# Chapter 6

# Polynomials

## Vocabulary

- Term
- Polynomial
- Coefficient
- Degree of a term
- Degree of a polynomial
- Leading term
- Descending order
- Like terms
- Scientific notation
- Distributive law

## 6.1 Introduction to polynomials

Up to this point, we have been looking at algebra from the point of view of solving equations and inequalities. Indeed, this is a major point of distinction between arithmetic and algebra. In arithmetic, there is no such thing as a conditional statement: every equation or inequality is either true or false. In algebra, a typical equation or inequality may be true or false, depending on the values of the variables involved. For that reason, the word "solve," in the sense of finding all solutions, only has meaning in the context of algebra.

There is another way of looking at algebra, however, that has nothing to do with solving equations or inequalities. In this view, algebra is a kind of "arithmetic of symbols." Variables, which up to this point have been treated as unknown numbers, will be viewed as symbols which can themselves by added, subtracted, multiplied or divided according to fixed rules. These rules should, of course, correspond to the rules of arithmetic of numbers when values are substituted for the variables. For instance, we would like the commutative and associative laws of addition and multiplication to apply, and we would like multiplication to distribute over addition.

We have already seen this arithmetic of symbols in the course of solving linear equations when we "combined like terms." In this chapter, we carry out this symbolic arithmetic in a more general setting.

We will start by setting up the terminology that we will use throughout this chapter.

**Terms**

A *term* is an algebraic expression which is not itself written as the sum (or difference) of two or more expressions. It may involve products or quotients of constants or variables.

An algebraic expression can be written as a sum of terms. In this case, a *term* is a quantity which appears in an algebraic expression as part of a sum (a "summand").

Phrased differently, terms are expressions which are *added*.

For example, the algebraic expression

$$x^2 + 4x + 5$$

has three terms: $x^2$, $4x$, and 5. One might say that "terms are separated by the addition symbol (+)."

From now on, it will be especially important to distinguish between subtraction and "adding the opposite." For example, we will consider the expression

$$x^3 - 3x^2 - 5x - 4$$

as being

$$x^3 + (-3x^2) + (-5x) + (-4).$$

In particular, this expression has four terms (which are added): $x^3$, $-3x^2$, $-5x$ and $-4$.

A word of caution in our terminology. We will encounter more complicated expressions like

$$(x - 3)(x^2 + 5x + 6).$$

*This expression has only one term!* It is formed by two grouped expressions, $(x-3)$ and $(x^2+5x+6)$, which are *multiplied*, not added. In a similar way, the expression

$$(2x+3)(4x-1)+(3x+2)(x-5)$$

has two terms: the first term is $(2x+3)(4x-1)$ and the second term is $(3x+2)(x-5)$. The point is that terms can be quite complicated, but they must appear as part of a sum.

A polynomial will be an algebraic expression having a particular form.

**Polynomials in one variable**

A *polynomial* in one variable (say $x$) is an algebraic expression, which can be written in such a way that each of its terms has the form

$$ax^n,$$

where $a$ represents any number and $n$ represents a whole number.

(It is easy to make a definition for polynomials with more than one variable. For example, a polynomial in two variables $x$ and $y$ should have terms of the form $ax^m y^n$, where both $m$ and $n$ are whole numbers.)

The most important feature of a polynomial is the exponent of the variable part of each term. To say that the exponent must be a whole number means, for instance, that the exponent cannot be negative, nor can it be fractional.

Here are some examples of polynomials:

- $x^2-6x-7$;
- $\dfrac{3x-5}{2}$;
- $t^5+1$;
- $x^7+x^6+x^5+x^4+x^3+x^2+x+1$.

Here are some examples of algebraic expressions that are not polynomials:

- $\dfrac{1}{x}$;
- $x^{-2}$ (we will see what this means shortly);
- $\dfrac{x+1}{x^2-3x+5}$;
- $\sqrt{x}$;

- $2^x + x^2$.

**Exercise 6.1.1.** *For each of the examples listed above of algebraic expressions which are not polynomials, identify what it is about them that prevents them from being considered polynomials.*

Notice that each term of a polynomial is formed by multiplying a "number part," called the *coefficient*, with a "variable part." The variable part is completely described by the exponents of the variables involved.

**Degree**

For a polynomial in one variable, the *degree of a term* is the exponent of the variable part of the term. The *degree of a polynomial* is the highest degree of any of its terms.

Pay attention to the fact that this definition is really *two* definitions: the degree of a term is (usually) different than the degree of the polynomial in which it appears.

In the case of polynomials in more than one variable, the definition requires a little more care. In that case, the degree of a term is defined to be the sum of all the exponents having a variable base. So the degree of the term $-5x^2y$ would be 3.

Notice that a polynomial of degree one corresponds to what we referred to in the preceding chapter as *linear*.

**Exercise 6.1.2.** *For each of the polynomials below, identify the terms. For each term, identify the coefficient and the degree. Then determine the degree of the polynomial.*

*(a)* $x^2 - 6x - 7$;

*(b)* $\dfrac{3x-5}{2}$;

*(c)* $t^5 + 1$;

*(d)* $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

A polynomial can be classified by the number of terms it involves. For example, a *monomial* is a polynomial with one term, a *binomial* is a polynomial with two terms, and a *trinomial* is a polynomial with three terms.

Normally, we will write a polynomial in *descending order*: terms with higher degree will be written to the left of terms with lower degree. In case a polynomial is not written in descending order, we will take the trouble to rewrite it by using the commutative law of addition, as the following example illustrates. The terms with the highest degree is called the *leading term.*

**Example 6.1.3.** *Write the polynomial* $4x^3 - 5x^6 + 6 - 3x + x^2$ *in descending order.*

**Answer.** *The polynomial not written in descending order. For example, a degree* 3 *terms* ($4x^3$) *is written to the left of a degree* 6 *term* ($-5x^6$).

*There are five terms:* $4x^3$, $-5x^6$, $6$, $-3x$ *and* $x^2$. *We arrange them in descending order, from highest degree term (the degree* 6 *term) to the lowest degree term (the degree* 0 *term).*

*Written additively, the polynomial is written*

$$(-5x^6) + (4x^3) + (x^2) + (-3x) + (6),$$

*where we have used parentheses only to highlight the separate terms. More simply, the polynomial would be written in descending order as*

$$-5x^6 + 4x^3 + x^2 - 3x + 6.$$

The above example emphasizes again the importance of the ability to interchange between subtracting and "adding the opposite."

### 6.1.1 Exercises

Decide whether or not the following algebraic expressions are polynomials.

1. $x^{100} - \pi x^2$
2. $6x - 18x^3$
3. $\dfrac{9x+2}{4}$
4. $x^2 + x + 3 - \frac{2}{x^2}$
5. $5^x - 2^x$

For each of the following polynomials, (a) list the terms; (b) for each term, identify the degree of the term and the coefficient of the term; (c) rewrite the polynomial in descending order; and (d) identify the degree of the polynomial.

6. $3 - x$
7. $5 + x^2 - 3x$
8. $1 - x + x^2 - x^3 + \dfrac{x^4}{2}$

## 6.2 Adding and subtracting polynomials

With this introduction to the terminology of polynomials, we now proceed to the "arithmetic of polynomials" that we discussed in the introduction. In other words, we will combine two polynomials using the basic operations of addition, subtraction, multiplication, and division, to obtain (most of the time!) a new polynomial.

The key idea to understand how to add and subtract polynomials is the idea of "like terms." We have already seen this concept several times. We saw it in the course of solving linear equations: we combined (meaning added or subtracted) variable terms with variable terms and constant terms with constant terms. We even saw the idea earlier: it is the reason why, when adding fractions, that we need to have a common denominator.

The basic idea of combining like terms can be seen by looking at some simple examples using whole number coefficients, where we can represent multiplication as repeated addition. For example, $3x^4$ simply means $x^4 + x^4 + x^4$. Thinking in this way, we could represent the sum $3x^4 + 2x^4$ as

$$(x^4 + x^4 + x^4) + (x^4 + x^4),$$

which (by virtue of the associative law) is the same as $x^4 + x^4 + x^4 + x^4 + x^4$, or just $5x^4$.

The equation[1] $3x^4 + 2x^4 = 5x^4$ has another justification, which is better suited to explaining a general rule. Applying the distributive law[2],

$$3x^4 + 2x^4 = (3 + 2)x^4 = 5x^4.$$

The advantage of this way of looking at the sum is that it holds for *any* coefficients, not just for whole number coefficients.

Of course, both of the above approaches depended on the fact that both terms being added were "$x^4$-terms." If we tried to apply either of the above strategies to the sum $3x^4 + 2x^3$, both would fail. In fact, these two terms cannot be "combined" at all.

The essence of the above discussion can be summarized in the following points:

- Two terms are called *like terms* if they have the same variable part. In other words, the terms should involve the same variables, and each variable should have the same exponent.

- Like terms can be *combined* by adding (or subtracting) their coefficients. Symbolically, $ax^n + bx^n = (a + b)x^n$. Notice the variable part does not change as a result of adding like terms.

[1] Notice that this equation, unlike most of the equations we saw in Chapter 2, is an *identity*: it is true for all values of $x$.

[2] Thinking of the variable as an (unknown) number, the distributive law needs no extra justification. Considering the variable as a symbol, however, the distributive law technically must be "extended" to apply in the setting of variables as well as of numbers.

- Terms that are not like terms cannot be combined. Their sum must be represented by two (unlike) terms.

### 6.2.1 Adding polynomials

With this said, addition of polynomials is nothing more than "combining like terms."

**Example 6.2.1.** *Add:* $(3x^2 - 2x + 1) + (4x^2 + 6x - 7)$.

**Answer.** *The parentheses in this problem are grouping symbols, written to emphasize that we are adding two polynomials,* $3x^2 - 2x + 1$ *and* $4x^2 + 6x - 7$. *However, thanks to the associative and commutative properties of addition, the parentheses have no impact whatsoever on the problem: we can rearrange and group the terms any way we like. In particular,*

$$\begin{aligned} (3x^2 - 2x + 1) + (4x^2 + 6x - 7) & \\ 3x^2 - 2x + 1 + 4x^2 + 6x - 7 & \quad \textit{Removing parentheses} \\ (3x^2 + 4x^2) + (-2x + 6x) + (1 - 7) & \quad \textit{Grouping like terms} \\ 7x^2 + 4x - 6 & \quad \textit{Combining like terms} \end{aligned}$$

*The answer is* $7x^2 + 4x - 6$.

It is common practice when adding polynomials to take advantage of "column notation." Just like when adding numbers with many digits, different columns represent different "place values" (powers of ten), columns can be used in adding polynomials so that different columns represent different like terms. We will always write polynomials in descending order when using column notation.

For example, in the previous example, we can write

$$\begin{array}{rrrrrrr} & 3x^2 & - & 2x & + & 1 \\ + & 4x^2 & + & 6x & - & 7 \\ \hline & 7x^2 & + & 4x & - & 6 \end{array}$$

(We will many times not write the + symbol in front of the second polynomial when using column notation, with addition being assumed.)

Notice also using column notation that all like terms are *added*. So in the second column, for example, we are adding the terms $-2x$ and $6x$. Speaking loosely, we could say "the minus sign applies to the coefficient of the term."

Using column notation, it is especially important to pay attention to "missing terms," as the following example illustrates.

**Example 6.2.2.** *Add:* $(x^4 - 5x^2 + 2x - 6) + (x^3 - 8x - 3)$.

**Answer.** *Writing the sum in column notation,*

$$\begin{array}{ccccccccc} x^4 & & & - & 5x^2 & + & 2x & - & 6 \\ & & x^3 & & & - & 8x & - & 3 \\ \hline x^4 & + & x^3 & - & 5x^2 & - & 6x & - & 9 \end{array}$$

*Notice again that*

- *The addition of the two polynomials is not explicitly written; it is assumed that the polynomials in the two rows are added;*
- *The columns are always added, with a minus sign considered as the sign of the coefficient of the term involved.*

*The answer is* $x^4 + x^3 - 5x^2 - 6x - 9$.

**Example 6.2.3.** *Add:* $(x^3 - x^2 + 4x - 7) + (3x^2 - 4x + 1)$.

**Answer.** *Rewriting the sum in column notation:*

$$\begin{array}{ccccccc} x^3 & - & x^2 & + & 4x & - & 7 \\ & & 3x^2 & - & 4x & + & 1 \\ \hline x^3 & + & 2x^2 & & & - & 6 \end{array}$$

*Notice that in the third column, the sum of* $4x$ *and* $-4x$ *is* $0x$, *which is simply* $0$ *(this property of* $0$, *familiar in the setting of numbers, extends to variables as well). Since adding* $0$ *does not change the quantity being added, we do not need to write the* $0$ *term, unless it is the only term in the polynomial remaining.*

*The answer is* $x^3 + 2x^2 - 6$.

### 6.2.2 Subtracting polynomials

To subtract polynomials, we will follow the same strategy that we used to subtract signed numbers: we will think of subtraction as “adding the opposite.” We only need to think carefully of what we mean by the opposite of a polynomial.

We have used the word ”opposite” in the sense that two numbers are opposites if their sum is zero. We will use the word in exactly the same way for polynomials: *two polynomials are opposites if their sum is zero.*

**Example 6.2.4.** *The following are examples of polynomials which are opposites:*

- *The opposite of* $2x - 4$ *is* $-2x + 4$.
- *The opposite of* $x^3 - 6x^2 - 7x + 2$ *is* $-x^3 + 6x^2 + 7x - 2$.
- *The opposite of* $1 - t^3$ *is* $t^3 - 1$.

**Exercise 6.2.5.** *For each pair of polynomials in the previous example, add the polynomials to show that their sum is zero in order to confirm that the polynomials are opposites.*

In the case of numbers, we used the symbol $-$ to represent "the opposite of." So $-2$ means the same as "the opposite of 2," and $-(-6)$ means the same as "the opposite of $-6$."

We will use the same understanding of the $-$ symbol in the case of polynomials. Rewriting the example above, we have

- $-(2x-4)$ means "the opposite of $2x-4$," or $-2x+4$.
- $-(x^3-6x^2-7x+2)$ means "the opposite of $x^3-6x^2-7x+2$," or $-x^3+6x^2+7x-2$.
- $-(1-t^3)$ means "the opposite of $1-t^3$," or $t^3-1$.

From even these few examples, you should be able to see an important pattern that we will always use in practice: *To find the opposite of a polynomial, change the sign of the coefficient of every term in the polynomial.*

With this background, we can interpret subtraction of polynomials.

**Subtraction of polynomials**

To subtract two polynomials, add the first polynomial to the opposite of the second polynomial.

**Example 6.2.6.** *Subtract:* $(3x-5)-(2x+1)$.

**Answer.** *We rewrite the subtraction problem as "adding the opposite:"*

$$(3x-5)+(-2x-1).$$

*Notice that the first polynomial remains the same; we add the opposite of the second polynomial, which was originally* $2x+1$.

*Now, combining like terms,*

$$\begin{aligned} &(3x-5)+(-2x-1) && \\ &3x-5+(-2x)+(-1) && \\ &[3x+(-2x)]+[-5+(-1)] && \textit{Grouping like terms} \\ &x-6 && \textit{Combining like terms} \end{aligned}$$

*The answer is* $x-6$.

**Example 6.2.7.** *Subtract:* $(t^3-2t^2+t-5)-(2t^3+t-4)$.

**Answer.** *Rewriting,*

$$(t^3-2t^2+t-5)+(-2t^3-t+4).$$

*In column notation, we add*

$$\begin{array}{rcrcrcr} t^3 & - & 2t^2 & + & t & - & 5 \\ -2t^3 & & & - & t & + & 4 \\ \hline -t^3 & - & 2t^2 & & & - & 1 \end{array}$$

*The answer is* $-t^3 - 2t^2 - 1$.

As a final example, we remind the reader that when a subtraction is indicated by a sentence of the form, "Subtract $X$ from $Y$," the first quantity appears second in the difference, as $Y - X$. Unlike addition, subtraction is not commutative—the order that we write the terms does affect the outcome.

**Example 6.2.8.** *Subtract* $x^2 - 2x - 5$ *from* $8x + 2$.

**Answer.** *Translated into algebra, the problem asks to perform the following subtraction problem:*

$$(8x + 2) - (x^2 - 2x - 5).$$

*Now, rewriting as an addition problem,*

$$\begin{aligned} &(8x + 2) + (-x^2 + 2x + 5) \\ &8x + 2 + (-x^2) + 2x + 5 \\ &-x^2 + (8x + 2x) + (2 + 5) && \textit{Grouping like terms} \\ &-x^2 + 10x + 7 && \textit{Combining like terms} \end{aligned}$$

*The answer is* $-x^2 + 10x + 7$.

### 6.2.3 Exercises

Perform the indicated operations.

1. $(x^2 - 5x - 6) + (2x^2 + 2x + 4)$
2. $(5x^3 + 2x^2 - 3x + 3) + (-x^3 - 3x^2 + 2x + 4)$
3. $(y^2 + 5y - 1) - (-3y^2 + 2y - 4)$
4. $(x^3 - x^2 - x + 1) - (4x^3 - 3x^2 + 2x + 4)$
5. $(3x^3 - 2x + 1) - (-x^3 - 3x^2 + 2x + 4)$
6. Subtract $4x + 2$ from $-x + 15$
7. Subtract $x^2 - 4x - 1$ from $2x^2 - 2x + 5$

## 6.3 Properties of exponents

Up to this point, the exponents with a variable base appearing in polynomials have mainly served to distinguish between like and unlike terms. Going further, however, we will need to pay more attention to how the terms involving exponents interact when multiplied or divided.

The following table summarizes key properties of exponents. Here, in general, $x$ and $y$ represent bases, which will be either a number or a variable. The exponents $a$ and $b$ will for now represent numbers, but we will be a little vague here about exactly what kind of numbers they are (in the case of polynomials, the exponents will be whole numbers).

**Properties of exponents**

(E1) $x^a \cdot x^b = x^{a+b}$.

(E2) $\dfrac{x^a}{x^b} = x^{a-b}$ (as long as $x \neq 0$).

(E3) $x^0 = 1$ (as long as $x \neq 0$).

(E4) $(x \cdot y)^a = x^a \cdot y^a$.

(E5) $(x^a)^b = x^{a \cdot b}$.

(E6) $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$ (as long as $y \neq 0$).

Before we provide examples to show these properties "in action," it is worth making some comments about these properties.

- Notice that the properties all involve the operations of multiplication or division. One might say, "Exponents behave nicely with multiplication and division." The interaction between exponents and addition (or subtraction) is more complicated, as we will see below.
- The key feature of properties (E1) and (E2) is that the factors being multiplied or divided have *the same base* (denoted by $x$).
- The key feature of properties (E4), (E5), and (E6) is that the base of the exponential on the left hand side involves only the operations of multiplication, division, or exponentiation.
- Most of the properties (with the exception of (E3)) are easy to justify when the exponents are positive whole numbers. In that case, writing

exponents as "repeated multiplication," the properties follow directly from the commutative and associative properties of multiplication along with the definition of division as an inverse operation to multiplication.

- Property (E3) is of a different nature than the others; for this reason, it is sometimes hardest to justify. The reason is that it has no interpretation as a "repeated multiplication," since the phrase, "Multiply the base $x$ by itself 0 times" is meaningless. Instead, (E3) follows from formally extending property (E2), in the following sense. Consider the expression $\frac{x^a}{x^a}$, where for the moment we will consider $a$ to be a positive whole number and $x$ any nonzero number (so that the denominator is not zero!). On the one hand, any nonzero number divided by itself is 1: $\frac{x^a}{x^a} = 1$. On the other hand, if we insist that (E2) must hold, we have $\frac{x^a}{x^a} = x^{a-a} = x^0$. For this reason, if (E2) is to hold, the only way to define $x^0$ in a consistent way is $x^0 = 1$, as in (E3).

Properties (E1)–(E6) are most useful when the bases involved are variables. In the following examples, we will use the common word "simplify" to mean "use the relevant properties of exponents to write in an equivalent, simpler form."

**Example 6.3.1.** *Simplify:* $(w^5x^8)(w^2x^3)$.

**Answer.** *The only operation involved is multiplication, so we can change the order and grouping of the factors at will, relying on the commutative and associative properties of multiplication.*

$$\begin{aligned} &(w^5x^8)(w^2x^3) && \\ &(w^5w^2)(x^8x^3) && \textit{Grouping factors with the same base} \\ &(w^{5+2})(x^{8+3}) && \textit{Property (E1)} \\ &w^7x^{11}. && \end{aligned}$$

*The answer is $w^7x^{11}$. Notice the the bases of the remaining exponentials ($w$ and $x$) are different, and so no further simplification is possible.*

**Example 6.3.2.** *Simplify:* $\frac{x^3y^5}{x^3y}$.

**Answer.** *This time we have an expression involving multiplication and division. The strategy will be the same: grouping factors with the same base.*

$$\begin{aligned}
&\frac{x^3y^5}{x^3y} \\
&\left(\frac{x^3}{x^3}\right)\cdot\left(\frac{y^5}{y}\right) && \textit{Grouping factors with the same base} \\
&(x^{3-3})(y^{5-1}) && \textit{Property (E2)} \\
&x^0y^4 \\
&1\cdot y^4 && \textit{Property (E3)} \\
&y^4.
\end{aligned}$$

*The answer is* $y^4$. *Notice that the in the factor* $\dfrac{y^5}{y}$, *the exponent of the denominator* $y$ *is* 1.

**Example 6.3.3.** *Simplify:* $\left(\dfrac{x^8y^4}{x^2}\right)^3$.

**Answer.** *In this case, we will simplify the expression inside the grouping symbols first.*

$$\begin{aligned}
&\left(\frac{x^8y^4}{x^2}\right)^3 \\
&\left(\frac{x^8}{x^2}\cdot\frac{y^4}{1}\right)^3 && \textit{Grouping factors with the same base} \\
&(x^{8-2}\cdot y^4)^3 && \textit{Property (E2)} \\
&(x^6y^4)^3 \\
&(x^6)^3\cdot(y^4)^3 && \textit{Property (E4)} \\
&x^{6\cdot 3}y^{4\cdot 3} && \textit{Property (E5)} \\
&x^{18}y^{12}.
\end{aligned}$$

*The answer is* $x^{18}y^{12}$.

**Example 6.3.4.** *Simplify:* $(3x^2y^7)^4$.

**Answer.** *In this example, the base is a product of three factors:* 3, $x^2$, *and* $y^7$. *Property (E4), applied to this situation, implies that each factor separately must be raised to the fourth power*[3].

$$\begin{aligned}
&(3x^2y^7)^4 \\
&(3)^4\cdot(x^2)^4\cdot(y^7)^4 && \textit{Property (E4)} \\
&81\cdot x^{2\cdot 4}\cdot y^{7\cdot 4} && \textit{Property (E5)} \\
&81x^8y^{28}.
\end{aligned}$$

[3] Property (E4) is stated for a base which is the product of two factors. However, in the case of three factors, we can apply the property twice: $(x\cdot(y\cdot z))^a = x^a\cdot(y\cdot z)^a = x^a\cdot y^a\cdot z^a$.

*The answer is* $81x^8y^{28}$.

**Example 6.3.5.** *Simplify:* $\dfrac{(2xy^4)^3}{x^2y^5}$.

**Answer.** *We will first simplify the numerator, then separate the factors according to common bases.*

$$\begin{aligned}
&\frac{(2xy^4)^3}{x^2y^5} && \\
&\frac{(2)^3(x)^3(y^4)^3}{x^2y^5} && \textit{Property (E4)} \\
&\frac{8x^3y^{12}}{x^2y^5} && \textit{Property (E5)} \\
&\left(\frac{8}{1}\right)\cdot\left(\frac{x^3}{x^2}\right)\cdot\left(\frac{y^{12}}{y^5}\right) && \textit{Grouping factors with the same base} \\
&(8)(x^{3-2})(y^{12-5}) && \textit{Property (E2)} \\
&8xy^7. &&
\end{aligned}$$

*The answer is* $8xy^7$. *Notice that in the last step, as usual, we do not write the exponent* 1*:* $x^1 = x$. *In the grouping step, we grouped the whole number* 8 *in the numerator, writing it as a fraction* 8/1 *for the sake of seeing the multiplication more clearly.*

### 6.3.1 Integer exponents

The properties of exponents listed above are enough for most of the work we will do with polynomials. However, it is worth pointing out that they also give a way to define negative exponents in a way consistent with our understanding of exponents as repeated multiplication, similar to the way that the definition $x^0 = 1$ is required if the other properties are to be satisfied.

Namely, we make the following definition:

For any nonzero base $x \neq 0$ and any exponent $a$, we define

$$x^{-a} = \frac{1}{x^a}.$$

In particular,

$$x^{-1} = \frac{1}{x}.$$

To see that this definition is consistent with the properties above, notice that on the one hand, $x^{0-a} = x^{-a}$. On the other hand, if properties (E2) and (E3) are to hold, we have $x^{0-a} = \dfrac{x^0}{x^a} = \dfrac{1}{x^a}$. In other words, to be consistent with the standard properties of whole number exponents, the only possible definition for negative exponents is the one we have stated, $x^{-a} = 1/x^a$.

Since this definition is less intuitive than our usual understanding of whole number exponents, we list some numerical examples of this definition.

**Example 6.3.6.** *Find the values of each of the following exponentials.*

*(a)* $3^{-1}$

*(b)* $2^{-5}$

*(c)* $10^{-4}$

*(d)* $(-2)^{-3}$

*(e)* $\left(\dfrac{2}{5}\right)^{-1}$

*(f)* $\left(\dfrac{3}{4}\right)^{-2}$

**Answer.** *(a)* $3^{-1} = \dfrac{1}{3^1} = \dfrac{1}{3}$.

*(b)* $2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$.

*(c)* $10^{-4} = \dfrac{1}{10^4} = \dfrac{1}{10000} = 0.0001$.

*(d)* $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$.

*(e)* $\left(\dfrac{2}{5}\right)^{-1} = \dfrac{1}{\left(\frac{2}{5}\right)^1} = \dfrac{1}{\left(\frac{2}{5}\right)} = \dfrac{1}{1} \cdot \dfrac{5}{2} = \dfrac{5}{2}$.

*(f)* $\left(\dfrac{3}{4}\right)^{-2} = \dfrac{1}{\left(\frac{3}{4}\right)^2} = \dfrac{1}{\frac{9}{16}} = \dfrac{1}{1} \cdot \dfrac{16}{9} = \dfrac{16}{9}$.

The previous examples provide evidence that negative exponents do not affect the sign of the result, but instead indicate a reciprocal. This is not surprising if we think of exponents as repeated multiplication; the "opposite" sign indicates the "opposite" in the sense of multiplication, which is the notion of reciprocal. Note especially in the last example that $\left(\dfrac{3}{4}\right)^{-2} = \left(\dfrac{4}{3}\right)^2$.

Since we have defined negative exponents in such a way to be consistent with the familiar properties of exponents, we can manipulate and simplify expressions involving negative exponents exactly as with positive exponents, as the following examples illustrate.

**Example 6.3.7.** *Simplify:* $\dfrac{x^7 \cdot x^{-9}}{x^{-4}}$. *Write the answer using only positive exponents.*

**Answer.**

$$\begin{aligned} &\frac{x^7 \cdot x^{-9}}{x^{-4}} \\ &\frac{x^{7+(-9)}}{x^{-4}} && \textit{Property (E1)} \\ &\frac{x^{-2}}{x^{-4}} \\ &x^{(-2)-(-4)} && \textit{Property (E2)} \\ &x^{(-2)+4} \\ &x^2. \end{aligned}$$

*The answer is* $x^2$.

*As usual, when we subtract negative numbers, we took the trouble to rewrite the subtraction as "adding the opposite."*

**Example 6.3.8.** *Simplify:* $\left(\dfrac{x^{-3}y}{x^{-5}y^8}\right)^{-2}$. *Write the answer using only positive exponents.*

**Answer.** *We will simplify inside the grouping symbols first:*

$$\begin{aligned} &\left(\frac{x^{-3}y}{x^{-5}y^8}\right)^{-2} \\ &\left(\frac{x^{-3}}{x^{-5}} \cdot \frac{y}{y^8}\right)^{-2} && \textit{Grouping common bases} \\ &\left(x^{(-3)-(-5)} \cdot y^{1-8}\right)^{-2} && \textit{Property (E2)} \\ &\left(x^{(-3)+(5)}y^{1+(-8)}\right)^{-2} && \textit{Subtraction as "adding the opposite"} \\ &\left(x^2y^{-7}\right)^{-2} \\ &(x^2)^{-2}(y^{-7})^{-2} && \textit{Property (E4)} \\ &x^{(2)(-2)}y^{(-7)(-2)} && \textit{Property (E5)} \\ &x^{-4}y^{14} \\ &\frac{1}{x^4} \cdot \frac{y^{14}}{1} && \textit{Rewriting negative exponent as a reciprocal} \\ &\frac{y^{14}}{x^4}. \end{aligned}$$

*The answer is* $y^{14}/x^4$. *Notice that while the equivalent expression* $x^{-4}y^{14}$ *is just as "simple" as the final answer, we took the extra effort to write the* *answer using only the more familiar whole number exponents, as requested in the problem.*

*The reader should be advised that the above approach to simplifying is not the only possible route to the final answer. For example, Property (E6) of exponents could be applied to the expression as a first step. (Check to see that the final answer is the same!)*

### 6.3.2 Exercises

Use properties of exponents to simplify the following algebraic expressions.

1. $(x^2)(x^5)$

2. $(z^4)^3$

3. $(4x^3)^{-2}$

4. $\dfrac{(a^2b^3)^2}{a^3b^5}$

5. $\dfrac{x^5 \cdot y^2}{(y^2)^3}$

6. $\dfrac{y^5 \cdot y^2}{(y^2)^3}$

The following exercises illustrate the error of the common mistake of applying a "rule" to equate $(a+b)^2$ with $a^2+b^2$. For the given values of $a$ and $b$ below, evaluate (a) $(a+b)^2$ and (b) $a^2+b^2$.

7. $a = 2$, $b = 3$

8. $a = -1$, $b = 2$

## 6.4 A detour: Scientific notation

The distance from the sun to Earth is, on average, approximately 93,000,000 miles. The speed of light (in a vacuum) is approximately 300,000,000 meters per second. The radius of a hydrogen atom is approximately 0.000000000053 meters.

In many fields of science, we are faced with either very large or very small quantities, like the ones in the previous paragraph. Scientific notation is a convenient way of treating such numbers. In this section, we briefly review scientific notation. Even though it is not related to polynomials (or even algebra) as such, it will give us an opportunity to practice using the properties of exponents.

**Scientific notation**

A number is written in *scientific notation* if it has the form

$$a \times 10^n,$$

where

- $a$ is a number whose magnitude is between 1 and 10, possibly being 1 but strictly less than 10. Symbolically, $1 \le |a| < 10$.
- $n$ is any *integer* (positive, negative or zero).

Notice the unfortunate but completely standard use of the "times" symbol $\times$ representing the multiplication involved in scientific notation. We will refer to the number $a$ as the "number part" (or coefficient) of the number written in scientific notation to distinguish it from the "exponential part." (Technically, of course, all "parts" of a number written in scientific notation are numbers.)

For example, the following numbers are written in scientific notation:

- $2 \times 10^8$;
- $-4.5 \times 10^{15}$;
- $3.14 \times 10^{-4}$.

The following numbers are not written in scientific notation (can you see why?):

- $22 \times 10^{-14}$;
- $755.88$;
- $10 \times 10^4$;
- $-8 \times 10^{3/2}$.

Converting between scientific notation and standard (decimal) notation is accomplished with the help of the nice properties of powers of ten.

**Example 6.4.1.** *Convert each of the following numbers in scientific notation into standard (decimal) notation.*

*(a)* $2 \times 10^8$*;*

*(b)* $-4.5 \times 10^{15}$*;*

*(c)* $3.14 \times 10^{-4}$.

**Answer.** *(a)* $2 \times 10^8 = 2(100000000) = 200,000,000$.

*(b)* $-4.5 \times 10^{15} = -4.5(1000000000000000) = -4,500,000,000,000,000.$

*(c)* $3.14 \times 10^{-4} = 3.14\left(\frac{1}{10^4}\right) = 3.14\left(\frac{1}{10000}\right) = 3.14(0.0001) = 0.000314.$

The clever reader can certainly see a way to describe a shortcut based on the three examples above in terms of "moving the decimal place."

**Example 6.4.2.** *Write each of the following numbers in scientific notation.*

*(a)* $93,000,000$

*(b)* $300,000,000$

*(c)* $0.000000000053.$

**Answer.** *In order to determine the correct power of ten in writing a number in scientific notation, first identify what we will call the* leading digit*, meaning the first non-zero digit appearing in the number reading from left to right. So the leading digit in (a) would be* $9$*; the leading digit in (b) would be* $3$*; and the leading digit in (c) would be* $5$*.*

*The exponent of* $10$ *of the number written in scientific notation will be determined by the place value of the leading digit. (This can be determined by counting the digits between the leading digit and the units digit, not including the units digit but including the leading digit if it is not the same as the units digit.) So in (a), the* $9$ *is in the position with place value* $10^7$*; in (b), the* $3$ *is in the position with place value* $10^8$*; and in (c), the* $5$ *is in the position with place value* $10^{-11}$*.*

*Putting this together, we obtain:*

*(a)* $93,000,000 = 9.3 \times 10^7$*;*

*(b)* $300,000,000 = 3 \times 10^8$*;*

*(c)* $0.000000000053 = 5.3 \times 10^{-11}$*.*

Keep in mind that "big" numbers have positive powers of ten in scientific notation, while "small" numbers have negative powers of ten.

### 6.4.1 Multiplication and division of numbers in scientific notation

Because scientific notation has multiplication "built in" to the notation, performing the operations of multiplication and division with numbers in scientific notation are particularly simple using the rules of exponents. We will illustrate this sentence with the following examples.

**Example 6.4.3.** *Multiply:* $(3 \times 10^7)(1.5 \times 10^{-2})$.

**Answer.** *The first thing to notice is that since the only operations appearing are multiplication, we can use the commutative and associative properties to re-order and re-group the factors:*

$$(3 \times 10^7)(1.5 \times 10^{-2}) = (3)(1.5)(10^7)(10^{-2}).$$

*Since scientific notation always involves multiples of the same base (ten), we can apply property (E1), adding the exponents:*

$$\begin{aligned}(3 \times 10^7)(1.5 \times 10^{-2}) &= (3)(1.5)(10^7)(10^{-2}) \\ &= (4.5)(10^{7+(-2)}) \\ &= 4.5 \times 10^5.\end{aligned}$$

*The answer is* $4.5 \times 10^5$.

Summarizing the previous example, multiplying numbers in scientific notation involves multiplying their "number part" and adding the exponents of the powers of ten.

The following example shows that the product of two numbers written in scientific notation will not automatically result in a number written in scientific notation.

**Example 6.4.4.** *Multiply:* $(5 \times 10^8)(3 \times 10^4)$.

**Answer.** *Following the previous strategy:*

$$\begin{aligned}(5 \times 10^8)(3 \times 10^4) &= (5)(3)(10^8)(10^4) \\ &= (15)(10^{8+4}) \\ &= 15 \times 10^{12}.\end{aligned}$$

*Unfortunately, the result is not in scientific notation, since the "number part"* 15 *has magnitude greater than* 10. *We will approach this problem by writing the number part in scientific notation, then again using the fact of having a common base of* 10 *to apply Property (E1) of exponents again:*

$$\begin{aligned}(5 \times 10^8)(3 \times 10^4) &= 15 \times 10^{12} \\ &= (1.5 \times 10^1)(10^{12}) \\ &= 1.5 \times 10^{1+12} \\ &= 1.5 \times 10^{13}.\end{aligned}$$

*The answer, written in scientific notation, is* $1.5 \times 10^{13}$.

Our approach to dividing numbers written in scientific notation is similar to multiplication, but we will use Property (E2) of dividing exponentials with a common base.

**Example 6.4.5.** *Divide:* $\dfrac{6 \times 10^{-2}}{4 \times 10^{-5}}$.

**Answer.** *The only difference between division and multiplication is that we need to group the numerators and denominators carefully:*

$$\begin{aligned}\frac{6\times 10^{-2}}{4\times 10^{-5}} &= \frac{6}{4}\cdot\frac{10^{-2}}{10^{-5}}\\ &= 1.5\times 10^{(-2)-(-5)}\\ &= 1.5\times 10^{(-2)+(5)}\\ &= 1.5\times 10^{3}.\end{aligned}$$

*The answer is* $1.5\times 10^3$. *Notice that we took the trouble to rewrite the subtraction of exponents as addition of the opposite.*

**Example 6.4.6.** *Divide:* $\dfrac{4\times 10^{-5}}{8\times 10^{12}}$.

**Answer.** *Following the procedure of the previous examples,*

$$\begin{aligned}\frac{4\times 10^{-5}}{8\times 10^{12}} &= \frac{4}{8}\cdot\frac{10^{-5}}{10^{12}}\\ &= 0.5\times 10^{(-5)-(12)}\\ &= 0.5\times 10^{(-5)+(-12)}\\ &= 0.5\times 10^{-17}.\end{aligned}$$

*Unfortunately, we again are in the situation where the result is not written in scientific notation; the "number part" has magnitude less than* 1. *Applying a similar strategy as the one in Example 6.4.4,*

$$\begin{aligned}\frac{4\times 10^{-5}}{8\times 10^{12}} &= 0.5\times 10^{-17}\\ &= (5\times 10^{-1})\times 10^{-17}\\ &= 5\times 10^{(-1)+(-17)}\\ &= 5\times 10^{-18}.\end{aligned}$$

*The answer is* $5\times 10^{-18}$.

### 6.4.2 Exercises

Write the following numbers in scientific notation.

1. $7,500,000,000,000,000,000$ (the approximate number of grains of sand on the planet)
2. 0.000000000275 (the approximate diameter of a water molecule, measured in meters)

Write the following numbers in standard (decimal) notation.

3. $6.022 \times 10^{23}$ (the approximate number of carbon atoms in 12 grams of pure carbon)
4. $1 \times 10^{-3}$ (the number of liters in one milliliter)

Perform the indicated operation. Write your answer in scientific notation.

5. Multiply: $(6 \times 10^4)(3 \times 10^3)$
6. Divide: $\dfrac{4 \times 10^6}{2 \times 10^8}$
7. Multiply: $(7 \times 10^{-4})(1.3 \times 10^{-10})$
8. Divide: $\dfrac{-1.53 \times 10^1}{-3 \times 10^{-3}}$
9. Multiply: $(3.4 \times 10^8)(2.1 \times 10^{-5})$
10. Divide: $\dfrac{3 \times 10^{-12}}{6 \times 10^{-4}}$
11. Multiply: $(1.55 \times 10^{10})(8.1 \times 10^{-10})$
12. Divide: $\dfrac{-2 \times 10^5}{8 \times 10^{-5}}$

## 6.5 Multiplying polynomials

Returning to the subject of the arithmetic of polynomials, we now turn to multiplication. Since polynomials (in one variable $x$) are made up of terms having the form $ax^n$, the results of the previous section will apply, especially Property (E1) of exponents.

Notice that multiplying a monomial by a monomial involves nothing more that applying Property (E1) directly, as the following example shows.

**Example 6.5.1.** *Multiply:* $(15x^3)(4x^2)$.

**Answer.** *The two polynomials being multiplied,* $15x^3$ *and* $4x^2$*, each have one term. To multiply them, we will apply the associative and commutative properties to regroup the factors, the apply Property (E1):*

$$\begin{array}{rl} (15x^3)(4x^2) & \\ (15 \cdot 4)(x^3 \cdot x^2) & \textit{(Re-grouping the factors)} \\ 60x^{3+2} & \textit{(Property (E1))} \\ 60x^5. & \end{array}$$

*The answer is* $60x^5$.

Normally, we will not illustrate the regrouping as a separate step.

In order to multiply polynomials with more than one term, we will need to remember the *distributive law.* The distributive law describes an important relationship between multiplication and addition (which, in the case of operations with whole numbers, is based on multiplication as repeated addition).

Symbolically, the distributive law is often summarized by the identity

$$a(b + c) = ab + ac. \tag{6.1}$$

It's worth paying a little more attention to what Equation 6.1 is really saying. On the left hand side, there are two operations, addition and multiplication; the order of operations dictates that the sum, which is grouped, is performed before the multiplication. On the right hand side, there are three operations: two multiplications and one addition. The order of operations on the right dictates that the two multiplications are performed first, followed by the addition. The distributive law gives a precise way that the order of operations between addition and multiplication can be changed. In words, the product of two factors, one of which is a sum of two terms, is the same as the sum of the product of the first factor with each of the two terms involved in the sum.

Before going further, let's apply the straightforward expression of the distributive law to the product of a monomial with a binomial.

**Example 6.5.2.** *Multiply:* $(3x^2)(2x + 7)$.

**Answer.**

$$\begin{aligned}(3x^2)(2x+7) &= (3x^2)(2x) + (3x^2)(7) && \textit{(The distributive law)}\\ &= 6x^{2+1} + 21x^2 && \textit{(Property (E1))}\\ &= 6x^3 + 21x^2.\end{aligned}$$

*The answer is* $6x^3 + 21x^2$.

It's also worth noting that even though the distributive law is written with only one of the factors involving a sum, it in fact applies more generally, keeping again the commutative and associative laws in mind. For example, try to justify each step in the following sequence of identities (each step involves applying one of either the commutative law, the associative law, or the distributive law as stated above):

$$\begin{aligned}(a+b+c)(x+y) &= (a+b+c)(x) + (a+b+c)(y)\\ &= (x)(a+b+c) + (y)(a+b+c)\\ &= (x)(a+(b+c)) + (y)(a+(b+c))\\ &= (x)(a) + (x)(b+c) + (y)(a) + (y)(b+c)\\ &= xa + (xb+xc) + ya + (yb+yc)\\ &= ax + bx + cx + ay + by + cy.\end{aligned}$$

What is most important about the above sequence identities is not really the in-between steps (although pointing out the different laws at work would make your fourth-grade math teacher smile!). We started with the product of a factor with three terms $a + b + c$ with a factor with two terms $x + y$. The final expression involves the sum of six ($= 3 \times 2$) multiplications. Each of the six multiplications involves one term from the first expression and one term from the second expression. Moreover, each term from the first expression is "matched" with each term in the second expression, which is why we ended up with six multiplications.

Let's summarize the distributive law in the following way:

**The distributive law**

The product of two factors, each of which is a sum of several terms, is the same as the sum of terms obtained by multiplying each term of the first factor by each term of the second factor.

Let's see a few examples of the distributive law in action in multiplying polynomials. You will notice that in these cases, after applying the distributive law and properties of exponents, like terms often appear—which can (and should) then be combined!

**Example 6.5.3.** *Multiply:* $(x + 4)(x + 2)$.

**Answer.** *Notice that in multiplying a polynomial with two terms by a polynomial with two terms will result, applying the distributive law, to* $2 \times 2 = 4$ *multiplications:*

$$\begin{aligned} &(x + 4)(x + 2) \\ &(x)(x) + (x)(2) + (4)(x) + (4)(2) && \textit{(Distributing)} \\ &x^2 + 2x + 4x + 8 && \textit{(Multiplying in each term)} \\ &x^2 + 6x + 8. && \textit{(Combining like terms)} \end{aligned}$$

*The answer is* $x^2 + 6x + 8$.

**Example 6.5.4.** *Multiply:* $(2x + 5)(3x + 2)$.

**Answer.**

$$\begin{aligned} &(2x + 5)(3x + 2) \\ &(2x)(3x) + (2x)(2) + (5)(3x) + (5)(2) && \textit{(Distributing)} \\ &6x^2 + 4x + 15x + 10 && \textit{(Multiplying in each term)} \\ &6x^2 + 19x + 10. && \textit{(Combining like terms)} \end{aligned}$$

*The answer is* $6x^2 + 19x + 10$.

The following example illustrates the way that we will approach the distributive law involving subtraction.

**Example 6.5.5.** *Multiply:* $(2x - 1)(x - 6)$.

**Answer.** *Both of the two binomials involve subtraction. However, as usual, we can consider this as a sum by writing the subtraction as "adding the opposite." In particular, the two terms in the first binomial are* $2x$ *and* $-1$*, while the two terms from the second binomial are* $x$ *and* $-6$*. We will write this explicitly when distributing.*

$$\begin{aligned} &(2x-1)(x-6) \\ (2x)(x) + (2x)(-6) + (-1)(x) + (-1)(-6) \quad &\textit{(Distributing)} \\ 2x^2 - 12x - x + 6 \quad &\textit{(Multiplying in each term)} \\ 2x^2 - 13x + 6. \quad &\textit{(Combining like terms)} \end{aligned}$$

*The answer is* $2x^2 - 13x + 6$.

*(Notice that in the third line, we switched back to writing the polynomial using subtraction, instead of writing* $2x^2 + (-12x) + (-x) + 6$ *. Either way of writing would be acceptable, since they both have the same terms, but it is typical to write it as we have done in the answer above.)*

**Example 6.5.6.** *Multiply:* $(x + 4)(x - 4)$.

**Answer.**

$$\begin{aligned} &(x+4)(x-4) \\ (x)(x) + (x)(-4) + (4)(x) + (4)(-4) \quad &\textit{(Distributing)} \\ x^2 - 4x + 4x - 16 \quad &\textit{(Multiplying in each term)} \\ x^2 - 16. \quad &\textit{(Combining like terms)} \end{aligned}$$

*The answer is* $x^2 - 16$*. The "middle" like terms "cancelled" (their sum is zero).*

When we have whole number powers of a polynomial, it is best to take the 10 seconds required to rewrite the problem as a repeated multiplication.

**Example 6.5.7.** *Multiply:* $(3x + 4)^2$.

**Answer.** *We start by making the multiplication explicit.*

$$\begin{aligned} &(3x+4)^2 \\ (3x+4)(3x+4) \quad &\textit{(Exponent as repeated multiplication)} \\ (3x)(3x) + (3x)(4) + (4)(3x) + (4)(4) \quad &\textit{(Distributing)} \\ 9x^2 + 12x + 12x + 16 \quad &\textit{(Multiplying in each term)} \\ 9x^2 + 24x + 16. \quad &\textit{(Combining like terms)} \end{aligned}$$

*The answer is* $9x^2 + 24x + 16$. *Notice that the answer is NOT the same as* $(3x)^2 + (4)^2$ *(refer to the last two exercises in Section 6.3.2)!*

**Example 6.5.8.** *Multiply:* $(2x - 1)(x^2 - 5x + 2)$.

**Answer.** *The distributive law, applied to this multiplication of a binomial with a trinomial, will involve* $2 \times 3 = 6$ *multiplications.*

$$\begin{aligned}
&(2x-1)(x^2-5x+2) \\
&(2x)(x^2)+(2x)(-5x)+(2x)(2)+(-1)(x^2)+(-1)(-5x)+(-1)(2) \quad \textit{(Distributing)} \\
&2x^3-10x^2+4x-x^2+5x-2 \quad \textit{(Multiplying in each term)} \\
&2x^3-11x^2+9x-2. \quad \textit{(Combining like terms)}
\end{aligned}$$

*The answer is* $2x^3 - 11x^2 + 9x - 2$.

The last example illustrates the fact that when multiplying a product of three (or more) factors, we still have to apply the distributive law "one multiplication at a time."

**Example 6.5.9.** *Multiply:* $(x - 3)(x + 2)(2x + 5)$.

**Answer.** *We will introduce square brackets to group the first multiplication:*

$$\begin{aligned}
&[(x-3)(x+2)]\,(2x+5) \\
&[(x)(x)+(x)(2)+(-3)(x)+(-3)(2)]\,(2x+5) \\
&\left[x^2+2x-3x-6\right](2x+5) \\
&(x^2-x-6)(2x+5) \\
&(x^2)(2x)+(x^2)(5)+(-x)(2x)+(-x)(5)+(-6)(2x)+(-6)(5) \\
&2x^3+5x^2-2x^2-5x-12x-30 \\
&2x^3+3x^2-17x-30.
\end{aligned}$$

*The answer is* $2x^3 + 3x^2 - 17x - 30$.

To summarize, multiplying polynomials involves the distributive law, property (E1) of exponents, and combining like terms.

### 6.5.1 Exercises

Multiply the following polynomials.

1. $(x - 3)(x + 2)$
2. $(3x - 4)(2x - 1)$
3. $(2x + 3)^2$

4. $(x-1)(x^2+x+1)$
5. $(x+3)(x^3+2x^2-3x-2)$
6. $(x^2-3x-1)(x^2+2x+3)$
7. $(3x-2)^2$.
8. $(4x-3)(2x-1)(x+2)$.
9. $(2x-5)(x^2+4x-6)$
10. (*) $(2x-1)^3$.
11. (*) Verify the identity $(a+b)^2 = a^2+2ab+b^2$ by applying the distributive law on the left hand side.
12. (*) For what values of $a$ and $b$ does the equality $(a+b)^2 = a^2+b^2$ hold?

## 6.6 Dividing a polynomial by a monomial

Just as in division of whole numbers, division of polynomials raises certain very fundamental issues. The most basic problem is the fact that *the quotient of two polynomials may not be a polynomial.* This can be seen from a very simple example. Consider the quotient of the polynomial $x^2$ divided by the polynomial $x^3$. By the Property (E2) of exponents, above, we have $x^2 \div x^3 = x^{2-3} = x^{-1}$. Even though we can now make perfect sense of the meaning of $x^{-1}$ (as $1/x$), it is not a polynomial, since the exponent is not a whole number.

We will not attempt a full treatment of division of polynomials here. In order to do so, we would need to consider "fractions of polynomials," what are known as *rational expressions* (rational since they are formed by ratios of polynomials). This is usual treated in an "intermediate algebra" course. A more detailed treatment would involve the *division algorithm* and a corresponding "long division" of polynomials. This is usually treated in a precalculus course.

We are interested in those division problems that can be handled using just the properties of exponents and the distributive law. It turns out that this can be done, provided that we divide a polynomial by a *monomial.*

We will be using the distributive law in the following form:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}.$$

In words, *the quotient of several terms (in the numerator) by a single term (in the denominator) is the same as the sum of the quotients of each term (in the numerator) by the term in the denominator.* As an exercise, you may derive this version of the distributive law from the usual one by expressing "division by $c$" as "multiplication by $1/c$."

Here are a few examples of division of polynomials, when the divisor (in the denominator) is a monomial.

**Example 6.6.1.** *Divide:* $\dfrac{27x^6 + 18x^4}{3x^2}$.

**Answer.**

$$\begin{aligned}
\frac{27x^6 + 18x^4}{3x^2} &= \frac{27x^6}{3x^2} + \frac{18x^4}{3x^2} && \textit{(Distributing)} \\
&= 9x^{6-2} + 6x^{4-2} && \textit{Dividing, using property (E2)} \\
&= 9x^4 + 6x^2.
\end{aligned}$$

*The answer is* $9x^4 + 6x^2$.

**Example 6.6.2.** *Divide:* $\dfrac{2y^3 - 12y^2 - 18y}{2y}$.

**Answer.**

$$\begin{aligned}
\frac{2y^3 - 12y^2 - 18y}{2y} &= \frac{2y^3}{2y} + \frac{-12y^2}{2y} + \frac{-18y}{2y} && \textit{(Distributing)} \\
&= y^2 - 6y - 9.
\end{aligned}$$

*The answer is* $y^2 - 6y - 9$. *Notice that we maintained our custom of writing the distributive law using addition, in this case involving adding terms with negative coefficients. Also notice that* $\dfrac{y^2}{y} = y^{2-1} = y^1 = y$, *while* $\dfrac{y}{y} = y^{1-1} = y^0 = 1$.

**Example 6.6.3.** *Divide:* $\dfrac{5x^5 - 10x^4 + 5x^2}{5x^2}$.

**Answer.**

$$\begin{aligned}
\frac{5x^5 - 10x^4 + 5x^2}{5x^2} &= \frac{5x^5}{5x^2} + \frac{-10x^4}{5x^2} + \frac{5x^2}{5x^2} && \textit{(Distributing)} \\
&= x^3 - 2x^2 + 1.
\end{aligned}$$

*The answer is* $x^3 - 2x^2 + 1$. *Pay special attention to the last term:*

$$\begin{aligned}
\frac{5x^2}{5x^2} &= \frac{5}{5} \cdot \frac{x^2}{x^2} \\
&= 1 \cdot x^{2-2} \\
&= 1 \cdot x^0 \\
&= 1 \cdot 1 = 1.
\end{aligned}$$

*Many times this phenomenon is referred to as "canceling." The main thing to remember is that "canceling" does not mean "disappearing!"*

**Example 6.6.4.** *Divide:* $\dfrac{11x^4 - 8x^3 + 10x^2}{4x^3}$.

**Answer.**

$$\begin{aligned}\frac{11x^4 - 8x^3 + 10x^2}{4x^3} &= \frac{11x^4}{4x^3} + -8x^3 4x^3 + 10x^2 4x^3 && \textit{(Distributing)}\\ &= \frac{11}{4}x^{4-3} + \frac{-8}{4}x^{3-3} + \frac{10}{4}x^{2-3}\\ &= \frac{11}{4}x - 2 + \frac{5}{2}x^{-1}.\end{aligned}$$

*The answer is* $(11/4)x - 2 + (5/2)x^{-1}$.
*Two things to notice about this example:*

1. *Because the division of whole number coefficients did not end up so "clean" this time, in the sense that the result did not end up being an integer, we wrote the division step more explicitly.*

2. *The fact that the result includes an* $x^{-1}$*-term (with a negative exponent) is another reminder: the quotient of two polynomials may not be a polynomial!*

### 6.6.1 Exercises

Perform the indicated division problems.

1. $\dfrac{10x^2 - 20x}{-2x}$

2. $\dfrac{x^2 - 3x + 9}{3x}$

3. $\dfrac{3x^5 - 12x^4 - 9x^2}{3x}$

4. $\dfrac{2x^8 - 2x^5 - 8x^4}{2x^4}$

5. $\dfrac{25x^3 - 35x^2 + 5x}{-5x}$

## 6.7 Chapter summary

- A polynomial is a special algebraic expression, all of whose terms involve only whole-number powers of the variable or variables.

- Exponents (originally defined in terms of repeated multiplication) have a number of properties which follow the principle, "Exponents work well with multiplication and division."

- Adding polynomials involves simply combining like terms.

- Subtracting polynomials (similar to subtracting signed numbers) is best accomplished by rewriting the difference as "adding the opposite," with the opposite of a polynomial understood as "changing the sign of every term."
- Multiplying polynomials involves both the distributive property and the simplest property of exponents, $x^a \cdot x^b = x^{a+b}$. After distributing and multiplying, be sure to check to see if there are any like terms which can be combined.
- Dividing a polynomial by a monomial can also be accomplished using a version of the distributive law, dividing each term of the dividend by the divisor and applying the rule of exponents $x^a/x^b = x^{a-b}$.
- The rules of exponents are also helpful in working with numbers written in scientific notation. In particular, two numbers written in scientific notation can be easily multiplied or divided, with the understanding that it sometimes takes an extra step to ensure that the final answer is written in scientific notation.