ or as $3x(2x^2 + 7x)$. (Check that these are all valid!) We have written the polynomial as a product of two factors, as required. However, these factorizations are not "complete," in the sense that one of the factors still has factors in common among its terms. From now one, we will always ask to factor *completely*, which in this context means to factor out not just any common factor, but the *greatest* common factor. We will have more to say about "factoring completely" below.

The following examples illustrate the procedure for factoring out the GCF, as well as a number of issues to watch out for.

Example 7.2.2. *Factor completely:* $12x^7 - 8x^5 + 16x^3$.

Answer. *The polynomial has three terms,* $12x^7$, $-8x^5$ *and* $16x^3$. *The coefficients have (positive) common factors* 1, 2 *and* 4. *The highest power of* x *that is common to all three terms is* x^3. *So the greatest common factor is* $4x^3$.

Dividing the original polynomial by the GCF of $4x^3$*:*

$$\frac{12x^7 - 8x^5 + 16x^3}{4x^3}$$
$$\frac{12x^7}{4x^3} + \frac{-8x^5}{4x^3} + \frac{16x^3}{4x^3}$$
$$3x^4 - 2x^2 + 4.$$

The answer is $4x^3(3x^4 - 2x^2 + 4)$.

Notice that the greatest common factor includes the *least* exponent appearing in any of the terms. Although this "rule" seems strange, keep in mind we are looking to what factors are in *common* to all terms.

The next example shows illustrates a basic feature of polynomials: Not every polynomial can be factored, at least in any way that will be considered in this text. (One should keep in mind the example of prime numbers from arithmetic.)

Example 7.2.3. *Factor completely:* $x^2 + 5$.

Answer. *The polynomial has two terms,* x^2 *and* 5*. The only positive common factor of the coefficients is* 1*. There is no common factor of* x*. So the GCF of these two terms is* 1.

Although we could conceivably divide the original polynomial by 1*, this will result in the same polynomial, and so as a factorization we would have to write* $(1)(x^2 + 5)$*. However, we have gained nothing in the sense that the new "factor" is the same as the original polynomial.*

The answer is: The polynomial cannot be factored.

In particular, from now on, we will be more precise about what we mean by the verb "to factor." To factor will mean: Write as a product of two or more factors, *none of which are 1.* (There will be one exception to this when we discuss factoring by grouping in Sections 7.5 and 7.6 below.)

CAUTION: We will see many examples in later sections of polynomials whose terms have no common factor, but that can be factored using other techniques. (For the record, the polynomial $x^2 + 5$ in the example above cannot be factored using *any* of the methods we will discuss.)

Example 7.2.4. *Factor completely:* $10x - 25$.

Answer. *The polynomial has two terms,* $10x$ *and* -25*. The coefficients have positive common factors* 1 *or* 5*. They do not have a common variable factor, since the second term does not involve* x*. So the greatest common factor is* 5.

Dividing the original polynomial by the GCF of 5*, we obtain*

$$\frac{10x - 25}{5}$$
$$\frac{10x}{5} + \frac{-25}{5}$$
$$2x - 5.$$

The answer is $5(2x-5)$.

Example 7.2.5. *Factor completely:* x^3-4x^2-2x.

Answer. *The polynomial has three terms,* x^3, $-4x^2$ *and* $-2x$. *The coefficients have only positive common factor* 1. *The highest power of* x *that is common to all three terms is* x. *So the greatest common factor is* x.

Dividing the original polynomial by the GCF of x:

$$\frac{x^3-4x^2-2x}{x}$$
$$\frac{x^3}{x}+\frac{-4x^2}{x}+\frac{-2x}{x}$$
$$x^2-4x-2.$$

The answer is $x(x^2-4x-2)$.

When the leading coefficient of a polynomial is negative, it is customary to "factor out" a negative number, so that the more complicated factor has positive leading coefficient. The next two examples in this section illustrate that point.

Example 7.2.6. *Factor completely:* $-4x^2+8x-6$.

Answer. *The polynomial has three terms,* $-4x^2$, $8x$ *and* -6. *The positive common factors of the coefficients are* 1 *and* 2. *There is no common factor involving* x. *Since the leading coefficient is negative, we will use* -2 *as the GCF.*

Dividing the original polynomial by the GCF of -2:

$$\frac{-4x^2+8x-6}{-2}$$
$$\frac{-4x^2}{-2}+\frac{8x}{-2}+\frac{-6}{-2}$$
$$2x^2-4x+3.$$

The answer is $-2(2x^2-4x+3)$. *Notice that the second, more complicated factor (the trinomial) has a positive leading coefficient of* 2.

Example 7.2.7. *Factor completely:* $-x^2-2x+4$.

Answer. *The polynomial again has three terms,* $-x^2$, $-2x$ *and* 4. *The only positive common factor of the coefficients is 1. There is no common factor involving* x. *Even though normally we might say that this polynomial cannot be factored, we will go to the trouble of "factoring out" the common factor of* -1 *because the leading coefficient is negative.*

Dividing the original polynomial by the GCF of -1*:*

$$\frac{-x^2-2x+4}{-1}$$
$$\frac{-x^2}{-1}+\frac{-2x}{-1}+\frac{4}{-1}$$
$$x^2+2x-4.$$

The factorization, according to what we have written so far, is $(-1)(x^2+2x-4)$*. However, it is typical in this case to suppress the multiplication by* -1*, which has the effect of "the opposite of." So we will simply write* $-(x^2+2x-4)$*.*

The answer is $-(x^2+2x-4)$*.*

We conclude with an example illustrating the fact that the principles of "factoring out" a variable common factor extend to polynomials with more than one variable.

Example 7.2.8. *Factor completely:* $4x^2y^3z^5 - 12x^5y^8z^3 + 16x^3y^4$*.*

Answer. *The polynomial has three terms,* $4x^2y^3z^5$*,* $-12x^5y^8z^3$ *and* $16x^3y^4$*. The coefficients have greatest common factor* 4*. The highest power of* x *that is common to all three terms is* x^2*. The highest power of* y *that is common to all three terms is* y^3*. Since the third term has no factor of* z*,* z *will not appear in the greatest common factor. Combining all this information, the greatest common factor of the three terms is* $4x^2y^3$*.*

Dividing the original polynomial by the GCF of $4x^2y^3$*:*

$$\frac{4x^2y^3z^5-12x^5y^8z^3+16x^3y^4}{4x^2y^3}$$
$$\frac{4x^2y^3z^5}{4x^2y^3}+\frac{-12x^5y^8z^3}{4x^2y^3}+\frac{16x^3y^4}{4x^2y^3}$$
$$z^5-3x^3y^5z^3+4xy.$$

The answer is $4x^2y^3(z^5-3x^3y^5z^3+4xy)$*.*

7.2.1 Exercises

Factor the following polynomials completely.

1. $6x^3-2x$
2. $4x^5-12x^3-8x^2$
3. $18x-9$
4. $-3x^4+15x^3-9x^2$

5. $6ab^3 - 12a^2b^2$
6. $-xy^4 - 2x^2y^3 - 15x^5y$

7.3 Differences of squares

Let's look at the polynomial

$$x^2 - 16.$$

Based on the previous section, we might be tempted to say that this polynomial cannot be factored. After all, the only positive factor x^2 and -16 have in common is 1.

But not so fast! Looking back at Example 6.34, we've seen this polynomial before—it was the result of the product $(x + 4)(x - 4)$. So once again, we can "cheat:" $x^2 - 16$ can be factored as $(x + 4)(x - 4)$!

We never would have noticed that $x^2 - 16$ could have been factored just by looking for common factors. The question is now: What made this polynomial so special, and is there a pattern that we can use?

The first thing to notice about this polynomial is that it has only two terms—it is a binomial. More important, though, each of the two terms (ignoring for a moment the signs) are *perfect squares*: x^2 is $(x)^2$ ("x squared") and 16 is $(4)^2$ ("four squared.") Finally, the two perfect squares are *subtracted*. For that reason, this example and those having these common features are called "differences of squares."

Notice, by the way, that the two quantities which are being squared—in this example, the x and the 4, play a key role in the factorization: $(x + 4)(x - 4)$. This pattern is at the heart of factoring a difference of squares.

Factoring a difference of squares

The factorization of a polynomial having the special form $a^2 - b^2$ is

$$(a + b)(a - b).$$

(Notice that since the answer is a product, the order that we write the factors is not important, thanks to the commutative property of multiplication.)

Exercise 7.3.1. *Show by multiplying $(a + b)(a - b)$ that this product is really the same as $a^2 - b^2$. You can look back at Example 6.34 if you need a hint.*

What this "formula" says is that once you see that you have a difference of squares, you are almost done. Just figure out what quantities are being squared (playing the roles of a and b in the formula), and fill them into the pattern

$$(\_\_ + \_\_)(\_\_ - \_\_).$$

The only thing that requires some care is recognizing a difference of squares. (If you aren't familiar with the perfect square numbers, write a list of the first 10 or 12: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, and so on.)

Example 7.3.2. *Factor completely:* $x^2 - 81$.

Answer. *Notice first that the two terms* x^2 *and* -81 *have no factors in common (except* 1*).*

However, x^2 *is a perfect square (x squared) and* 81 *is also a perfect square (*9 *squared). The perfect squares are being subtracted. It is a difference of squares!*

The answer is (applying the pattern) $(x + 9)(x - 9)$.

Example 7.3.3. *Factor completely:* $9x^2 - 25$.

Answer. *Again, the terms have no common factor (other than* 1*).*

The term $9x^2$ *is a perfect square:* $9x^2 = (3x)^2$*, using Property (E4) of exponents. Also,* 25 *(*$= (5)^2$*) is a perfect square. Since the perfect squares are subtracted, this is a difference of squares.*

The answer is $(3x + 5)(3x - 5)$.

Example 7.3.4. *Factor completely:* $x^2 + 4$.

Answer. *The two terms have no common factor (other than* 1*).*

Both terms x^2 *and* 4 *are again perfect squares:* $x^2 = (x)^2$ *and* $4 = (2)^2$*. However, the terms are not subtracted! This example is not a difference of squares, and so the "formula" approach we have been using cannot be applied. In fact, there is no simple factorization for* x^2+4*, at least with polynomials with integer coefficients.*

The answer is: $x^2 + 4$ *cannot be factored.*

The previous example is an example of a general fact: **a sum of squares** $a^2 + b^2$ **cannot be factored** using polynomials with integer coefficients. (The reader refer to Exercise 11 to see that $(a + b)(a + b)$ is not in general the same as $a^2 + b^2$.)

The next example shows that sometimes, differences of squares may appear "in disguise."

Example 7.3.5. *Factor completely:* $x^6 - 25y^4$.

Answer. *The two terms have no common factor other than* 1.

On the surface, the exponents for the variables are not 2*, so this may not appear to be a difference of squares. However, because both exponents are* ***even****,* *we can use Property (E4) and (E5) of exponents (Section 6.3) to express them as perfect squares. In particular,* $x^6 = (x^3)^2$ *and* $25y^4 = (5y^2)^2$*. Since the terms are subtracted, this is a difference of squares!*

The answer is $(x^3 + 5y^2)(x^3 - 5y^2)$*.*

In the previous examples, we have never had to worry about common factors. In fact, the reader might wonder why bother making note in all those examples that there were no common factors other than one. The next example shows that a difference of squares might not be apparent until common factors are "factored out."

Example 7.3.6. *Factor completely:* $3x^3 - 27x$.

Answer. *In this example, the two terms* $3x^3$ *and* $-27x$ *have a common factor of* $3x$*. Our first step will be to factor out the common factor, as in the previous section:*

$$3x(x^2 - 9).$$

We have obtained a factorization of $3x^3 - 27x$*, in the sense that we have written it as a product of two factors. However, the factor* $x^2 - 9$ *is a difference of squares, and so can itself be factored. In other words, we have not factored* completely*. The factor* $x^2 - 9$ *factors as* $(x + 3)(x - 3)$*.*

The answer is $3x(x + 3)(x - 3)$*. Notice that the common factor that we factored out first must appear in the final factorization.*

We close this section with another reminder about factoring *completely*.

Example 7.3.7. *Factor completely:* $16x^4 - 1$.

Answer. *The two terms have no common factor (other than* 1*).*
Both terms $16x^4$ *and* 1 *are perfect squares:* $16x^4 = (4x^2)^2$ *and* $1 = (1)^2$*. Since they are being subtracted, we have a difference of squares, and we write*

$$(4x^2 + 1)(4x^2 - 1).$$

We need to make sure that we have factored completely*. Looking carefully at the two remaining factors, we see that the first factor* $4x^2 + 1$ *is a sum of squares, and as mentioned above, cannot be factored. However, the second factor* $4x^2 - 1$ *is again a difference of squares:* $4x^2 = (2x)^2$ *and* $1 = (1)^2$*. In other words,* $4x^2 - 1$ *can be factored as* $(2x + 1)(2x - 1)$*.*

The answer is $(4x^2 + 1)(2x + 1)(2x - 1)$*.*

7.3.1 Exercises

1. $a^2 - 9$
2. $x^2 - 25$

3. $x^2 + 64$
4. $4x^2 - 36$
5. $25x^4 - 81y^6$
6. $x^4 - 4x^2$
7. $3x^3 - 75x$
8. (*) (Other special products) Use the formula $a^2 + 2ab + b^2 = (a + b)^2$ from Exercise 11 to factor the following polynomials:
   (a) $x^2 + 2xy + y^2$
   (b) $x^2 + 10x + 25$
   (c) $x^2 + 12x + 36$
9. (*) (Other special products) Show that $a^2 - 2ab + b^2 = (a - b)^2$ by multiplying the right side. In words, this is a "formula" which says that if we have a sum of squares *with an additional term that is the opposite of twice the product of the two quantities being squared*, it can be factored as the square of the difference of the two quantities. (You will notice that this is actually just a version of the formula in the previous exercises, replacing $-b$ for b.)
10. (*) Use the formula in the previous exercise to factor the following polynomials:
    (a) $x^2 - 2x + 1$
    (b) $x^2 - 18x + 81$
    (c) $x^2 - 8x + 16$
11. (*) (Difference of cubes) Show that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ by multiplying the right side. This is a "formula" which says that if we have a difference of perfect cubes (quantities raised to the third power), the expression can be factored as the product of the difference of the two quantities and the sum of the squares of the two quantities and the product of the two quantities.
12. (*) Use the formula in the previous exercise to factor the following polynomials completely:
    (a) $x^3 - 8$
    (b) $x^3y - 125y^4$
13. (*) (Sum of cubes) Show that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ by multiplying the right side. Unlike the case of squares, sums of perfect cubes can be factored! The "formula" says that if we have a sum of perfect cubes, the expression can be factored as the product of the sum of the two quantities and the sum of the squares of the two quantities minus the product of the two quantities.

14. (*) Use the formula in the previous exercise to factor the following polynomials:
    (a) $x^3 + 64$
    (b) $8x^3 + 27$

7.4 Quadratic trinomials I. Monic trinomials

Many times, we will encounter polynomials having the form $ax^2 + bx + c$, where a, b, and c represent numerical coefficients. When a (the coefficient of x^2) is not zero, this polynomial is **quadratic**, meaning that it is a polynomial of degree 2, or, what is the same, that the highest degree of any term is 2. We will call polynomials of the form $ax^2 + bx + c$ **quadratic trinomials**, given that there are in general three terms whose highest degree is 2. (Notice though that if any of the coefficients are zero, there may be less than three terms.)

We will follow the custom from now on of always using the letter a to represent the coefficient of the degree 2 term (the *quadratic term*), b for the the coefficient of the degree 1 term (the *linear term*), and c for the degree 0 term (the *constant term*).

In the next two sections, we will discuss methods to factor quadratic trinomials. To make the presentation easier, we will first consider an easier case, when $a = 1$. (Polynomials whose leading coefficient is 1 are called *monic* polynomials.) Then, in the next section, we will take up the general case.

As we have in the past sections, let's start with an example. We will try to factor the quadratic trinomial

$$x^2 + 6x + 8.$$

(In the notation of the previous paragraphs, $a = 1$, $b = 6$ and $c = 8$.)

We first check that this polynomial cannot be factored using any of our two prior methods. There is no factor (except 1) common to all three terms. Also, it is clearly not a difference of squares—it has three terms, after all. So both of our methods so far fail.

Let's cheat! Looking back, in Example 6.31, $x^2 + 6x + 8$ happened to have been the result of the multiplication $(x+4)(x+2)$. In other words, $(x+4)(x+2)$ is the factorization for $x^2 + 6x + 8$.

As usual, we can't always hope that every polynomial we want to factor will have been the result of some multiplication problem we had previously done. However, as has been our pattern, let's see if we can find some key features of this example to help us find a general method for factoring (monic) quadratic trinomials.

Let's first set a goal of factoring a quadratic trinomial as a product of two linear (degree 1) polynomials. In fact, if we are factoring a *monic* quadratic trinomial, we will attempt to factor our quadratic trinomial $x^2 + bx + c$ into a product of the form

$$(x + \underline{\quad})(x + \underline{\quad}),$$

where the blanks will represent some numbers that we have to "fill in." (In our example, these numbers were 4 and 2.) Notice that this special form will guarantee that the result will be a monic polynomial, since the only degree 2 term from distributing will be $(x)(x) = x^2$.

How can we find numbers to "fill in the blanks" so that, when we multiply them, we obtain the correct product? Let's try to use our example above for clues. Was there any relationship between the numbers 4 and 2 in the factorization, on the one hand, and the coefficients 6 and 8 in the original trinomial? Actually, there are two relationships that you could notice: first, $6 = 4 + 2$, and second, $8 = (4)(2)$. Our method for factoring monic quadratic trinomials is based on these two important relationships: If a quadratic trinomial $x^2 + bx + c$ can be factored as $(x + p)(x + q)$ for some numbers p and q, then $p \cdot q = c$ and $p + q = b$.

These relationships are the key to the following method.

Factoring a monic quadratic trinomial

If a monic quadratic polynomial $x^2 + bx + c$ with integer coefficients b and c can be factored, the factorization has the form

$$(x + p)(x + q),$$

where p and q are integers satisfying $p \cdot q = c$ and $p + q = b$. To find p and q:

1. List all integer factors of c, positive and negative, in pairs;
2. From this list, find a pair of factors whose sum is b.

If no such integers exist, then the quadratic trinomial cannot be factored.

This technique, like the technique involving difference of squares, amounts to a kind of "fill in the blank"-type formula, where the p and q in this technique are exactly the numbers to "fill in the blanks" in the formula

$$(x + \underline{\quad})(x + \underline{\quad}).$$

Although it may not be obvious from the description, *the signs of p and q are crucial to the method.* The remaining examples of the section will illustrate this point.

Example 7.4.1. *Factor completely: $x^2 + 7x + 12$.*

Answer. *The terms have no factor in common (other than 1). It is not a difference of squares. It is, however, a monic quadratic trinomial, with $b = 7$*

and $c = 12$. According to the strategy, we will look for factors of 12 that add to 7.

Below we list the factors of 12 in pairs, along with the sums.

$1, 12$	$(1 + 12 = 13)$	$-1, -12$	$((-1) + (-12) = -13)$
$2, 6$	$(2 + 6 = 8)$	$-2, -6$	$((-2) + (-6) = -8)$
$3, 4$	$(3 + 4 = 7)$	$-3, -4$	$((-3) + (-4) = -7)$

The pair we are looking for is 3 and 4, since their product $(3)(4)$ is 12 and their sum $3 + 4$ is 7. These will be the values we will use to "fill in the blanks."

The answer is $(x + 3)(x + 4)$. (Again, we remind the reader that the orders of the factors is not important, thanks to the commutative property of multiplication. We could have also written the answer as $(x + 4)(x + 3)$.)

Now that we have illustrated our method with an example, we turn to an example involving negative coefficients.

Example 7.4.2. *Factor completely: $x^2 - 8x + 7$.*

Answer. *We check whether we can factor out a common factor or apply the difference of squares formula; neither apply. The polynomial is a quadratic trinomial. In this case $b = -8$ and $c = 7$. Notice, as always with polynomials, we are considering the polynomial as $x^2 + (-8x) + 7$, and the coefficient of the x-term is negative.*

Listing the factors of 7 (there are less this time, since 7 is prime!):

$1, 7$	$(1 + 7 = 8)$	$-1, -7$	$((-1) + (-7) = -8)$

We see that the pair of factors whose product is 7 and whose sum is -8 is -1 and -7. When we use these numbers to "fill in the blanks," we obtain $(x + (-1))(x + (-7))$. Normally, however, we will rewrite the "adding the opposite" as subtraction.

The answer is $(x - 1)(x - 7)$.

Example 7.4.3. *Factor completely: $x^2 - 2x - 8$.*

Answer. *The terms of the polynomial have no common factor other than 1, and it is not a difference of squares. It is a quadratic trinomial, with $b = -2$ and $c = -8$.*

Listing the pairs of factors of -8:

$1, -8$	$(1 + (-8) = -7)$	$-1, 8$	$((-1) + 8 = 7)$
$2, -4$	$(2 + (-4) = -2)$	$-2, 4$	$((-2) + 4 = 2)$

(Notice that when c is negative, we should choose our pairs of factors with opposite signs.)

The pair of factors of -8 *we are looking for is* 2 *and* -4*, since their sum is* -2*.*

The answer is $(x+2)(x-4)$*. (This is NOT the same as* $(x-2)(x+4)$*, as you can see by multiplying!)*

Example 7.4.4. *Factor completely:* $x^2 + 2x + 6$*.*

Answer. *The terms of the polynomial have no common factor other than* 1*, and it is not a difference of squares. It is a quadratic trinomial, with* $b = 2$ *and* $c = 6$*.*

Listing the pairs of factors of 6*:*

$1, 6$	$(1 + 6 = 7)$	$-1, -6$	$((-1) + (-6) = -7)$
$2, 3$	$(2 + 3 = 5)$	$-2, -3$	$((-2) + (-3) = -5)$

In this example, none of the pairs add up to the value of b*, which was* 2*.*
The polynomial cannot be factored.

Example 7.4.5. *Factor completely:* $2x^3 - 10x^2 - 48x$*.*

Example 7.4.6. *In this example, the three terms do have a common factor: the GCF is* $2x$*. The first step then will be to factor out the GCF:*

$$2x(x^2 - 5x - 24).$$

However, we cannot yet say that the polynomial is factored completely*, since the second factor is a quadratic trinomial (with* $b = -5$ *and* $c = -24$*). The next step will be to attempt to factor* $x^2 - 5x - 24$*.*

Listing the pairs of factors of -24*:*

$1, -24$	$(1 + (-24) = -23)$	$-1, 24$	$((-1) + 24 = 23)$
$2, -12$	$(2 + (-12) = -10)$	$-2, 12$	$((-2) + 12 = 10)$
$3, -8$	$(3 + (-8) = -5)$	$-3, 8$	$((-3) + 8 = 5)$
$4, -6$	$(4 + (-6) = -2)$	$-4, 6$	$((-4) + 6 = 2)$

We see that the pair we are looking for is 3 *and* -8*, since their sum is* -5 *. We can use these to "fill in the blank" and factor the quadratic trinomial, not forgetting about the factor of* $2x$ *we already found.*

The answer is $2x(x+3)(x-8)$*.*

7.4.1 Exercises

Factor the following quadratic trinomials.

1. $x^2 - 4x - 32$
2. $y^2 + 3y - 18$

3. $x^2 + 8y + 12$
4. $x^2 - 12x + 20$
5. $x^2 + 3x + 15$
6. $y^2 - y - 6$
7. $t^2 + 6t + 9$
8. $x^2 - 6x + 5$

Factor the following polynomials using any technique.

9. $3x^3 - 9x^2 - 12x$
10. $5x^2 - 80$
11. $6x^3 - 2x$
12. $2x^2 + 8$
13. $4x^4 - 12x^3 - 8x^2$
14. $3x^3 - 12x$
15. (*) Apply the method of this section to factor the following polynomials that have "quadratic form." In each case, an appropriate substitution will help. Don't forget to factor completely!
    (a) $x^4 + 5x^2 + 6$ (Hint: Substitute $u = x^2$).
    (b) $x^4 - x^2 - 12$
    (c) $x^6 + 2x^3 - 8$ (Hint: Substitute $u = x^3$).
    (d) $x^{64} - 10x^{32} + 9$

7.5 Quadratic trinomials II. The ac-method

The previous section showed that, at least for monic quadratic trinomials, the coefficients of the terms give important information as to how to factor the trinomial into a product of linear factors.

Why was it so important in the previous section that the quadratic trinomial be monic? Since the leading coefficient was 1, the coefficients of x in the linear factors were also forced to be 1, and so there were only two numbers left to find in the "formula" $(x + \underline{\quad})(x + \underline{\quad})$.

In the case of a non-monic quadratic trinomial $ax^2 + bx + c$ $(a \neq 1)$, we have no guarantee about the coefficients of x in the linear factors. One way to proceed would be to try to make a more elaborate "fill-in-the-blank" strategy, now having the form

$$(\underline{\quad}\, x + \underline{\quad})(\underline{\quad}\, x + \underline{\quad}).$$

In fact, this would be a reasonable approach. Of course, it would have to involve a, b and c, not just b and c like in the monic case.

We will follow a different strategy, however. Instead of the brute-force guessing and checking that the fill-in-the-blank approach would involve, we will follow a strategy that is designed to genuinely reverse the distributive law involved in multiplying two binomials. This method will be a little longer, but it involves no guessing at all. It is called the ac-method for reasons that will be clear shortly.

We will illustrate this method with an example. Let's try to factor the quadratic trinomial

$$6x^2 + 19x + 10.$$

The terms have no factors in common, and it is clearly not a difference of squares. And while the polynomial is a quadratic trinomial, it is not monic, since $a = 6$. A quick check will reveal that the fill-in-the-blank procedure of the previous section will lead nowhere in this case.

We will present the steps in the context of this example, then summarize the steps at the end.

Example 7.5.1. *Factor completely:* $6x^2 + 19x + 10$.

Answer. *We follow a four-step approach.*

Step 1. Form the product ac. *In this case,* $a = 6$ *and* $c = 10$*, so the product* ac *is* $(6)(10) = 60$.

Step 2. Find a pair of factors of ac ***whose sum is*** b. *We are looking for factors of* 60 *(from Step 1) whose sum is* $b = 19$. *This is exactly the process we used in the previous section; a little work will show that the pair of numbers we are looking for is* 4 *and* 15 *(since* $(4)(15) = 60$ *and* $4 + 15 = 19$*).*

Step 3. Use the pair of factors from Step 2 to "split" the x***-term.*** *We are going to rewrite the middle term using the two numbers we found in Step 2:*

$$6x^2 + 4x + 15x + 10.$$

Notice we have not changed the polynomial in any way, since $15x + 4x$ *is* $19x$. *We have only changed the way the polynomial is written.*

Step 4. Factor by grouping. *The heart of the ac-method is the following procedure. First, we will group the four terms from Step 3 into two groups:*

$$(6x^2 + 4x) + (15x + 10).$$

We are going to try to factor each group separately. *For example, the first group* $6x^2 + 4x$ *has a common factor of* $2x$*, which we can factor out:* $2x(3x + 2)$. *Likewise, the second group has a common factor of* 5*, which can be factored out to obtain* $5(3x + 2)$. *In other words, our polynomial now has the form*

$$2x(3x + 2) + 5(3x + 2).$$

Written in this way, the polynomial has two terms, one from each group. Notice that these two terms have a common factor of $(3x+2)$*! Even though this common* *factor looks more complicated that our usual monomial common factors, we treat it the same way: we factor it out. We will write the common factor* $3x + 2$ *outside, and we will be left with a factor of* $2x$ *(from the first term) and* 5 *(from the second term):*

$$(3x + 2)(2x + 5).$$

The answer is $(3x + 2)(2x + 5)$*.*

At this point, the reader should look back at Example 6.32, where we performed the multiplication $(2x + 5)(3x + 2)$ (the same as our final answer with the order of factors reversed) to obtain $6x^2 + 19x + 10$. The steps involved in that multiplication example were exactly the same as the steps of this method of factorization, but in reverse! The ac-method is designed to mimic (in reverse) the process of distributing in the case of multiplying a binomial by a binomial.

One thing should be pointed out right away. The order of the pair in splitting the middle term in Step 3 does not matter. The reader can verify this fact in the previous example, writing $6x^2+15x+4x+10$ instead of $6x^2+4x+15x+10$. The result should be the same, with the order of factors reversed. (We will see, however, that sometimes one way of splitting the middle term will give an easier result than the other.)

For the reader's reference, we repeat the four-step ac-method here.

Factoring a quadratic trinomial: The ac-method

If a quadratic polynomial $ax^2 + bx + c$ with integer coefficients a, b and c can be factored as a product of linear factors having integer coefficients, then the following procedure will give the factorization:

1. Form the product ac;
2. Find a pair of factors of ac whose sum is b;
3. Use the pair from Step 2 to "split" the x-term into a sum of two terms having the pair of numbers as coefficients;
4. Factor the resulting polynomial by grouping.

If there is no pair of factors of ac whose sum is b, then the quadratic trinomial cannot be factored into a product of linear factors with integer coefficients.

For the rest of this section, we will write our quadratic polynomials in descending order (as we usually do anyway). For this reason, we will sometimes refer to the x-term as the "middle term."

We will now illustrate the ac-method with several examples. Along the way, we will point out three "tips" to make using the ac-method easier.

Example 7.5.2. *Factor completely:* $3x^2 - 8x + 4$.

Answer. *First, notice that the three terms have no common factor other than* 1*, and that the polynomial is not a difference of squares. It is a quadratic trinomial, and it is not monic, since* $a = 3$*. We will use the ac-method.*

The product ac in this example is 12*. So we need to find a pair of factor of* 12 *whose sum is* -8*. The pair is* -6 *and* -2*.*

We use this pair to split the middle term:

$$3x^2 - 6x - 2x + 4.$$

There is a small but important difference in this example from the previous one: the coefficient of the second x-term is negative (of course, so is the coefficient of the first x-term, but that matters less). In this case, we are going to factor out a negative number.

We group the polynomials:

$$(3x^2 - 6x) + (-2x + 4).$$

The first group has a common factor of $3x$*. Factoring out we obtain* $3x(x-2)$*. The second group has a common factor of* 2*. However, as we mentioned, we will factor out* -2 *instead to obtain* $-2(x - 2)$*. (Be careful of the signs when factoring out a negative number!) In other words, we obtain:*

$$3x(x - 2) - 2(x - 2).$$

Because we made the effort to factor out a negative number from the second group, we see the factor $(x - 2)$ *in common to the two groups, giving a factorization of* $(x - 2)(3x - 2)$*.*

The answer is $(x - 2)(3x - 2)$*.*

The previous example contains an important lesson:

Helpful hint # 1: When factoring a polynomial whose leading coefficient is negative, it is usually a good idea to factor out a negative common factor.

The next example shows that this hint also leads to another tactic to make factoring simpler.

Example 7.5.3. *Factor completely:* $2x^2 - x - 10$.

Answer. *We check to see that the three terms have no common factor, and that the polynomial is not a difference of squares. It is a quadratic trinomial which is not monic (since* $a = 2$*), suggesting the ac-method.*

We first form the product ac, with $a = 2$ *and* $c = -10$*, so* $ac = -20$*.*

We now try to find a pair of factors of -20 *whose sum is* -1*. Listing the factors if necessary, we find that* 4 *and* -5 *are factors of* -20 *whose sum is* -1*, as required.*

As mentioned earlier, the order of this pair does not matter when splitting the middle term. This time, though, the factors have different signs. As we saw in the previous example, if we write the term with the negative coefficient second (to obtain $2x^2 + 4x - 5x - 10$*), we should aim to factor out a negative factor. Since this requires special care about the sign of the other term, we will instead write the term with the negative coefficient first:*

$$2x^2 - 5x + 4x - 10.$$

Grouping the terms as $(2x^2 - 5x) + (4x - 10)$*, we see that the first group has a common factor of* x*, while the second group has a common factor of* 2*. Factoring the two groups separately, we obtain*

$$x(2x - 5) + 2(2x - 5).$$

As we expect in the ac-method, we see that the two resulting terms have a common factor of $2x - 5$*. Factoring it out, we obtain*

$$(2x - 5)(x + 2).$$

The answer is $(2x - 5)(x + 2)$*.*

Exercise 7.5.4. *For practice, re-do the previous example, splitting the middle term as*

$$2x^2 + 4x - 5x - 10.$$

The lesson of the previous example can be summarized in the following tip.

Helpful hint # 2: If the pair of factors used to split the middle term in the *ac*-method have different signs, it is usually more convenient to write the term with the negative coefficient *first*.

Example 7.5.5. *Factor completely:* $4x^3 + 4x^2 + 2x$*.*

Answer. *Notice first that this polynomial is* not *a quadratic trinomial. It is a trinomial, of course, but it is not quadratic since the leading term has degree* 3*.*

However, the three terms have a common factor of $2x$*. So we immediately factor out the (greatest) common factor to obtain*

$$2x(2x^2 + 2x + 1).$$

Although we now have a factorization, since we have written the polynomial as a product of two factors, we need to decide whether the polynomial is factored completely*. In particular, since the second factor* $2x^2 + 2x + 1$ *is a quadratic* *trinomial which is not monic (since $a = 2$), we should try to apply the ac-method to determine whether it can be factored further.*

To apply the ac-method to factor $2x^2 + 2x + 1$, we see that the product ac is 2, since $a = 2$ and $c = 1$. So we need to find factors of 2 whose sum is 2 (since $b = 2$). It shouldn't take long to check that there is no pair of factors that satisfy this property. In this case, the ac-method determines that the polynomial $2x^2 + 2x + 1$ cannot be factored into a product of linear factors.

The answer is $2x(2x^2 + 2x + 1)$.

The final example of this section will lead to one last hint to keep in mind when applying the *ac*-method.

Example 7.5.6. *Factor completely: $12x^2 - 33x - 9$.*

Answer. *Looking at the three terms, there is a common factor of 3. The first step will be to factor it out:*

$$3(4x^2 - 11x - 3).$$

As in the previous example, we need to determine whether the remaining quadratic trinomial $4x^2 - 11x - 3$ can be factored further as a product of linear factors. We will apply the ac-method, using $a = 4$, $b = -11$ and $c = -3$. (Notice that to do this, the common factor of 3 no longer needs to be considered, although it will remain in the final factorization.)

The product ac in this case is -12. We will look for a pair of factors of -12 whose sum is -11; such a pair is -12 and 1. Using this pair to split the middle term (writing the factor with the negative coefficient first), we obtain

$$4x^2 - 12x + x - 3.$$

Grouping the factors as $(4x^2 - 12x) + (x - 3)$, we see that the first group has a common factor of $4x$. The second group, however, normally would not be factored, since the only common factor of x and -3 is 1. However, to make the factorization more clear, we are going to factor out the common factor of 1! In other words, factoring the two groups separately we obtain

$$4x(x - 3) + 1(x - 3).$$

Written in this way, we see that the two terms have a common factor of $x - 3$. Factoring out this common factor, we obtain $(x - 3)(4x + 1)$. In other words, the factor $4x^2 - 11x - 3$ can be factored as a product of linear factors.

The answer is $3(x - 3)(4x + 1)$.

There are two things to notice about the previous example. First, taking the time to factor out the common factor first, apart from being good general practice, made the *ac*-method much smoother. After all, if we had applied the

ac-method to the quadratic trinomial $12x^2 - 33x - 9$ without factoring out the 3 first, we would need to find factors of -108 whose sum is -33. That can be done, but who wants to go through the list of 12 pairs of factors of -108? Even if we did that, we would still see that one of the linear (binomial) factors would have a common factor of 3 that we would still have to factor out in order to factor completely. Always look for common factors first!

The second thing to notice is more particular to the ac-method.

Helpful hint # 3: When factoring by grouping using the ac-method, if one of the groups has no common factor other than 1, take the time to factor out the common factor of 1.

To close this section, we point out that even though we have developed the ac-method as a method of factoring non-monic quadratic trinomials, the method also works for the monic trinomials in the previous section. (We don't normally use the method for monic trinomials, though, since the "shortcut" presented in the previous section is so much faster.) In the challenge exercises, we give some other examples of situations where the ac-method can help.

7.5.1 Exercises

Factor the following polynomials completely.

1. $2x^2 - x - 55$
2. $3x^2 + 4x + 1$
3. $6x^2 + x - 2$
4. $15x^2 + x - 2$
5. $5x^2 - 3x - 1$
6. $2x^2 - x - 10$
7. $6x^2 - 22x + 20$
8. $2x^4 - 4x^3 - 16x^2$
9. (*) Factor completely the following polynomials that have "quadratic form." In each case, an appropriate substitution will help.
   (a) $2x^4 + 5x^2 + 3$ (Hint: Substitute $x^2 = u$, so $x^4 = u^2$).
   (b) $4x^4 + x^2 - 5$
   (c) $3x^6 - 10x^3 + 3$ (Hint: Substitute $u = x^3$).
   (d) $2x^6 + 5x^3 - 7$
   (e) $4x^{1000} - 9x^{500} - 9$

10. (*) Use the *ac*-method to factor the following quadratic trinomials *in two variables.*

    (a) $x^2 - xy - 12y^2$
    (b) $x^2 + 3xy + 2y^2$
    (c) $2x^2 - 5xy - 3y^2$
    (d) $3x^2 - 2xy - y^2$

7.6 Factoring by grouping

We will end our discussion of factoring by noticing that the technique of factoring by grouping, which we used as a key component of the *ac*-method, can be applied in a wider setting.

In each of the following examples, you will notice that none of the methods we have discussed so far can be applied: they have no factors (other than 1) common to all terms, they are not differences of squares, and they are not quadratic trinomials. However, because they have four terms involving pairs of variables, there is hope that they may be factored into a product of two binomials.

Example 7.6.1. *Factor completely:* $3ax + 2ay + 12bx + 8by$.

Answer. *The reader should first check that none of the preceding factoring techniques can be applied to this polynomial (in four variables!).*

Let's try to factor by grouping. Write the polynomial in two groups:

$$(3ax + 2ay) + (12bx + 8by).$$

We notice that the first group has a common factor of a*, while the second group has a common factor of* $4b$*. So factoring the two groups separately, we obtain*

$$a(3x + 2y) + 4b(3x + 2y).$$

Since the two terms now have a common factor of $3x + 2y$*, we can factor it out to obtain* $(3x + 2y)(a + 4b)$.

The answer is $(3x + 2y)(a + 4b)$.

The reader should notice that the procedure in this example is exactly the same as the one we encountered every time we apply the *ac*-method. There is an important difference, though. In applying the *ac*-method, the two terms obtained by factoring a common factor from the two groups separately *will always have a common factor* (as long as the middle term is split by using the factors of ac whose sum is b). For arbitrary polynomials like the ones we are looking at now, the two terms might not have a common factor *even if the two groups can be factored separately.*

Example 7.6.2. *Factor completely:* $6np + 4nq - 15mp - 10mq$.

Answer. *First check that none of the preceding factoring techniques can be applied to this polynomial.*

Again, we group the terms:

$$(6np + 4nq) + (-15mp - 10mq).$$

We see that the first group $6np+4nq$ *has a common factor of* $2n$*. In the second group* $-15mp-10mq$*, notice that the first term is negative. As in Example 7.5.2 above, we will factor out a common factor with a negative coefficient—in this case* $-5m$*. So we obtain*

$$2n(3p + 2q) - 5m(3p + 2q).$$

(Notice again that factoring out the -5 *in the second group changes the signs of both terms in the group.)*

Like last time, the two terms have a common factor of $3p + 2q$*. Factoring out, we obtain*

$$(3p + 2q)(2n - 5m).$$

The answer is $(3p + 2q)(2n - 5m)$*.*

There is one thing worth mentioning about the preceding example. Instead of grouping the first two terms and the last two terms, we could have rearranged the terms first as $6np - 15mp + 4nq - 10mq$. The reader should check that the result after factoring by grouping is the same as the result above (with the order of the factors possibly different). The difference is that written in this different order, there is no need to factor out a negative factor in the second group, which might eliminate some difficulty with signs.

Reordering the terms in the last example might have been helpful, but it was optional. The next example shows that sometimes reordering the terms is essential to apply the method of factoring by grouping we have described.

Example 7.6.3. *Factor completely:* $3sx + 2ty - 3tx - 2sy$.

Answer. *As usual, we check to see that none of the preceding factoring techniques can be applied to this polynomial.*

This time, if we try to group in the most obvious way, as $(3sx + 2ty) + (-3tx - 2sy)$*, neither term has any common factor at all (except* 1 *of course). However, before giving up, let's try to rearrange the terms. For example, let's try rewriting the polynomial as*

$$3sx - 3tx + 2ty - 2sy.$$

Now, grouping as usual, we get

$$(3sx - 3tx) + (2ty - 2sy).$$

Written this way, we see that the first group has a common factor of $3x$ *while the second group has a common factor of* $2y$*. Factoring the two groups separately, we get*

$$3x(s-t)+2y(s-t).$$

These two terms have a common factor of $s-t$*. Factoring out this common factor, we get*

$$(s-t)(3x+2y).$$

The answer is $(s-t)(3x+2y)$*.*

We end this section on a pessimistic note.

Example 7.6.4. *Factor completely:* $6ax+2ay+5bx-3by$*.*

Answer. *This polynomial has no factor common to all four terms. It is not a difference of squares or a quadratic trinomial. Since there are four terms with pairs of variables, we will try to factor by grouping.*

Grouped in the obvious way

$$(6ax+2ay)+(5bx-3by),$$

we see that the first group has a common factor of $2a$ *while the second group has a common factor of* b*. Factoring the groups separately, we get*

$$2a(3x+y)+b(5x-3y).$$

However, the two resulting terms have no factor in common! This obstacle is serious, since even though the groups have been factored, we still have not written the whole original polynomial as a product of two factors, since there are still two terms.

Before we give up, we remember from the last example that sometimes reordering the terms can be helpful. So let's try rewriting the original polynomial as $6ax+5bx+2ay-3by$*. Now, grouping as*

$$(6ax+5bx)+(2ay-3by),$$

we see that the first group has a common factor of x *and the second group has a common factor of* y*. Factoring the two groups separately, we obtain*

$$x(6a+5b)+y(2a-3b).$$

Again, these two terms have no common factor.

The reader should try other ways to reorder the terms to convince themself that in no case can we obtain two terms with a common factor, as we have above.

The polynomial cannot be factored.

7.6.1 Exercises

Factor by grouping, if possible.

1. $6xz + 9xw - 8yz - 12yw$
2. $45cw + 63cz - 20dw - 28dz$
3. $20ax - 24ay + 15bx + 18by$
4. $4ac - 9bd - 3ad + 12bc$
5. $20ax - 15ay - 8bx + 6by$
6. $3mx + 6my - 2nx - 4ny$

7.7 Chapter summary

- To factor a polynomial means to write it as a product of two or more factors, none of which are 1.
- Not every polynomial can be factored.
- In order to factor a polynomial, we have the following checklist to apply:

Factoring checklist

To factor a polynomial, answer the following questions in the given order:

1. Do the terms have any factors in common?
   - If so, "factor out" the greatest common factor.
2. Is the polynomial a difference of squares?
   - If so, apply the "formula" $a^2 - b^2 = (a + b)(a - b)$.
3. Is the polynomial a quadratic trinomial of the form $ax^2 + bx + c$?
   - If so, and $a = 1$, find factors of c whose sum is b to "fill in the blanks"
     $$(x + \_\_\_)(x + \_\_\_).$$
   - If so, and $a \neq 1$, apply the ac-method.
4. Can the method of factoring by grouping be applied?

If the above list gives a factorization of the polynomial, make sure to apply the checklist to the each of the factors to make sure the polynomial is factored *completely*.