Chapter 4

Linear equations and inequalities in one variable

Vocabulary

- Algebraic statement
- Equation
- Inequality
- Strict inequality
- Conditional statement
- Solution
- Ordered pair
- Solve
- Linear equation
- Coefficient
- Equivalent equations
- Addition principle
- Multiplication principle
- Like terms
- Coefficient
- Identity
- Contradiction
- Literal equation

4.1 Algebraic statements and solutions

In the last chapter, we considered algebraic expressions: expressions formed by combining numbers and variables with the operations of addition, subtraction, multiplication, division, exponents and roots. However, the only thing that we did with them was to evaluate them for given values of the variables involved. As soon as we took one step into the world of algebra, we quickly went right back into the world of arithmetic, evaluating expressions by performing operations with numbers.

In this chapter, we will begin see some important differences between the world of algebra and that of arithmetic.

A *mathematical statement* is a comparison of two expressions by means of the relations of equality (denoted with the symbol =), greater than (with the symbol $>$), less than (with the symbol $<$), and the compound relations "greater than or equal to" (denoted with the symbol $\geq$) and "less than or equal to" (with the symbol $\leq$). They can be broadly categorized as *equations* (those involving equality) and *inequalities* (all the rest). The basic inequalities $<$ and $>$ are called *strict* inequalities, while the compound inequalities $\leq$ and $\geq$ are sometimes called non-strict inequalities.

The first thing to notice about mathematical statements, unlike mathematical expressions, is that *a statement may be true or false.* For example, $1+2=3$ is an example of a *true* equation, while $1+1=5$ is an example of a *false* equation. Both examples are equations, but one is true and one is false.

In arithmetic, whether a statement is true or false can be completely decided by performing all operations, then comparing the resulting numbers. In algebra, however, this is not usually true.

An algebraic statement is a mathematical statement involving algebraic expressions. A typical algebraic equation or inequality may be true or false, depending on the values of the variables involved. Such a statement is called *conditional.* By contrast, there are no conditional statements in arithmetic: every statement is simply true or false, since there are no unknown quantities.

Example 4.1.1. *Consider the algebraic equation* $x+1=4$.

- *The equation* $x+1=4$ *is true when the value of* x *is* 3.
- *The equation* $x+1=4$ *is false when the value of* x *is* -2.

These two statements show that the equation $x+1=4$ *is a conditional equation, since whether it is true or false depends on the value of the variable* x.

The fact that the typical algebraic statement is conditional prompts the following definition. We will come back to this definition over and over again throughout the text.

A solution of an algebraic statement

A *solution* of an algebraic equation or inequality is a value for each of the variables which, when substituted into the statement, make the statement true.

Using this language, we can reformulate the results of Example 4.1.1: 3 is a solution of the equation $x+1=4$, but -2 is not a solution of $x+1=4$.

Example 4.1.2. *Determine whether* -2 *is a solution of the equation*

$$x^2+5x+6=0.$$

Answer. *We will substitute* -2 *for* x*:*

$$\begin{aligned}(-2)^2 + 5(-2) + 6 &= 0\\ 4 + 5(-2) + 6 &= 0\\ 4 + (-10) + 6 &= 0\\ -6 + 6 &= 0\\ 0 &= 0.\end{aligned}$$

The equation is true when x *is* -2*. So* -2 *is a solution of the equation* $x^2 + 5x + 6 = 0$*.*

Example 4.1.3. *Determine whether* -3 *is a solution of the equation*

$$x^2 + 5x + 6 = 0.$$

Answer. *We will substitute* -3 *for* x*:*

$$\begin{aligned}(-3)^2 + 5(-3) + 6 &= 0\\ 9 + 5(-3) + 6 &= 0\\ 9 + (-15) + 6 &= 0\\ -6 + 6 &= 0\\ 0 &= 0.\end{aligned}$$

The equation is true when x *is* -3*. So* -3 *is a solution of the equation* $x^2 + 5x + 6 = 0$*.*

Notice that in the previous two examples, we found two *different* solutions to the same equation.

In the case of statements with more than one variable, we need to specify a value for *each* variable in order to specify a solution. Many times, we will encounter equations or inequalities in two variables x and y.

Convention: When specifying a solution of an equation or inequality in two variables x and y, we will use ordered pair notation. An *ordered pair* is a pair of two numbers grouped with parentheses. For example, a typical ordered pair might be $(4, 7)$. In this case, the first number will always represent a value for x, while the second value will always represent a value for y. A full treatment of two-variable statements is found in the next chapter. However, we can already get a feeling for them in the context of deciding whether a given ordered pair is a solution or not.

Example 4.1.4. *Determine whether* $(1, -2)$ *is a solution of the equation*

$$3x + 4y = 11.$$

Answer. *The ordered pair* $(1, -2)$ *indicates that the value of* x *is* 1 *and the value of* y *is* -2*. Substituting:*

$$\begin{aligned}3(1) + 4(-2) &= 11\\ 3 + (-8) &= 11\\ -5 &= 11.\end{aligned}$$

The equation is false. So $(1, -2)$ *is NOT a solution of the equation*

$$3x + 4y = 11.$$

Example 4.1.5. *Determine whether* $(5, -1)$ *is a solution of the equation* $3x + 4y = 11$.

Answer. *The ordered pair* $(5, -1)$ *represents the case that the value of* x *is* 5 *and the value of* y *is* -1. *Substituting:*

$$\begin{aligned} 3(5) + 4(-1) &= 11 \\ 15 + (-4) &= 11 \\ 11 &= 11. \end{aligned}$$

The equation is true. So $(5, -1)$ *IS a solution of the equation* $3x + 4y = 11$.

Notice that in the previous two examples (with the same equation), we have found ONE solution of the equation, namely $(5, -1)$. The one solution is made up of two numbers—the values for x and for y both need to be specified for a single solution.

Example 4.1.6. *Determine whether* $-1/2$ *is a solution of the inequality*

$$5x + 1 \geq -1.$$

Answer. *The inequality has only one variable, so a solution is simply a number. We substitute the given candidate:*

$$\begin{aligned} 5\left(-\frac{1}{2}\right) + 1 &\geq -1 \\ \left(\frac{-5}{2}\right) + 1 &\geq -1 \\ \left(\frac{-5}{2}\right) + \frac{2}{2} &\geq -1 \\ -\frac{3}{2} &\geq -1. \end{aligned}$$

The inequality is false. So $-1/2$ *is NOT a solution of the inequality* $5x + 1 \geq -1$.

We close this section with one of the most important definitions in this text:

Solving an algebraic statement

To **solve** an algebraic equation or inequality means to **find all solutions**.

Every word in this definition is important:

1. **Solutions**: To solve an equation or inequality, we need to keep in mind what a typical solution looks like. Is it a number, as is the case for statements with one variable? Is it an ordered pair, as is the case for statements with two variables?

2. **All**: To solve an equation or inequality, we need to know *how many solutions* the statement has. Does it have one solution? Two? Infinitely many? None?

3. **Find**: Rather than guess solutions and then check whether in fact they are solutions or not, we would like to have a procedure that produces (or "finds") the solutions for us.

4.1.1 Exercises

1. Is -2 a solution of $4x + 3 = -5$?

2. Is 0 a solution of $3(x + 2) - 2 = x - 4(x - 1)$?

3. Is $11/3$ a solution of $5(x - 2) = 2x + 1$?

4. Is 3 a solution of $3x - 2 = -x - 4$?

5. Is $-1/4$ a solution of $x + 2 = 3x - 2(x - 1)$?

6. Is -1 a solution of $3x + 2 < -2$?

7. Is $1/3$ a solution of $3x + 2 < -2$?

8. Is 3 a solution of $2x^2 - 5x = 3$?

9. Is $-1/2$ a solution of $2x^2 - 5x = 3$?

10. Is -3 a solution of $2x^2 - 5x = 3$?

11. Is $-1/2$ a solution of $4x^3 + 12x^2 - x = 3$?

12. Is $(2, 3)$ a solution of $x - 3y = -7$?

13. Is $(-7, 0)$ a solution of $x - 3y = -7$?

14. Is $(1, 1)$ a solution of $x - 3y \geq -7$?

15. Is $(-4, 1)$ a solution of $x - 3y \geq -7$?

4.2 Solving linear equations in one variable

In this section, our goal will be to develop a method that will "find" all the solutions for certain equations in one variable.

Let's start off with a very, very simple example.

Example 4.2.1 (A very, very simple example). *Solve:* $x = -2$.

Answer. *One solution of the equation is obvious:* -2. *Just to make sure, substituting,* $-2 = -2$ *is a true statement.*

On the other hand, it is just as obvious that any value of x *which is not* -2 *will NOT be a solution of the equation.*

In other words, the equation has only one solution.

The solution is -2.

If only it were so easy to find the solutions to every equation! In fact, our first goal of producing a method to find all solutions to a given equation will be to convert it to an *equivalent equation* that has the same easy form as our very, very easy example. We say that two equations are **equivalent** if they have exactly the same solutions.

In order to do this, let's first specify what made our very, very easy example so easy:

- One side of the equation is a number. All operations involving numbers have been performed.
- The other side of the equation is an algebraic expression containing just one variable and no operations. (One sometimes says: "x is by itself on one side of the equation.")

A consequence of the second item is that the equation is *linear*. **A linear equation is an equation where the only operations performed on a variable are addition, subtraction, and multiplication by a constant (called a coefficient).**[1] So, for example, the equation $3x + 4y = 11$ is a linear equation (with two variables). The equation $x^2 + 5x + 6 = 0$ is not linear, because the x^2 term involves a variable raised to a power different from 1.

The remainder of this section will be devoted to the following sentence:

Every linear equation (almost[2]) in one variable, let's say x, is equivalent to an easy equation of the form $x =$ ___. (The blank will be a number.)

There is an important consequence of this fact: **(Almost[2]) Every linear equation in one variable has exactly one solution.**

[1] We will see a more concise way of stating this definition in Chapter 5.

[2] See section 4.2.3 below.

4.2.1 The rules of the game

Given any linear equation in one variable x, no matter how complicated, we will develop a method to find an equivalent equation of the form $x =$ ___. Then we will be able to read off the solution to the easy equation; it will also be the solution of the (maybe more complicated) original equation also.

The method will be based on the following two properties of equations:

- **The addition principle**: Adding (or subtracting) the same quantity to both sides of an equation will produce an equivalent equation.
- **The multiplication principle**: Multiplying (or dividing) the same *nonzero* quantity to both sides of an equation will produce an equivalent equation.

These properties can be summarized in the loose slogan: "Doing the same thing to both sides of an equation does not change the solutions of the equation." As usual with loose slogans, though, we should be aware of the fine print. For example, multiplying both sides of an equation (which may be false) by 0 will yield the equation $0 = 0$, which is always true!

4.2.2 Applying the rules: Solving linear equations

We will use the properties in the previous section to attempt to start with any linear equation in one variable (say x) and obtain an equivalent equation of the form

$$x = \underline{\quad}.$$

Since the equations have the same solution, this solution will be obvious by considering the simple equation.

Recall that one of the special features of the easy equation $x =$ ___ is that the expression with the variable has no operations involved. **Our guiding strategy for solving linear equations will be to identify the operations involved in the expression involving the variable, and then to "undo" them, one by one, by using the addition and multiplication principles**.

Example 4.2.2. *Solve:* $x - 41 = 36$.

Answer. *Notice that the left-hand side of the equation (which involves the variable* x*) involves just one operation: subtraction. To "undo" the operation of subtracting* 41*, we will do the opposite: add* 41 *to both sides:*

$$\begin{aligned} x - 41 + 41 &= 36 + 41 && \textit{Addition principle} \\ x + 0 &= 77 && \textit{Performing addition on both sides} \\ x &= 77. \end{aligned}$$

In other words, the equation $x - 41 = 36$ *is equivalent to the (easy!) equation* $x = 77$.

The solution is 77.

It is common to write the addition principle "vertically." The preceding example would be written:

$$\begin{array}{rcrcr} x & - & 41 & = & 36 \\ & + & 41 & \vdots & +41 \\ \hline x & & & = & 77. \end{array}$$

Notice the way that the terms are carefully lined up. The dots $\vdots$ under the equal sign are meant to remind us that we must add the same quantity on both sides.

Example 4.2.3. *Solve:* $5x + 18 = 12$.

Answer. *This time, the left hand side (involving the variable x) involves two operations: addition and multiplication. We will "undo" them in the order* opposite *the order of operations. First, we will undo the addition, then the multiplication.*

$$\begin{array}{rcrcr} 5x & + & 18 & = & 12 \\ & - & 18 & \vdots & -18 \\ \hline 5x & & & = & -6 \\ \frac{5x}{5} & & & = & \frac{-6}{5} \\ x & & & = & -\frac{6}{5}. \end{array}$$

The solution is $-6/5$*.*

Sometimes, we will encounter statements with variables on *both* sides of the equation or inequality. In this case, we will need to take an extra step to make sure that the equivalent equation has variables on only one side of the statement, like our easy equation. This can be done using the same addition principle that we have been using so far. But we need to make one thing that we have been using behind the scenes more explicit.

In a linear equation in one variable, **like terms** are identified according to whether they involve the variable or not. In other words, terms involving the variable will be like terms, and terms not involving the variable will be also be called like terms.

The most important feature of like terms for now is that **like terms can be added** (or "combined"). When we add like terms involving variables, we add the *coefficients* of the terms, but leave the variable part the same. For example, in the expression $5x + 2 + 3x + 9$, the terms $5x$ and $3x$ are like terms, and $5x + 3x = 8x$. Likewise, 2 and 9 are like terms, and $2 + 9 = 11$. So $5x + 2 + 3x + 9 = 8x + 11$.

When we use the addition principle, we will make a habit of writing like terms in the same column.

Example 4.2.4. *Solve:* $3x - 9 = 8x + 7$.

Answer (First method)**.** *This equation has variables on both sides of the equation. As mentioned, our first job will be to write an equivalent equation which only has a variable on one side of the equation. Which side? It doesn't really matter, as we will illustrate here.*

One popular line of thinking goes like this: Our easy equation looks like $x = \_\_\_$*, so let's try to write the variable on the left hand side, just like our easy model.*

In that case, we need to eliminate the $8x$ *from the right-hand side. We do that by adding the opposite, which is* $-8x$*, to both sides.*

$$\begin{array}{rcrcrcr} 3x & - & 9 & = & 8x & + & 12 \\ -8x & & & \vdots & -8x & & \\ \hline -5x & - & 9 & = & & & 12 \\ & + & 9 & \vdots & & + & 9 \\ \hline -5x & & & = & & & 21 \\ \frac{-5x}{-5} & & & = & & & \frac{21}{-5} \\ x & & & = & & & -\frac{21}{5}. \end{array}$$

The solution is $-21/5$*.*

Answer (Second method)**.** *Let's see what would have happened if we had written the equivalent equation with the variable on the* right hand side*. In that case, we would like to eliminate the* $3x$ *term from the left-hand side.*

$$\begin{array}{rcrcrcr} 3x & - & 9 & = & 8x & + & 12 \\ -3x & & & \vdots & -3x & & \\ \hline & - & 9 & = & 5x & + & 12 \\ & - & 12 & \vdots & & - & 12 \\ \hline & & -21 & = & 5x & & \\ & & \frac{-21}{5} & = & \frac{5x}{5} & & \\ & & -\frac{21}{5} & = & x & & . \end{array}$$

Even though this equation is not exactly in the form $x = \_\_\_$*, it's just as easy—the variable is "by itself" on one side of the equation.*

The solution is $-21/5$*.*

Looking at the two methods above, the second method has the advantage that the coefficient of the variable term is positive after using the addition principle to obtain an equation with the variable on one side only. That is because in the original equation, the variable term with the larger coefficient was on the right. From now on, we will follow the custom of writing our equivalent equation with the variable on the side where the original coefficient of the variable term was larger.

The only thing that can make a linear equation in one variable more complicated than the examples we have seen above is if there are more operations on one or both sides of the equation. For example, we may have to perform multiplication (using the distributive law if necessary), or we may have to combine like terms one one or both sides separately. The following example illustrates this situation. Notice that before we begin to apply the addition or multiplication principles ("doing the same thing to both sides") we will perform operations on *each side separately.*

Example 4.2.5. *Solve:* $4(x-3)+2x=7x-9$.

Answer. *Notice that on the left hand side, the* 4 *will distribute over both terms in the parentheses. In addition, there is more than one term on the left-hand side with a variable; we will have to combine like terms.*

$$\begin{aligned} 4(x-3)+2x &= 7x-9 \\ 4(x)+4(-3)+2x &= 7x-9 \\ 4x-12+2x &= 7x-9 \\ 6x-12 &= 7x-9. \end{aligned}$$

Now that both sides are simplified, we can apply the addition and multiplication principles as above. Notice that since there are variables on both sides of the equation, and the higher coefficient of x is on the right-hand side, we will write our equivalent equation with the variable on the right.

$$\begin{array}{cccccc} 6x & - & 12 & = & 7x & - & 9 \\ -6x & & & \vdots & -6x & & \\ \hline & - & 12 & = & x & - & 9 \\ & + & 9 & \vdots & & + & 9 \\ \hline & & -3 & = & x & & . \end{array}$$

The solution is -3.

Summary: Solving linear equations in one variable x

1. Simplify each side of the equation separately:
   - Perform any multiplication, using the distributive property if necessary;
   - Combine like terms.
2. Use the addition property to form an equivalent equation with all variables on one side of the equation;
3. Use the addition property to form an equivalent equation with all constant terms on the side of the equation not containing the variable term;
4. Use the multiplication property to form an equivalent equation of the form $x = \_\_\_$.

4.2.3 Some unusual cases: Linear equations in one variable that do not have exactly one solution

We mentioned above that for *almost* every linear equation in one variable x, there is an equivalent equation of the form $x = \_\_\_$, and so *almost* every linear equation has exactly one solution.

In this section, we will illustrate what can go wrong.

Example 4.2.6. *Solve:* $2(x - 3) + 4 = 2x - 2$.

Answer. *We will apply the general method as usual, simplifying first and then writing an equivalent equation with variables on the left-hand side.*

$$\begin{aligned} 2(x-3)+4 &= 2x-2 \\ 2x-6+4 &= 2x-2 \end{aligned}$$

$$\begin{array}{rrrcrrr} 2x & - & 2 & = & 2x & - & 2 \\ -2x & & & \vdots & -2x & & \\ \hline & - & 2 & = & & - & 2. \end{array}$$

Wait! What happened to the variable? Since the coefficients of the variable were the same on both sides, eliminating from one side (using the addition principle) actually eliminated the variables from both sides.

Actually, that's not bad at all. The equivalent equation, $-2 = -2$, is an arithmetic equation which is always true—no matter the value for x. *(You may try substituting several different values for x into the original equation $2(x - 3) + 4 = 2x - 2$ to confirm that they will all give a true statement.)*

All real numbers *are solutions for this equation. In particular, the equation has infinitely many solutions, not just one. An equation which is true for all values of the variables involved is called an* ***identity****.*

Here is another example where a similar problem occurs.

Example 4.2.7. *Solve:* $x + (2x - 9) = 3(x + 1)$.

Answer.

$$x + (2x - 9) = 3(x + 1)$$
$$x + 2x - 9 = 3x + 3$$

$$
\begin{array}{rcrcrcr}
3x & - & 9 & = & 3x & + & 3 \\
-3x & & & \vdots & -3x & & \\
\hline
 & - & 9 & = & & & 3.
\end{array}
$$

Again, eliminating the variable from one side had the effect of eliminating the variable entirely from the equation. But this time, the original equation is equivalent to an equation which is false*—no matter the value of* x*.*

The equation has no solution*. An equation which has no solution is called a* ***contradiction****.*

We can summarize the results about solutions to linear equations in the following general fact (which is not *almost always* true, but is always true!):

For every linear equation in one variable, one and only one of the following statements must hold:

- The equation has exactly one solution; OR
- It is an identity, and every number is a solution; OR
- It is a contradiction, and it no solution.

This has what might be a startling conclusion: If a linear equation in one variable has two different solutions, then it must have infinitely many.

4.2.4 Another use of the multiplication principle: Equations involving fractions

In this section we illustrate how to use the multiplication principle in order to "clean up" equations involving fractions. While this is an extra step, compared to the four-step process for solving linear equations in two variables above, it does reduce the need to perform arithmetic with fractions that requires extra care and attention to detail.

The extra step that we are going to introduce when we attempt to solve an equation involving fractions is to *multiply both sides of the equation by a common denominator of all fractions appearing in the equation.* By the distributive property, this amounts to multiplying every term on both sides of the equation by the common denominator. Since by definition, every denominator will be a factor of the common denominator, the multiplication will have the effect of ensuring that every term will only involve integers.

Example 4.2.8. *Solve:* $\frac{2x}{3} + \frac{1}{4} = 4.$

Answer. *The least common denominator of the two fractions involved is* 12. *We will multiply both sides of the equation by* 12*:*

$$\begin{aligned} (12)\left(\frac{2x}{3} + \frac{1}{4}\right) &= (12)(4) \\ (12)\left(\frac{2x}{3}\right) + (12)\left(\frac{1}{4}\right) &= 48 \\ \left(\frac{12}{1}\right)\left(\frac{2x}{3}\right) + \left(\frac{12}{1}\right)\left(\frac{1}{4}\right) &= 48 \\ \frac{24x}{3} + \frac{12}{4} &= 48 \\ 8x + 3 &= 48. \end{aligned}$$

Notice that even though we aren't done yet, the new equivalent equation $8x + 3 = 48$ *is much simpler to work with than the original equation.*

Now that both sides are simplified, we can apply the addition and multiplication principles as above.

$$\begin{array}{cccccc} 8x & + & 3 & = & 48 \\ & - & 3 & \vdots & -3 \\ \hline 8x & & & = & 45 \\ \frac{8x}{8} & & & = & \frac{45}{8} \\ x & & & = & \frac{45}{8}. \end{array}$$

The solution is $45/8$.

In the previous example, we wrote out all the steps involved in applying the distributive law after multiplying both sides by a common denominator. In the future, we will omit the step showing the multiplication of both sides, and apply the distributive law directly by *multiplying every term on both sides of the equation by a common denominator.*

Example 4.2.9. *Solve:* $\frac{x-2}{3} + \frac{1}{2} = \frac{x}{6}.$

Answer. *The least common denominator of all three fractions in* 6*. Pay careful attention to the fact that the first fraction on the left-hand side has a numerator involving a group with two terms.*

$$\begin{aligned}(6)\left(\frac{x-2}{3}\right)+(6)\left(\frac{1}{2}\right)&=(6)\left(\frac{1}{6}\right)\\ \frac{6(x-2)}{3}+\frac{6}{2}&=\frac{6}{6}\\ 2(x-2)+3&=1.\end{aligned}$$

Now we proceed as usual:

$$\begin{array}{ccccccc} 2(x-2) & & & + & 3 & = & 1\\ 2x & - & 4 & + & 3 & = & 1\\ 2x & & & - & 1 & = & 1\\ & & & + & 1 & \vdots & +1\\ \hline 2x & & & & & = & 2\\ \frac{2x}{2} & & & & & = & \frac{2}{2}\\ x & & & & & = & 1.\end{array}$$

The solution is 1.

4.2.5 Some word problems

In this section we will apply the techniques of solving linear equations in one variable to some mathematical word problems. In doing so, we will pay special attention to the problem of translating between English and algebra. The technique outlined here emphasizes the translation aspect by using words (like "dictionary") which are essential ingredients in translating from one language to another.

For each word problem, we will follow the following four-step process:

A four-step strategy to approach word problems

1. **Create a "dictionary" for the problem**. The dictionary consists of a list of all unknown quantities, each expressed both in English and as an algebraic expression.
2. **Write an algebraic equation that represents the problem**. This involves using the dictionary from the previous step.
3. **Solve the equation**. This is just using the technique we have been discussing in this chapter. It is the one step where the techniques of algebra are used.
4. **Answer the question**. At this point, the solution of the equation (from the previous step), along with the dictionary, should allow us to answer the question asked in the original problem.

Let's see how this four-step technique works in a few examples.

Example 4.2.10. *The sum of three consecutive integers is* -237*. Find the integers.*

Answer. ***Step 1: Create a dictionary***. *Notice that there are three unknown quantities: the first integer, the second integer, and the third integer. So our dictionary will need three entries, one for each.*

We can always call one of the unknown quantities x*. Since our techniques so far have been with equations in one variable, we will try to write the other two unknown quantities in terms of* x*. For example, in this problem, we know the integers are* consecutive*. So if we call the first integer* x*, then the next two consecutive integers will be* $x + 1$ *and* $x + 2$*. We will write the dictionary as:*

Dictionary

First integer	x
Second integer	$x+1$
Third integer	$x+2$

Step 2: Write an equation. *Many times, looking for an equation in the original problem amounts to finding the word "is." In this problem, we see an equation in the sentence, "The sum of three integers IS* -237*." The word "sum" tell us that we will be adding—what? The three integers, which we translate using our dictionary. In other word, we can write:*

$$x + (x + 1) + (x + 2) = -237.$$

(Note that we have introduced parentheses in order to clearly see the three unknown quantities.)

Step 3: Solve the equation*.*

$$\begin{aligned} x+(x+1)+(x+2) &= -237 && \\ 3x+3 &= -237 && \textit{Combining like terms} \\ 3x &= -240 && \textit{Adding } -3 \textit{ to both sides} \\ x &= -80. && \textit{Dividing both sides by } 3. \end{aligned}$$

The solution is -80.

Step 4: Answer the question*. Notice that solving the equation in Step 3 does not completely answer the question. In particular, the problem asked to find all three integers. The first integer, represented by* x*, is* -80*. The second, represented by* $x+1$*, is* $-80+1$*, or* -79*. The third integer, represented by* $x+2$*, is* $-80+2$*, or* -78*.*

The three integers are -80*,* -79*, and* -78*.*

Example 4.2.11. *The length of a rectangle is* 3 *more than twice the width. Find the dimensions of the rectangle if the perimeter is* 75 *inches. (Use the fact that the perimeter is given by the formula* $P = 2L + 2W$*.)*

Answer. ***Step 1: Create a dictionary****. In this problem, we have two unknown quantities: the length and the width. As usual, we will call one of them* x*, and then try to write the other in terms of* x*. Notice that in this problem, one of the unknowns (the length) is expressed in terms of the other (the width): "The length is ... the width." In cases like this, it is much easier to call the unknown appearing second (in this case, the width) as* x*. That way, we can translate the expression for the first directly:* 3 *more than twice the width will be written as* $2x + 3$*. So the dictionary will be:*

Dictionary

Width	x
Length	$2x+3$

Step 2: Write an equation*. In this case, the equation will come from the formula for the perimeter, along with the dictionary:*

$$2(x) + 2(2x + 3) = 75.$$

Step 3: Solve the equation*.*

$$\begin{aligned} 2(x)+2(2x+3) &= 75 && \\ 2x+4x+6 &= 75 && \textit{Multiplying} \\ 6x+6 &= 75 && \textit{Combining like terms} \\ 6x &= 69 && \textit{Adding } -6 \textit{ to both sides} \\ x &= \frac{69}{6} && \textit{Dividing both sides by } 6 \\ x &= \frac{23}{2} && \textit{Reducing to lowest terms} \end{aligned}$$

The solution is $23/2$.

Step 4: Answer the question. *The variable* x *represented the width of the rectangle, so the width is* $23/2 = 11.5$ *inches. The length was given by* $2x + 3$. *Substituting* $23/2$ *for* x, *we obtain* $2(23/2) + 3 = 23 + 3 = 26$ *so the length is* 26.

4.2.6 Exercises

Solve each of the following linear equations.

1. $2x - 5 = -1$
2. $3(x - 2) = 4$
3. $2x + 1 = 7x + 6$
4. $3(2x - 1) - 3 = x - 5$
5. $3(x - 2) + 4(2x - 1) = 11x - 17$
6. $2(x - 1) + 3(2x - 3) = x + 4$
7. $2(x - 3) = 3(x - 2) - x$
8. $5(3x + 2) + 3(x - 7) = 6x - 11$
9. $\frac{2}{3}x - \frac{1}{2} = \frac{3}{4}$
10. $\frac{x - 1}{4} - \frac{1}{2} = \frac{x}{6}$
11. $\frac{3}{4}x - 3 = \frac{1}{2}$
12. $\frac{2x - 1}{3} - \frac{1}{2} = \frac{x}{6}$

For each of the following, set up an equation representing the problem. Then solve the equation to answer the question.

11. The sum of three consecutive integers is 93. Find the integers.

12. The sum of two consecutive odd integers is 264. Find the integers.

13. One number is three less than seven times another number. If the sum of the two numbers is 15, find the two numbers.

14. Joshua brings home \$ 1,080 per week in net pay. If his deductions amount to 28 % of his gross pay, what is his weekly gross pay?

15. Suppose that for three consecutive odd integers, the sum of the first two and twice the third is 6. Find the integers.

16. One number is two less than three times a number. If the sum of the two numbers is 15, find the two numbers.

4.3 A detour: "Solving" literal equations

The method outlined above in Section 4.2 gives an effective procedure, or algorithm[3], to solve any linear equation in one variable. It turns out that this algorithm is also effective in a more general symbolic setting.

A *literal equation*, or *formula*, is an equation relating two or more variables. For example,

$$F = \frac{9}{5}C + 32, \qquad I = Prt, \qquad h^2 = a^2 + b^2, \qquad y = x^2$$

are four different literal equations. The first relates the two variables F and C (representing the temperature in degrees Fahrenheit and Celsius). The second relates the four variables I, P, r and t (representing interest, principal, rate and time, respectively). The third relates the three variable h, a, and b (representing the lengths of the hypotenuse and the two legs of a right triangle). The fourth relates the two variables x and y (representing the coordinates of a point on a particular parabola). Such relationships are common in the world, and formulas give a concise way of describing them.

A literal equation is *linear in a given variable* if the only operations involving that variable are addition, subtraction, and multiplication by a constant or another different variable. For example, the equation $I = Prt$ is linear in each of the four variables. The equation $y = x^2$ is linear in y, but not linear in x. The equation $h^2 = a^2 + b^2$ is not linear in any of the three variables.

You will notice that three of the four examples of literal equations are written so that one of the variables appears "by itself" on one side of the equation, with

[3]It is not coincidental that the English word *algorithm* derives from the Arabic *al-Khwarizmi*, a title referring to the 9th century mathematician Muhammad ibn Musa al-Khwarizmi. Al-Khwarizmi is best known for his textbook *Al-kitab al-mukhtasar fi hisab* ***al-gabr*** *wa'l-muqabala* ("The compendious book on calculation by completing and balancing"), from which the English word *algebra* derives.

no algebraic operations indicated. (In the equation $h^2 = a^2 + b^2$, the variable h is not "by itself" since it is being raised to the second power.) Many formulas are written in this format. It indicates that the value of the variable appearing "by itself" can be determined knowing the values for all the variables on the other side.

For example, given the "simple interest" formula $I = Prt$, suppose we are given that the principal amount is \$2,000 ($P = 2000$), the annual interest rate is 0.1% ($r = 0.001$) and the time invested is 3 years ($t = 3$). The formula implies that knowing this information, we should be able to determine the interest earned. Substituting the given values into the formula,

$$\begin{aligned} I &= (2000)(0.001)(3) \\ I &= 6. \end{aligned}$$

This is a linear equation whose only solution is 6. The interest earned on a principal amount of \$2,000 over three years at an annual interest rate of 0.1% is \$6.

You will notice in the example in the preceding paragraph that the mathematics involved in using a formula to determine the value of one variable when the others are given does not involve much algebra at all—at least when the variable to be determined is "by itself" on one side of the equation. All that remains, given values for the other variables, is evaluation.

However, for many reasons, it will often be convenient to rewrite a literal equation so that a given variable appears "by itself." When the equation is linear is the given variable, we can apply the method of Section 4.2 to "solve the equation in terms of the given variable." (Notice that this is an abuse of the meaning of the word "solve." In fact we are not solving the equation at all, which would involve finding values of all variables for which the equation is true. Nevertheless, the terminology is so common that it would be futile to avoid it.)

Example 4.3.1. *Solve for t: $I = Prt$.*

Answer. *The instructions, "Solve for t" mean, "Write an equivalent equation with the t by itself on one side of the equation." Notice that the equation is linear in t. The variable t is not "by itself," since it is multiplied by the variables P and r. Applying the multiplication principle, assuming that neither P nor r have the value 0:*

$$\begin{aligned} I &= Prt \\ \frac{I}{Pr} &= \frac{Prt}{Pr} \\ \frac{I}{Pr} &= t. \end{aligned}$$

Since the order of the equality does not matter, we can write it as

$$t = \frac{I}{Pr}.$$

The answer is $t = I/(Pr)$.

Example 4.3.2. *Solve for* y*:* $3x + 4y = 12$.

Answer. *The equation is linear in* y*. The variable* y *is not "by itself:" it is multiplied by* 4 *with* $3x$ *added to the result. We apply the algorithm of Section 4.2:*

$$\begin{array}{rrrcrr} 3x & + & 4y & = & & 12 \\ -3x & & & \vdots & -3x & \\ \hline & & 4y & = & -3x & +12 \\ & & \frac{4y}{4} & = & \frac{-3x+12}{4} & \\ & & y & = & \frac{-3x}{4} & +\frac{12}{4} \\ & & y & = & -\frac{3}{4}x & +3. \end{array}$$

The answer is $y = -\dfrac{3}{4}x + 3$.
A few things to notice about our use of the algorithm:

- *At the second-to-last step, dividing both sides of the equation by* 4*, we used the distributive property to divide each term on the right by* 4*.*
- *At the last step, we wrote the coefficient* $-3/4$ *of* x *more plainly. Notice that* $-\dfrac{3}{4}x = -\dfrac{3}{4} \cdot \dfrac{x}{1} = -\dfrac{3x}{4}$*. Also keep in mind that* $\dfrac{-3}{4} = \dfrac{3}{-4} = -\dfrac{3}{4}$.

Example 4.3.3. *Solve for* y*:* $2x - 5y = 8$.

Answer. *The equation is linear in* y*. Again we apply the algorithm of Section 4.2:*

$$\begin{array}{rrrcrr} 2x & - & 5y & = & & 8 \\ -2x & & & \vdots & -2x & \\ \hline & & -5y & = & -2x & +8 \\ & & \frac{-5y}{-5} & = & \frac{-2x+8}{-5} & \\ & & y & = & \frac{-2x}{-5} & +\frac{8}{-5} \\ & & y & = & \frac{2}{5}x & -\frac{8}{5}. \end{array}$$

The answer is $y = \dfrac{2}{5}x - \dfrac{8}{5}$.
Pay careful attention to the signs in studying this example!

4.3.1 Exercises

1. Solve for y: $3x - 2y = 6$.

2. Solve for y: $5x + 4y = 10$.

3. Solve for r: $I = Prt$.

4. Solve for C: $F = \frac{9}{5}C + 32$

5. (*) Solve for m: $y - y_0 = m(x - x_0)$.

4.4 Solving linear inequalities in one variable

A linear inequality, as its name implies, is an inequality in which the only operations involving the variables are addition, subtraction, and multiplying by a constant. In this section, we consider inequalities with just one variable.

Like equations, inequalities can be true or false. Solving an inequality involves finding all values for the variables which make the statement true.

A look at a very easy example of a linear inequality shows a significant difference compared to linear equations. Consider for example the linear inequality $x \leq 2$. (Compare this with Example 4.2.1.) Notice that the variable is "by itself" on one side of the inequality, with no other operations involved. We can see by inspection that 2 is a solution: $2 \leq 2$ is true. (Notice that $x \leq 2$ is a compound statement: it is true when EITHER $x < 2$ is true OR when $x = 2$ is true.) But 2 is not the ONLY solution! For example, 1 is also a solution: $1 \leq 2$ is true. -4 is another solution: $-4 \leq 2$ is true. 1.9999 is another solution: $1.9999 \leq 2$ is true. You can convince yourself pretty quickly that the very simple linear inequality $x \leq 2$ in fact has *infinitely many* solutions. This is typical for algebraic inequalities.

Most linear equations in one variable have exactly one solution. We have seen at least one situation (Example 4.2.6) where a linear equation might have infinitely many solutions, but in that case *every* real number is a solution. The "easy" inequality we were considering, $x \leq 2$, has infinitely many solutions, but not every real number is a solution. For example, 3 is not a solution: $3 \leq 2$ is false.

Even the simplest linear inequality poses the following question for us: How can we solve the inequality—find ALL solutions—when there are infinitely many of them? How can we indicate which numbers are solutions and which are not?

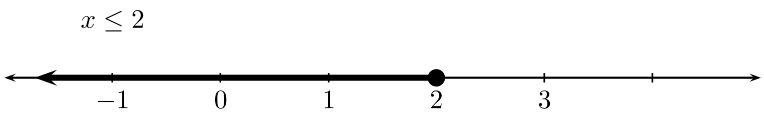
To answer these questions, we will introduce a technique that will be useful in a number of situations: graphing. To *graph* an algebraic statement means to draw a picture of all solutions. Typically our "picture" will involve our standard method of visualizing the real numbers: the number line. "Solving" and "graphing" are really the same type of problem (finding all solutions), but the answer is written differently.

Our basic method for graphing algebraic statements will be to indicate solutions on a number line with a solid circle (•). In the case that we have many solutions "infinitesimally close to each other," which will look like a solid (shaded) line. Here is the simple example we have been considering so far:

Example 4.4.1. *Graph all solutions of the inequality:* $x \leq 2$.

Answer. *Every number less than (on a number line, to the left of)* 2 *will be a solution, so we will shade the region of the number line to the left of two. In addition, we will use a solid circle for the "border" solution* 2*, to indicate that* 2 *is in fact a solution. We will use the term "border value" to denote the fact that for all values on one side of the value on the number line, the inequality is true, while for all values on the other side, the inequality is false.*

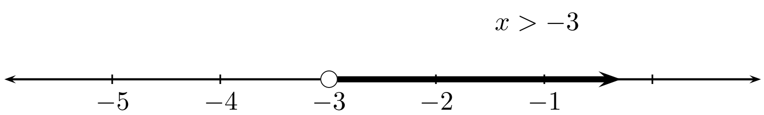
The graph of all solutions of $x \leq 2$ *is:*



Notice that the picture really describes ALL solutions of the inequality $x \leq 2$*; we have "solved" the inequality. We see whether a number is a solution or not by whether or not it is in the shaded region on the number line.*

In the graphical method for solving linear inequalities, special attention must be given to the "border" value. Consider for example the strict inequality $x > -3$. In this case, any number to the right of -3 on the number line will be a solution. The border value -3 is NOT a solution, however: $-3 > -3$ is false. In order to deal with the problem of shading every value to the right of -3 but not including -3 itself, we will indicate the border value with an empty circle (◦).

Hence the graph of all solutions of $x > -3$ is:



Summarizing:

Border values for linear inequalities in one variable

- For strict inequalities ($<$, $>$), the border value is indicated with an open circle ($\circ$);
- For non-strict inequalities ($\leq$, $\geq$), the border value is indicated with a solid circle ($\bullet$).

In order to solve more complicated linear inequalities in one variable, we will present two slightly different methods. Each has some advantages. Both will rely on the methods we have already developed to solve linear equations in one variable, namely the addition and multiplication principles. The main difference between the two methods will be in deciding which side of the border value to shade.

4.4.1 Solving linear inequalities in one variable: Test value method

As the above simple examples show, graphing an inequality has two key steps: finding the border value and deciding which side of the border value are solutions to the inequality. Our first method of solving inequalities separates these two steps. The main idea in this method is that the border value of a linear inequality divides the number line into a "greater than" side and a "less than" side (relative to this inequality); the border value corresponds to the solution of an equation. We will determine the appropriate side to shade by choosing a test value, which will determine which side is which.

Example 4.4.2. *Graph all solutions of the inequality:* $3(x+2) - 4 > 2x + 8$.

Answer. ***Step 1. Find the border value*** *In order to find the border value, we consider the corresponding equation:*

$$3(x+2) - 4 = 2x + 8.$$

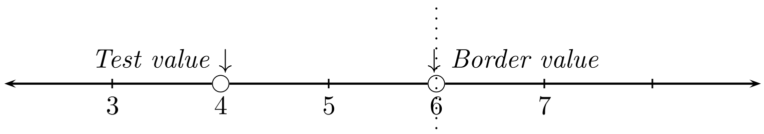
This is a linear equation in one variable. The border value is the solution of this equation.

$$\begin{array}{cccccccccc}
3(x+2) & & & - & 4 & = & 2x & + & 8 \\
3x & + & 6 & - & 4 & = & 2x & + & 8 \\
3x & & & + & 2 & = & 2x & + & 8 \\
-2x & & & & & \vdots & -2x & & \\
\hline
x & + & 2 & & & = & & & 8 \\
 & - & 2 & & & \vdots & & - & 2 \\
\hline
x & & & & & = & & & 6.
\end{array}$$

The border value is 6. *Notice that we will represent the border value with an open circle* (○) *since the inequality we are solving is strict.*

Step 2. Use a test value to determine which side to shade. *For a test value, we choose any number EXCEPT the border value. We "test" this value by substituting it into the original inequality to determine whether it is a solution. If it is a solution, all values on the same side of the border value will be solutions; if it is not a solution, all values on the opposite side of the border value will be solutions.*

Suppose we choose 4 *as a test value.*



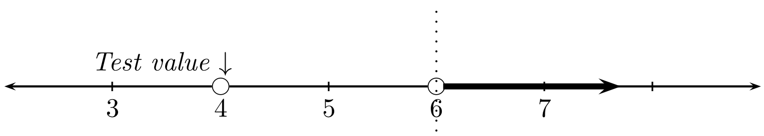
If 4 *is a solution of the inequality, we will shade all values* on the same side *of the border value* 6 *as the test value; if* 4 *is not a solution, we will shade all values* on the opposite side *of* 6 *as* 4.

Is 4 *as solution of* $3(x+2)-4>2x+8$?

$$\begin{aligned}3((4)+2)-4&>2(4)+8\\3(6)-4&>8+8\\18-4&>16\\14&>16.\end{aligned}$$

The inequality is false; the test value 4 *is not a solution of the inequality. Therefore we will shade all values on the* opposite *side of the border value as* 4.

The graph of all solutions of $3(x+2)-4>2x+8$ *is:*



$3(x+2)-4>2x+8$

For convenience, we summarize the test-value method:

Test-value method for graphing linear inequalities in one variable

- Find the border value by solving the corresponding linear *equation.*
- Determine which side of the border value to shade by choosing a test value and deciding whether it is a solution or not.

Represent the border value with an open or closed circle according to whether the inequality is strict or not.

Example 4.4.3. *Graph all solutions of the inequality:* $2(2x-3)+3x \leq 10x-5$.

Answer. ***Step 1: Find the border value.***

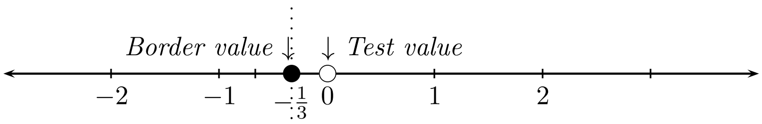
To find the border value, we solve the equation $2(2x-3)+3x = 10x-5$*:*

$$
\begin{array}{ccccccccc}
2(2x-3) & & & + & 3x & = & 10x & - & 5 \\
4x & - & 6 & + & 3x & = & 10x & - & 5 \\
7x & - & 6 & & & = & 10x & - & 5 \\
-10x & & & & & \vdots & -10x & & \\
\hline
-3x & - & 6 & & & = & & & -5 \\
 & + & 6 & & & \vdots & & & +6 \\
\hline
-3x & & & & & = & & & 1 \\
\frac{-3x}{-3} & & & & & = & & & \frac{1}{-3} \\
x & & & & & = & & & -\frac{1}{3}.
\end{array}
$$

The border value is $-1/3$. *We will indicate the border value with a solid circle (*$\bullet$*) since the original inequality (*$\leq$*) is not strict.*

Step 2. Use a test value to determine which side to shade.

Let's choose 0 *as our test value.*

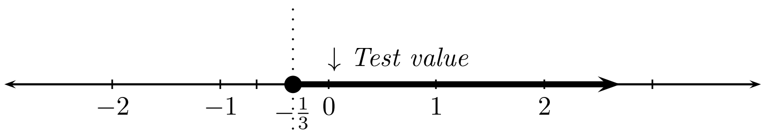


We test whether 0 *is a solution of* $2(2x-3)+3x \le 10x-5$*:*

$$\begin{aligned} 2(2(0)-3)+3(0) &\le 10(0)-5 \\ 2(0-3)+0 &\le 0-5 \\ 2(-3) &\le -5 \\ -6 &\le -5. \end{aligned}$$

The inequality is true; the test value 0 *is a solution of the inequality. Therefore we will shade all values on the* same *side of the border value as* 0.

The graph of all solutions of $2(2x-3)+3x \le 10x-5$ *is:*



$$2(2x-3)+3x \le 10x-5$$

We close with one final example.

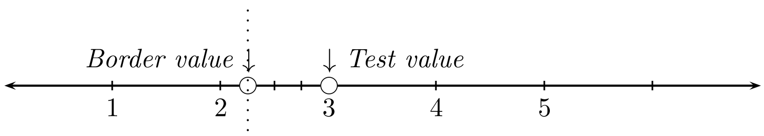
Example 4.4.4. *Graph all solutions of the inequality:* $4x-7<2$.

Answer. ***Step 1: Find the border value.*** *We solve the equation* $4x-7=2$*:*

$$\begin{array}{cccccc} 4x & - & 7 & = & 2 \\ & + & 7 & \vdots & +7 \\ \hline 4x & & & = & 9 \\ \frac{4x}{4} & & & = & \frac{9}{4} \\ x & & & = & \frac{9}{4}. \end{array}$$

The border value is $9/4$*, which we will indicate with an open circle (*$\circ$*) since the original inequality is strict.*

Step 2. Use a test value to determine which side to shade. *Let's choose* 3 *as a test value:*

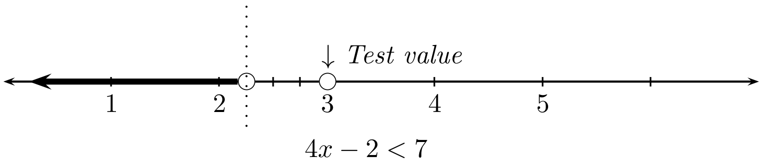


We test whether 3 *is a solution of the inequality* $4x - 7 < 2$*:*

$$\begin{aligned} 4(3) - 7 &< 2 \\ 12 - 7 &< 2 \\ 5 &< 2 \end{aligned}$$

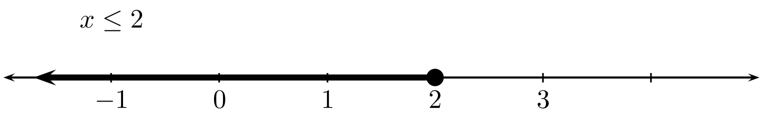
The inequality is false; 3 *is not a solution of* $4x - 2 < 7$*. Hence we shade on the opposite side of the border value.*

The graph of all solutions of $4x - 7 < 2$ *is:*

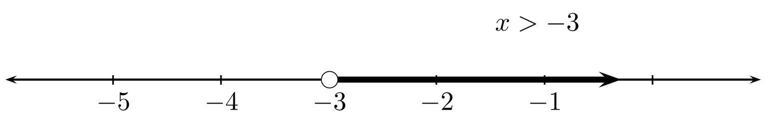


4.4.2 Solving linear inequalities in one variable: Standard form method

Another way of deciding which half of the number line to shade for a typical linear inequality in one variable is to take advantage of the special form $x <$ ___ or $x >$ ___. We will call this form "standard form," in which the variable appears on the LEFT side of the inequality by itself with no operations. We opened this section by considering an example of this form: $x \leq 2$. The graph consisted of shading all values less than (to the *left of*) 2, along with the border value 4 which we indicated with a solid circle (•):



On the other hand, we considered the inequality $x > -3$, also in standard form. The graph consisted of shading all values greater than (to the *right of*) -3, with the border value -3 indicated with an open circle (○) to indicate that it is not a solution of the inequality:



Our second method of solving linear inequalities in one variable will be to convert the inequality into an equivalent one in the standard form with the variable by itself *on the left side of the inequality*, in almost exactly the same way as we did to solve a linear equation. In that case, we can follow the rules:

- For inequalities in the standard form $x \leq a$ or $x < a$, always shade *to the left of* the border value a;
- For inequalities in the standard form $x \geq a$ or $x > a$, always shade *to the right of* the border value a.

As usual, the border value a will be indicated with a solid circle (•) or an open circle (◦) depending on whether or not the inequality is strict.

The standard form method has the advantage of the word "always." In particular, because we take the trouble to write the inequality in the standard form, there is no need to choose a test value to determine which side of the border value to shade. There are two important points that need to be kept in mind, however.

The first is that the standard form presumes that the variable is by itself on the LEFT side of the inequality. When we solved equations, we were free to write the variable by itself on EITHER side of the equation. In fact, we saw examples, where it was more convenient to write an equation in an equivalent form with the variable by itself on the right side of the equation.

This problem is easy to solve as long as we keep in mind that writing $a < b$ is exactly the same as writing $b > a$. In other words, writing the inequality from right to left "changes the sense" of the inequality (in other words, the inequality symbol "points in the opposite direction.") So if we were to rewrite a linear inequality into the form $5 \geq x$, we would simply rewrite it in the standard form $x \leq 5$.

The second point to keep in mind is more serious. The careful reader might have noticed that this method was described as rewriting the inequality in standard form "in almost exactly the same way" as for linear equations. More specifically, that means using the addition and multiplication principals to "do the same thing to both sides" to obtain a simpler statement. In fact, the addition principle can be used in exactly the same way: Adding (or subtracting) the same quantity from both sides of an inequality produces an equivalent inequality (an inequality with the same solutions).

The multiplication principal requires some adjustment, however. To see this, let's start with a true inequality $2 < 3$. Multiplying both sides by 5, for example, we obtain the inequality $10 < 15$, which is still true. However, if we multiply both sides of the same inequality by -5, we obtain $-10 < -15$—which is false! Clearly, multiplying (and dividing, as you might guess) by a negative quantity has a different effect than multiplying (or dividing) by a positive quantity. Without going into a lengthy explanation for why this is so, just keep in mind that multiplying by a negative quantity involves some sense of an "opposite," and additive opposites on a number line are "mirror images" of each other, reflected across the point representing 0.

For this reason, we need to adjust the multiplication principal for inequalities:

The multiplication principal for inequalities

- Multiplying (or dividing) both sides of an inequality by a positive quantity will produce an equivalent inequality;
- Multiplying (or dividing) both sides of an inequality by a negative quantity will produce an equivalent inequality *after changing the sense of the original inequality.*

For example, dividing both sides of the inequality $-3x > 6$ by -3 gives the equivalent inequality $\frac{-3x}{-3} < \frac{6}{-3}$, or $x < -2$. The sense of the original inequality has changed from $>$ to $<$.

With these two small adjustments to our method of approaching linear equations, let's go back to the same examples as we saw in the test value method.

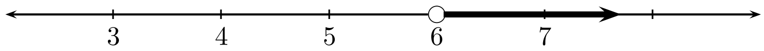
Example 4.4.5. *Graph the inequality:* $3(x+2) - 4 > 2x + 8$.

Answer. *We apply the addition principal and the revised multiplication principle to obtain an equivalent inequality in standard form:*

$$\begin{array}{ccccccccc}
3(x+2) & & & - & 4 & > & 2x & + & 8 \\
3x & + & 6 & - & 4 & > & 2x & + & 8 \\
3x & & & + & 2 & > & 2x & + & 8 \\
-2x & & & & & \vdots & -2x & & \\
\hline
x & + & 2 & & & > & & & 8 \\
 & - & 2 & & & \vdots & & - & 2 \\
\hline
x & & & & & > & & & 6.
\end{array}$$

Notice that at no point did we need to multiply or divide by a negative number.

The new, equivalent inequality is of the form $x > a$ *(where here* a *is* 6*). We will shade* to the right of 6*, indicating* 6 *with an open circle since the inequality is strict:*



$3(x+2) - 4 > 2x + 8$

Summarizing the standard-form method:

Standard form method for graphing linear inequalities in one variable

- Use the addition principle and (modified) multiplication principle to re-write the inequality in the standard form, with the variable by itself on the *left* side of the inequality.
- For inequalities involving "less than" ($<$ or $\leq$), shade to the left of the border value. For inequalities involving "greater than" ($>$ or $\geq$), shade to the right of the border value.

Represent the border value with an open or closed circle according to whether the inequality is or is not strict.

Let's apply this method to the other two examples we saw previously.

Example 4.4.6. *Graph all solutions of the inequality:* $2(2x-3)+3x \leq 10x-5$.

Answer. *We again rewrite the inequality in standard form:*

$$
\begin{array}{rrrrrrrrr}
2(2x-3) & & & + & 3x & \leq & 10x & - & 5 \\
4x & - & 6 & + & 3x & \leq & 10x & - & 5 \\
7x & - & 6 & & & \leq & 10x & - & 5 \\
-10x & & & & & \vdots & -10x & & \\
\hline
-3x & - & 6 & & & \leq & & & -5 \\
 & + & 6 & & & \vdots & & & +6 \\
\hline
-3x & & & & & \leq & & & 1 \\
\frac{-3x}{-3} & & & & & \geq & & & \frac{1}{-3} \\
x & & & & & \geq & & & -\frac{1}{3}.
\end{array}
$$

This time, in our final step, we had to divide by a negative number, so the sense of the inequality changed from $\leq$ *to* $\geq$.

Since the final inequality has the standard form $x \geq a$ *(where* a *is* $-1/3$*), we will shade to the right of the border value; the border value* $-1/3$ *will be indicated with a solid circle since the inequality is not strict.*

The graph of all solutions of $2(2x-3)+3x \leq 10x-5$ *is:*

$2(2x-3)+3x \leq 10x-5$

Example 4.4.7. *Graph all solutions of the inequality:* $4x-7<2$.

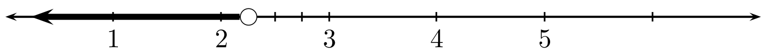
Answer. *Rewriting the inequality in standard form:*

$$
\begin{array}{rrrcr}
4x & - & 7 & < & 2 \\
 & + & 7 & \vdots & +7 \\
\hline
4x & & & < & 9 \\
\frac{4x}{4} & & & < & \frac{9}{4} \\
x & & & < & \frac{9}{4}.
\end{array}
$$

At no point did we multiply or divide by a negative number; the sense of the inequality remained the same.

Since the equivalent inequality has the standard form $x < a$ *(where here* a *is* $9/4$*), we will shade* to the left of *the border value* $9/4$*, indicating the border value with an open circle since the inequality is strict.*

The graph of all solutions of $4x - 7 < 2$ *is:*



$4x - 2 < 7$

To summarize:

- A typical linear inequality in one variable will have infinitely many solutions. For this reason, the solutions are typically indicated with a graph on a number line.
- The typical graph of a linear inequality will be half of a number line, all points to the left of or to the right of the border value, with the shaded region representing solutions to the inequality.
- The border value is indicated with either an open circle ($\circ$), in the case of a strict inequality ($<$ or $>$), or a solid circle ($\bullet$), in the case of a non-strict inequality ($\leq$ or $\geq$).
- Which side of the border value to shade can be determined either by the test value method or by the standard form method.

Keep in mind that an inequality, like an equation, can be an identity (and so all real numbers are solutions) or a contradiction (and so has no solution).

4.4.3 Exercises

In the problems below, "solve" means "graph all solutions." For each one, list five individual solutions.

1. Solve: $3x - 4 > 6$.

2. Solve: $2(x - 3) + 4 \leq x - 5$.

3. Solve: $3(2x - 1) + 4(3x + 5) > 2(x - 6)$.

4. Solve: $x - 5(2x + 1) \leq 6$.

5. (*) Solve: $-2(x - 3) \geq -4(x + 1) + 2x$.

6. (*) Solve: $2x + 3(x - 2) > 5(x - 1) - 1$.

4.5 Chapter summary

- A solution of an algebraic equation or inequality is a value for each variable which, when substituted, makes the statement true.
- To solve means to find all solutions.
- A linear equation is one in which the only operations involving variables are addition, subtraction, and multiplication by a constant.
- A typical linear equation in one variable has one solution. The exceptions are contradictions (statements that have no solution) and identities (statements which are true for all values of the variable).
- Solving a linear equation in one variable involves using the addition and multiplication principles to find an equivalent equation for the form

$$x = \underline{\quad}.$$

- "Solving" a literal equation for a given variable means writing an equivalent equation with the given variable "by itself" on one side of the equation.
- A typical linear inequality in one variable will have infinitely many solutions. To solve them, the solutions are graphed on a number line.