Chapter 3

Introduction to algebra

Vocabulary

- The order of operations
- Parentheses (and other grouping symbols)
- Variables
- Constants
- Algebraic expression
- Evaluate
- Function notation

3.1 The order of operations

In the previous chapters, we reviewed the basic arithmetic operations in the context of fractions and signed numbers. In a typical problem that we are going to encounter, however, there will be more than just one operation to perform. For that reason, the order in which we perform the operations is important. Usually, if we do operations in a different order, we will obtain different answers. For example, consider the expression with two operations (both division):

$$8 \div 4 \div 2.$$

If we perform the first division first, we obtain $2 \div 2 = 1$. If we perform the second division first, we obtain $8 \div 2 = 4$. The answers are different. Which is correct?

The order of operations is largely a matter of convention that has developed in history, and in fact more a question of typography than of mathematics. That being said, it is for the most part agreed upon as standard. We describe the order of operations here as a rule.

The order of operations

When an expression involves more than one operation, the operations are performed according to the following order:

1. **Operations inside grouping symbols**, from inside to outside. Parentheses are the most common grouping symbols, but there are many others.
2. **Exponents and roots**. When more than one exponent or root occur in the same expression, it is standard to evaluate them from left to right.
3. **Multiplication and division**. When there are more than one multiplication or division in the same expression, they must be performed in order from left to right.
4. **Addition and subtraction**. When there are more than one addition or subtraction in the same expression, they must be performed in order from left to right.

Referring to the example at the beginning of the section, the expression $8 \div 4 \div 2$ involves two division operations. Since both are at the same "level" in the order of operations, the divisions should be performed left-to-right:

$$8 \div 4 \div 2 = 2 \div 2 = 1.$$

If we had wanted to write an expression where the second division was performed first, we would need to use grouping symbols: $8 \div (4 \div 2)$.

It is worth noting at this point that both the associative properties of addition and of multiplication, as well as the distributive property of multiplication over addition, are all properties that concern the order of operations. For example, the associative property of addition says that repeated addition actually does not need to be performed from left to right, but can be performed in any order at all. The same holds, of course, for multiplicattion. This is *not* true for subtraction or division, however! Neither subtraction nor division is commutative or associative.

Important note: The unfortunate slogan "PEMDAS," sometimes taught to help memorize the order of operations, might imply that the first operation performed is "parentheses." This short-hand terminology is misleading, first and foremost because *parentheses do not represent an operation.* In particular, we draw special attention to a fact that often causes confusion:

Parentheses never mean multiplication.

Parentheses are grouping symbols, indicating that what is inside the parentheses should be considered as a single expression.

However, when no operation symbol is indicated between two expressions, the assumed operation will be multiplication.

The expression 2(3) involves multiplication ("two times three") for the same reason that that we will see that $2x$ involves multiplication ("two times x"). In both cases, the multiplication is indicated by the fact that there is no symbol between the two factors (of 2 and 3 in the first case, 2 and x in the second.) Of course, 23 is simply the number twenty-three, so the parentheses in the expression 2(3) serve to separate the three, as its own "group," from the two.

Keep in mind also that parentheses are not the only grouping symbols. In addition to various shaped parentheses, like brackets $[,]$ and braces $\{,\}$ some commonly encountered grouping symbols include the bar used to write a fraction $\frac{\cdots}{\cdots}$ (where the numerator and the denominator are both considered as two separate groups) and the bar over an expression in a radical symbol $\sqrt{\cdots}$ (where everything "inside" the radical sign under the bar is considered as one group.)

The following examples illustrate the order of operations, especially involving grouping symbols.

Example 3.1.1. *Perform the indicated operations:* $3 - 2(-4 + 11)$.

Answer. *There are are three operations: An addition, a subtraction and a multiplication. Since the addition is grouped with parentheses, it will be performed first. Of the remaining two, multiplication takes priority over subtraction. Hence we will perform the operations as indicated below:*

$$3 \overset{3}{-} 2 \overset{2}{\cdot} (-4 \overset{1}{+} 11).$$

$$\begin{aligned} &3 - 2(-4 + 11) \\ &3 - 2(7) && 1 \\ &3 - 14 && 2 \\ &3 + (-14) && \textit{changing subtraction to "adding the opposite"} \\ &-11 && 3 \end{aligned}$$

The answer is -11.

Example 3.1.2. *Perform the indicated operations:* $\dfrac{(-2) - (-6)}{1 - 5}$.

Answer. *There are three operations: two subtractions (one each in the numerator and denominator) and one division (indicated by the fraction bar). Since the fraction groups the numerator and the denominator separately, we perform the subtractions first, followed by the division, as indicated here:*

$$\frac{(-2) \overset{①}{-} (-6)}{1 \underset{②}{-} 5} ③.$$

$$\begin{aligned}
&\frac{(-2)-(-6)}{1-5} \\
&\frac{(-2)+6}{1+(-5)} && \textit{Changing both subtractions to “adding the opposite”} \\
&\frac{4}{1+(-5)} && ① \\
&\frac{4}{-4} && ② \\
&-1 && ③
\end{aligned}$$

The answer is -1.

Example 3.1.3. *Perform the indicated operations:* $\sqrt{(-5)^2 - 4(1)(-6)}$.

Answer. *This time there are five operations: one exponent, two multiplications, a subtraction and a square root. Since the square root symbol groups everything inside, we perform those (four) operations first, and the square root last. Within the group, the “usual” order of operations apply: first, the exponent, followed by the two multiplications, followed by the subtraction. The order is indicated here:*

$$\overset{⑤}{\sqrt{(-5)^{\overset{①}{2}} \overset{④}{-} 4 \overset{②}{\cdot} (1) \overset{③}{\cdot} (-6)}}.$$

$$\begin{aligned}
&\sqrt{(-5)^2 - 4(1)(-6)} \\
&\sqrt{25 - 4(1)(-6)} && ① \\
&\sqrt{25 - 4(-6)} && ② \\
&\sqrt{25 - (-24)} && ③ \\
&\sqrt{25 + 24} && \textit{Changing subtraction to “adding the opposite”} \\
&\sqrt{49} && ④ \\
&7 && ⑤
\end{aligned}$$

The answer is 7.

We close with the reminder that the commutative and associative properties give great flexibility with the order of operations involving addition and multiplication. For example, consider the expression

$$22 - 75 + (-18) - 52 - (-16) + 48 + (-12).$$

According to the order of operations, the six operations would be performed from left to right. However, changing all subtractions to "adding the opposite" and using the commutative property (to change the order) and the associative property (to re-group positive and negative terms together), it is much easier to perform the operations as

$$(22 + 16 + 48) + [(-75) + (-18) + (-52) + (-12)]\,.$$

3.1.1 Exercises

For each of the problems below: (1) Count the number of operations; (2) list the operations in order; and (3) perform the operations.

1. $3 - 5(4 - 1)$

2. $\sqrt{(-3)^2 + (-4)^2}$

3. $\dfrac{(-3) - (-1)}{-2 - (-1)}$

4. $-(-3)^2 + (2)(1)$

5. $3\left(-\dfrac{1}{4}\right)^2 - 2\left(-\dfrac{1}{4}\right) + 1$

6. $\sqrt{(-1)^2 - 4(2)(-3)}$

7. $\left(\dfrac{1}{2}\right)\left(\dfrac{2}{3}\right) - \left(\dfrac{1}{4}\right)(-2)$

8. $\dfrac{-(-1) + \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)}$

3.2 Algebraic expressions

All numbers are symbols. The number "5" is a symbol that indicates quantity, answering the question "how many." More abstractly, the number "5" represents the quality that all collections of five objects have in common.

There are times when it is convenient to introduce other symbols that represent numbers. For example, you might read in an astronomy book that light travels at the speed of nearly $300,000$ kilometers per second. This is actually a rule that tells you that if you know how long a light ray has been traveling, then you actually also know how far it has traveled. In 1 second, the light ray has traveled $300,000$ kilometers (more than half the distance from the earth to the moon). In 2 seconds, a light ray will travel $300,000 \times 2 = 600,000$ kilometers. In 750 seconds (12.5 minutes) , a light ray travels $300,000 \times 750 = 225,000,000$ kilometers, which is the average distance from the earth to the planet Mars.

You might summarize this rule as follows: If t represents the number of seconds that a light ray travels, then the distance it will travel is $300,000 \times t$, or simply $300000t$. (Remember the convention: When no operation is indicated, there is an assumed multiplication!)

For our purposes, we will call a *variable* any symbol (we will always use letters) which is meant to represent a number whose value is not specified. A variable generally indicates that the value of the number is either unknown or, as in the example above, changing with time. Because of this, we will sometimes call numbers *constants* to distinguish them from variables.

From this point of view, algebra will be the study of expressions formed by combining both numbers and variables by using the standard operations of addition, subtraction, multiplication, division, (numerical) exponents, and roots.[1]

The main feature which distinguishes algebra from arithmetic, then, is the use of variables.

Convention: By far the most popular symbol to represent an unknown quantity[2] is the letter x. Because of this, we will avoid using the symbol $\times$ to represent multiplication from now on.

3.3 Evaluating algebraic expressions

Given an algebraic expression, there is not much that we can do with it apart from identifying the variables and the operations involved in the expression.

Example 3.3.1. *Consider the algebraic expression $3x^2-5x+4$. This expression involves one variable x, and five operations: an exponent, two multiplications, a subtraction, and an addition. (Locate them!)*

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[1] There are other operations, like logarithms for example, which from this point of view are technically not algebraic, even though they are often treated in algebra courses.

[2] It is in this spirit that Malcolm X undoubtedly chose his name, to indicate that his true family name was unknown as a legacy of slavery.

If, on the other hand we are given values for *all* variables appearing in a given algebraic expression, we can evaluate[3] the expression for those given values. We do this by *substituting* the given values at every instance of the variable.

In the context of specifying values for a variable, we will use a short-hand notation with the equal sign "=". For example, we will say, "Evaluate an expression when $x = 1$." This means, "Evaluate the expression when x has the value 1." The use of the equal sign in this context is unfortunate, because its meaning is very different from the way we will use it for the rest of the book. However, like many unfortunate things, it is standard, and we will use it in this section.

Example 3.3.2. *Evaluate* $3x^2 - 5x + 4$ *when* $x = -2$.

Answer. *The variable x appears in the expression twice. We will substitute the value -2 in both instances. Then we will proceed according to the order of operations. We can indicate this order schematically as follows:*

$$3 \overset{②}{\cdot} x^{\overset{①}{2}} \overset{④}{-} 5 \overset{③}{\cdot} x \overset{⑤}{+} 4.$$

In other words, the exponential is evaluated first, then the first multiplication, etc.

$$\begin{aligned}
3(-2)^2 - 5(-2) + 4 &\quad \textit{Substituting } -2 \textit{ for } x \\
3(4) - 5(-2) + 4 &\quad ① \\
12 - 5(-2) + 4 &\quad ② \\
12 - (-10) + 4 &\quad ③ \\
12 + 10 + 4 &\quad \textit{Changing subtraction to "adding the opposite"} \\
22 + 4 &\quad ④ \\
26. &\quad ⑤
\end{aligned}$$

Notice the use of parentheses when we substitute a value for the variable. We can think of the variable as a placeholder, for which we insert the given value. In other words, we can think of the expression as being $3(\cdots)^2 - 5(\cdots) + 4$, and we will substitute the given value into the parentheses.

Example 3.3.3. *Evaluate* $\dfrac{y_2 - y_1}{x_2 - x_1}$ *when* $x_1 = 4$, $x_2 = -6$, $y_1 = -3$ *and* $y_2 = -18$.

Answer. *Notice that this algebraic expression has four variables, each of which appear once. (Be careful! These variable have* subscripts*, which should not be*

[3] The word *evaluate* means "find the value of"

mistaken for exponents or any other kind of operation. The subscripts belong with the symbol for the variable.)

There are three operations involved in this expression. We will first perform the subtraction on top, then the subtraction on bottom, and finally the division.

$$\frac{y_2 \overset{①}{-} y_1}{x_2 \underset{②}{-} x_1} ③.$$

$$\frac{(-18)-(-3)}{(-6)-(4)} \qquad \textit{Substituting}$$

$$\frac{(-18)+(3)}{(-6)+(-4)} \qquad \textit{Changing both subtractions to “adding the opposite”}$$

$$\frac{-15}{(-6)+(-4)} \qquad ①$$

$$\frac{-15}{-10} \qquad ②$$

$$\frac{3}{2} \qquad ③.$$

The answer is $3/2$. *Notice that the final answer, as the quotient of two negative numbers, is positive. Also, in performing the division, we write the fraction in reduced form since the answer is not an integer.*

3.3.1 Function notation

A function is a mathematical concept that is meant to express a relationship between two or more quantities. Students who will go on to study calculus will be expected to become familiar with these mathematical objects. The concept of a function is the main concept that is introduced in "precalculus" classes, and is central for any understanding of higher mathematics and many applications.

Very roughly, we can think of a function as a rule that takes one quantity and assigns to it another quantity. For our purposes, the quantities involved can be understood to be numbers.

Here, we are only going to discuss the most basic notation associated to functions. Functions will be given a "name," which we will denote with a letter, usually f but also g, h, etc. If the variable x represents a (numerical) quantity, the symbol $f(x)$ will represent the quantity that the function f assigns to the value x. *The notation $f(x)$ should be understood as a single symbol which represents a value.* In particular, *there is no multiplication implied in the notation $f(x)$.*

Many times, a function will be defined by means of an algebraic expression. For example, we might encounter a function described as

$$f(x) = 3x^2 - 5x + 4.$$

This just means that for any value of x, the function f assigns to x the value of $3x^2 - 5x + 4$ (evaluated with a given value for x). Notice the only variable appearing in the expression $3x^2 - 5x + 4$ is x. The notation $f(x)$ is meant to indicated that the function named f depends only of the value of the variable x.

When a function is defined algebraically using function notation, $f(x)$ can also be thought of as the "value" of f for the variable x. Since x is presumed unknown, the "value" of f is also unknown, but it is given in terms of x by the given algebraic expression. If the value of x is known, the value of f can be also be found by substituting the given value of the variable into the algebraic expression defining f. The notation $f(1)$, for example, means "the value of f when $x = 1$." Likewise, $f(-3)$ means "the value of f when $x = -3$.

Example 3.3.4. *Find $f(-1)$ if $f(x) = x^2 - 3x + 1$.*

Answer. *In this example, $f(x)$ is represented by the expression $x^2 - 3x + 1$. So $f(-1)$ is simply the value of $x^2 - 3x + 1$ when $x = -1$. We proceed exactly like we did in the previous section.*

$$\begin{aligned}
f(-1) &= (-1)^2 - 3(-1) + 1 && \textit{Substituting}\\
&= (1) - 3(-1) + 1 && \textit{Exponent first}\\
&= 1 - (-3) + 1 && \textit{Multiplication second}\\
&= 1 + 3 + 1 && \textit{Changing subtraction to ``adding the opposite''}\\
&= 4 + 1 && \textit{The first addition (formerly subtraction)}\\
&= 5. && \textit{The second addition}
\end{aligned}$$

The answer is 5.

3.3.2 Exercises

1. Evaluate $-3x^2 + 7x - 5$ when $x = -5$.

2. Evaluate $\sqrt{x^2 + y^2}$ when $x = -5$ and $y = 12$.

3. Evaluate $b^2 - 4ac$ when $a = 1$, $b = -5$, $c = 6$.

4. Evaluate $\dfrac{I}{RT}$ when $I = 150$, $R = 0.04$, $T = 2$.

5. Evaluate $2y^2 - 9y - 1$ when $y = -\dfrac{2}{3}$.

6. Evaluate $\dfrac{y_2 - y_1}{x_2 - x_1}$ when
   (a) $x_1 = 0$, $y_1 = -4$, $x_2 = 2$, $y_2 = 0$
   (b) $x_1 = -2$, $y_1 = 4$, $x_2 = 4$, $y_2 = -8$
   (c) $x_1 = 1$, $y_1 = -1$, $x_2 = 4$, $y_2 = 5$
   (d) $x_1 = -1/3$, $y_1 = 0$, $x_2 = 0$, $y_2 = -5$
   (e) $x_1 = 3$, $y_1 = 5$, $x_2 = -2$, $y_2 = -8$

7. Evaluate $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ when:
   (a) $a = 1$, $b = -1$, $c = -6$
   (b) $a = 2$, $b = -7$, $c = 3$
   (c) $a = 1$, $b = 2$, $c = -8$
   (d) $a = 1/2$, $b = -1/3$, $c = -1/6$
   (e) $a = 4$, $b = 16$, $c = 15$

8. Evaluate $\dfrac{9}{5}C + 32$ when:
   (a) $C = -40$
   (b) $C = -15$
   (c) $C = 10$
   (d) $C = 25$
   (e) $C = 32$

9. For the function $f(x) = x^2 - 4x + 3$, evaluate
   (a) $f(-3)$
   (b) $f(1/2)$
   (c) $f(2)$

10. For the function $f(x) = 3x^3 + 2x^2 - 5x - 12$, evaluate
    (a) $f(0)$
    (b) $f(-1)$
    (c) $f(2)$

3.4 Translating algebraic expressions

Sometimes, we want to translate between an algebraic expression, represented symbolically with variables, numbers, and operations, and a verbal expression with the same meaning. This is most useful when it comes to so-called "word problems" involving applications of algebra.

The first thing to identify in translating an algebraic expression are the variable or unknown quantities, which will be represented with a letter. When there are more than one unknown quantity, or when an unknown quantity occurs more than once in the expression, it is important to distinguish between using the same variable or different variables.

Example 3.4.1. *Translate the following expression into words:* $3x^2 + 2x + 4$.

Answer. *Notice that the expression involves only one variable, which we will translate as "an unknown number." Also notice that the expression involves five operations.*

Here is one possible translation:

"Three times the square of an unknown number, increased by twice the same number, increased again by four."

Can you identify the operations in the sentence above?

Subtraction sometimes causes confusion in translation, because of the importance that the order plays. For example, if someone ask you, "What is four less than ten?", or tells you to "subtract four from ten," you will perform the operation $10 - 4$. The same operation, however, could be expressed by, "Ten decreased by four."

Example 3.4.2. *Translate into an algebraic expression: Five less than three times a number.*

Answer. *We right away identify the unknown quantity, in this case "a number," and represent it by the variable x.*

The phrase "less than" is one of the phrases for subtraction that reverses the normal left-to-right order. Hence, one translation would be:

$$3x - 5.$$

Another issue to be aware of in translating is the presence of implied grouping.

Example 3.4.3. *Translate: "Twice the sum of a number and five, decreased by the difference of three times another number and two."*

Answer. *Notice right away that there are two unknown quantities, "a number" and "another number." We will call them s and t.*

This sentence is more complicated than the others because there is implied grouping. You might see this by noticing that the phrase has the following structure:

"Twice SOMETHING decreased by SOMETHING ELSE."

The SOMETHING and the SOMETHING ELSE are groups, each involving alegbraic expressions themsleves. However, right away, we can expect the answer to have the form

$$2(\qquad) - (\qquad).$$

What is the SOMETHING? "The sum of a number and five," translated as $s + 5$.

What is the SOMETHING ELSE? "The difference of three times another number and two," translated as $3t - 2$.

So one translation for the expression would be:

$$2(s + 5) - (3t - 2).$$

Let's summarize a few common issues to watch out for in translating between algebra and words:

- Make sure to represent the same unknown quantity with the same variable name, no matter how many times it appears in the expression, while different unknown quantities should be represented by different variables.
- Subtraction has several verbalizations that reverse the usual left-to-right order. "Subtract something from something else," or "Something less than something else," both reverse the usual left-to-right order, while "Something decreased by something else," or, "The difference of something and something else," maintain the usual left-to-right order.
- Grouped operations are often implied, for example, by phrases like "the quantity of." But they are also expressed in phrases like, "Three times the difference..."

3.4.1 Exercises

Translate the following phrases into algebraic expressions:

1. Twice the sum of an unknown quantity and 8.
2. Seven less than half of a number.
3. One-fourth of the difference of some number and 12.

Translate the following algebraic expressions into words:

4. $7y - 30$
5. $3(x + 2)$
6. $(x - 5)^2 + 3x$
7. $\dfrac{2x - 1}{3}$
8. $\dfrac{7x - 3}{x - 3}$

3.5 Chapter summary

- In expressions involving several operations, the operations are performed according to the order of operations.
- Parentheses and other grouping symbols indicate the expressions inside the symbols are to be treated as one group.
- Parentheses and other grouping symbols prioritize the operations within groups.
- Algebraic expressions consist of variables and numbers combined using the operations of addition, subtraction, multiplication, division, (numerical) exponents and roots.
- Algebraic expressions can be evaluated if values are given for all of the variables involved. In this case, the values are substituted for the variables and the operations are performed according to the order of operations.
- It is often necessary to translate back and forth between algebraic expressions and their English-language translation.