Chapter 2

Signed numbers

Vocabulary

- Magnitude
- Absolute value
- Positive
- Negative
- Opposite of a number
- Base (of an exponential expression)

2.1 Introduction

When a child learns to count, numbers only go in one direction—they "get bigger." In fact, for thousands of years, civilizations rose and fell using only positive numbers. After all, numbers first arise in answer to the question, "How many?" How many what? How many things—the things being things which might be picked up, looked at, put on a shelf, etc.

Negative numbers are more complicated. It doesn't make very much sense to say, "I have -5 books." Historically, negative numbers arose to take into account losses or debt, and was undoubtedly connected to the emergence of money or coin. In this way, there is a meaning to the sentence, "I have -5 dollars"—it simply means that I owe 5 dollars, instead of having 5 dollars. In this context, "to owe" is the opposite of "to have," and the negative numbers will be the opposite of the the more familiar positive numbers, in a way that will be made precise below.

For now, let's say that every positive number has an opposite, and this opposite (of a positive number) will be a negative number. Zero will be special, in that it is neither positive nor negative; it is "neutral," and we will say that zero "is its own opposite."

Our goal in this chapter will be to describe how to perform the basic arithmetic operations—adding, subtracting, multiplying, dividing—with these negative numbers.

As soon as we allow negative numbers, we will need to take into account two aspects of every nonzero number: its *sign*, which can be either positive or negative, and its *magnitude*, which is a positive numerical value. A positive number is indicated by a "+" symbol along with a magnitude. For example, the symbol $+5$ will represent the number whose sign is positive and whose magnitude is 5. A negative number is indicated by a "−" symbol along with a magnitude. For example, the symbol -5 will represent the number whose sign is negative and whose magnitude is 5. Notice that *the magnitude of a nonzero number is always positive.* Operations with signed numbers will have to take both of these aspects into account.

(The magnitude of a number is represented symbolically by means of the *absolute value* symbol $| \cdot |$. For example, we can summarize the preceding paragraph with $|5| = 5$ and $|-5| = 5$.)

We have already seen that zero is a special number when it comes to signs. In fact, the very idea of "opposite" that we have used to motivate the negative numbers will be defined relative to the number zero. Along with the fact that the number 0 will be neither positive nor negative, we will say that 0 has magnitude zero.

Warning: Do not confuse the meaning of symbols for the sign of a number with the meaning of the symbols for addition and subtraction. It is an unfortunate fact of history that the symbols are in fact the same, but the meanings are very different, as we will see below.

Convention: When a sign is not indicated for a number, it will be assumed to be positive. For example, the symbol "5" will have the same meaning as "+5."

2.2 Graphical representation and comparison of signed numbers

While there are several ways to understand negative numbers, the graphical representation of numbers on a number line is particularly helpful. Recall that a number line has three essential components: it extends infinitely (from left to right), it has a special point representing zero, and it has a specified unit length. In this representation, positive numbers will be those numbers represented by points *to the right of zero*, while negative numbers will be those numbers represented by points *to the left of zero*.

When we represent a signed number on a number line, the number's sign will tell us on which side of zero it will be represented, while its magnitude will tell us the distance (in terms of the specified unit length) from the representative point to the point representing zero. Thinking of the magnitude as the "distance from zero" corresponds to the convention that magnitudes of nonzero numbers, like distances, are always positive quantities.

The number line representation of signed numbers also gives us an easy was to visualize comparisons of signed numbers. By comparison, we mean either "less than," "equal to," or "greater than." Symbolically, these three possible comparisons are written as $<$ ("is less than"), $=$ ("is equal to") and $>$ ("is greater than").

Comparing positive numbers corresponds to our standard notions of quantity. So for example, $15 < 27$. Comparing positive numbers in decimal or fraction notation is only a little more challenging, in that we first need to see them as like quantities before comparing. So $0.043 > 0.0099$ (since $0.043 = 0.0430$ and $430 > 99$) and $\frac{3}{11} < \frac{2}{7}$ (since $\frac{3}{11} = \frac{21}{77}$, $\frac{2}{7} = \frac{22}{77}$, and $21 < 22$). But which is bigger, -10 or -15?

Using the number line representation and comparison of positive numbers as our guide, we will translate "less than" as "to the left of," and "greater than" as "to the right of." In this way, $-10 > -15$ since the point representing -10 is to the right of the point representing -15 on the number line.

This reasoning can be summarized in the following guide for comparing signed numbers. Note that the signs and the magnitudes are both important in comparing two signed numbers.

- The lesser of *two positive numbers* is the positive number with the lesser magnitude.
- The lesser of *one positive and one negative number* is the negative number.
- The lesser of *two negative numbers* is the negative number with the greater magnitude (the "most negative" number).

2.3 Operations with signed numbers: Addition and subtraction

How much money do you have at the end of the following situations? Think of debt as being represented by negative numbers and money you have as positive numbers.

- You have \$100. Your partner hands you \$250.
- You have \$100. Your partner hands you an \$80 phone bill.
- You have an \$80 phone bill. Your partner hands you \$250.
- You have an \$80 phone bill. Your partner hands you a \$100 electric bill.

In all four scenarios, you have something and your partner adds to what you have. But the way you treat the four cases is different.

The goal in this section is to arrive at rules for adding and subtracting signed numbers. Because we now have to keep track of two aspects of each number—its sign and its magnitude—the rules will be more complicated than the rules for adding and subtracting positive numbers that we learned in grade school.

2.3.1 Adding signed numbers

Before listing the rules for addition, let's give another illustration using the graphical representation of numbers on a number line. If you had to draw a picture, using the number line model of numbers, of the familiar equation "$2 + 3 = 5$," perhaps the best way to do it would be as follows:

First, draw an arrow starting at 0 and stretching for 2 units in the positive direction—to the right. Then, draw another arrow starting where the first arrow ended (at the point representing 2) and stretching for 3 units, also in the positive direction. The sum is represented by the point where the second arrow ends: at the point representing 5. See Figure 2.1.

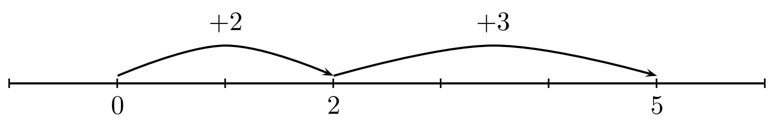

Figure 2.1: Adding 3 to 2.

The advantage of this graphical representation of addition is that it is very easily adapted to take into account negative numbers. We will simply draw negative numbers using arrows pointing in the negative direction–arrows pointing to the left. For example, Figure 2.2 is a number line representation of the sum $4 + (-5)$.

The result indicates that $4 + (-5) = -1$.

If you take a few minutes to draw a few more of these number line pictures, a few things should become clear. First, the pattern depends on whether the two numbers being added have the *same sign* or *different sign*. Depending on that, we will end up wither adding or subtracting the magnitudes.

Here is a summary of the conclusions of this discussion.

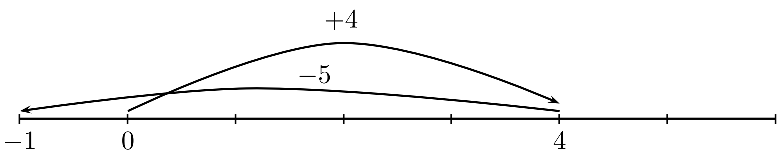

Figure 2.2: Adding −5 to 4.

Rules for adding signed numbers

- *To add signed numbers with the* ***same sign***:
  - The sum has the *same sign* as the sign of the two original numbers;
  - The sum has magnitude which is the *sum* of the magnitudes of the two original numbers.
- *To add signed numbers with* ***different signs***:
  - The sum has the *sign of the number with the larger magnitude* of the two orginal numbers;
  - The sum has magnitude which is the *difference* of the magnitudes of the two original numbers (subtracting the smaller magnitude from the larger).

Here are some examples.

Example 2.3.1. *Add:* $(-12) + (-15)$.
Notice that we are adding two numbers with the same sign—both are negative. This tells us two things:

- *The sum will have the same sign—it will also be negative.*
- *The magnitude will be the sum of the magnitudes:* $12 + 15 = 27$. *(Remember: magnitudes are always positive quantities!)*

So $(-12) + (-15) = -27$.
The answer is -27.

The parentheses in the preceding example are grouping symbols, meant to indicate that the symbol representing the sign of the numbers "goes with" the number, and are "separate from" the symbol representing addition. It is in mathematical "bad taste" to write expressions like $-12 + -15$.

Example 2.3.2. *Add:* $(-4) + 12$.

This time we are adding numbers with different signs. Notice the $+$ *sign represents addition, not a sign. However, reading the phrase as "negative four plus twelve," we see that the number* 12*, since it does not have a sign specified, is in fact positive.*

- *The sum will have the sign of the number with the bigger magnitude, which is positive. (After all,* $+12$ *has magnitude* 12*, and* -4 *has magnitude* 4*.)*
- *The sum will be the difference of the larger magnitude and the smaller magnitude:* $12 - 4 = 8$.

So $(-4) + 12 = 8$.
The answer is 8.

We illustrate one more example using fractions.

Example 2.3.3. *Add:* $-\dfrac{4}{5} + \dfrac{3}{10}$.

As usual when adding fractions, we will need to rewrite the problem using equivalent fractions with a common denominator. This will also allow us to compare the magnitudes of the fractions.

We are adding two numbers with opposite signs. The magnitude of $-\frac{4}{5}$ *is* $\frac{4}{5}\left(=\frac{8}{10}\right)$*. The magnitude of* $\frac{3}{10}$ *is* $\frac{3}{10}$*. The sign of the final answer will be negative, since the number with greater magnitude is negative.*

Subtracting the magnitudes:

$$
\begin{array}{rl}
\dfrac{4}{5} - \dfrac{3}{10} & \\
\dfrac{4^{\times 2}}{5^{\times 2}} - \dfrac{3^{\times 1}}{10^{\times 1}} & \textit{Writing with common denominator } 10 \\
\dfrac{8}{10} - \dfrac{3}{10} & \\
\dfrac{8-3}{10} & \\
\dfrac{5}{10} & \\
\dfrac{1}{2} & \textit{Reducing.}
\end{array}
$$

Hence: $-\dfrac{4}{5} + \dfrac{3}{10} = -\dfrac{1}{2}$.
The answer is $-\dfrac{1}{2}$.

What should be clear from the above rules and examples is that addition of signed numbers is a two-step process, corresponding to the two "parts" of signed numbers. We first establish the sign of the result. We then determine the magnitude of the results by either adding or subtracting the magnitudes, depending on the signs of the original two numbers.

Warning! Notice that these rules do NOT say that "a negative and a negative gives a positive," which is a famous distortion of a correct rule (see below). In fact, the sum of negative numbers is negative, not positive!

2.3.2 Subtracting signed numbers

We will not make a new list of rules for subtracting signed numbers. Instead, we will illustrate a procedure to rewrite subtraction problems as addition problems, and then rely on the same rules for adding signed numbers that we outlined in the previous section.

When subtraction is first presented in grade school, it is usually described as the operation of "taking away." Unfortunately, while this analogy still holds for negative numbers, it can be more confusing. Is it obvious to you that taking away debt has the same effect as giving you cash?

Instead, we revert to the number line representations that gave us a clue to our rules of adding signed numbers above. What would the number line picture look like for the subtraction problem $6 - 2 = 4$?

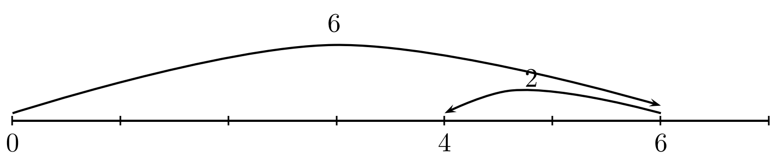

Figure 2.3: Subtracting 2 from 6.

Subtracting (positive!) 2 is represented with an arrow going in the opposite direction ("taking away from") the positive direction. See Figure 2.3.

Now compare Figure 2.3 to the number line representation of the addition problem $6+(-2)$ in Figure 2.4. The two pictures are identical! We have labelled them slightly differently to take into account that the second diagram depicts addition, where the first diagram depicts subtraction.

These two number line diagrams reflect a very important mathematical fact:

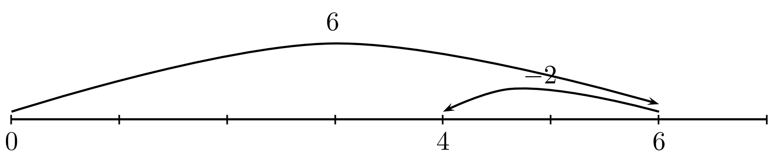

Figure 2.4: Adding −2 to 6.

Subtraction is the same as "adding the opposite."

To be more precise, we need to understand what "the opposite" means. In the context of addition, the opposite of a number is the number with the same magnitude, but opposite sign. So the opposite of 3 is -3, whereas the opposite of -8 is 8. To be even more precise, we will say that *two numbers are opposites if their sum is* 0. Notice that $3 + (-3) = 0$ and $(-8) + 8 = 0$.

Notation: Many times, it will be convenient to think of a negative sign as representing "the opposite of." So the symbol -5 can be thought of interchangeably as "negative five" and "the opposite of (positive) five." More importantly, the symbol $-(-12)$ should be understood as "the opposite of negative twelve"—which is of course positive twelve!

We are now going to outline a procedure for subtracting signed numbers. Keep in mind that in any subtraction problem, the order of the numbers matter! For example, $7-5$ is not the same as $5-7$. (In mathematical terms, subtraction is not *commutative*.)

Procedure for subtracting signed numbers

1. Rewrite the subtraction problem as the sum of the first number and the *opposite of the second number*;
2. Then apply the rules for adding signed numbers in the preceding section.

Important note: In applying this procedure for subtracting signed numbers, it is important to emphasize the difference in meaning between the symbol for subtraction and the symbol for "negative" (or "opposite of"). Most times, reading the problem out loud will indicate whether the $-$ symbol represents subtraction ("minus") or opposite ("negative").

Exercise 2.3.4. *Subtract:* $-10 - 12$.

Answer. *The problem reads: "Negative ten minus twelve." The numbers that are being subtracted are* -10 *and (positive)* 12*. Rewriting the subtraction as "Negative ten plus negative twelve,"*

$$\begin{aligned} -10 - 12 \\ -10 + (-12) \\ -22. \end{aligned}$$

The answer is -22*.*

Notice that this procedure involves two separate changes: the operation changes from subtraction to addition, and the sign of the second number changes. The first number does not change.

Exercise 2.3.5. *Subtract:* $(-15) - (-8)$.

Answer. *The problem reads: "Negative fifteen minus negative eight." The numbers being subtracted are* -15 *and* -8*. Rewriting,*

$$\begin{aligned} -15 - (-8) \\ -15 + (+8) \\ -7. \end{aligned}$$

The answer is -7*.*

Here is one more example, this time with fractions:

Exercise 2.3.6. *Subtract:* $\frac{1}{4} - \frac{5}{6}$.

Answer. *The problem reads: "(Positive) one-fourth minus (positive) five-sixths." Rewriting:*

$$\begin{aligned} \frac{1}{4} - \frac{5}{6} \\ \frac{1}{4} + \left(-\frac{5}{6}\right) \\ \frac{3}{12} + \left(-\frac{10}{12}\right) && \textit{(rewriting with least common denominator)} \\ -\frac{7}{12}. \end{aligned}$$

We will comment more on the placement of the negative sign in the next section, after discussing dividing signed numbers. Here, we obtained the last step by noting that the number with larger magnitude ($-\frac{10}{12}$) was negative, so the sum must be negative.

The answer is $-\dfrac{7}{12}$.

2.3.3 Exercises

Perform the indicated operations.

1. $(-4) + (-3)$
2. $(-5) + (-2)$
3. $3.25 + (-1.8)$
4. $(-12) + (-2)$
5. $10 + (-10)$
6. $\left(-\dfrac{2}{5}\right) + \left(-\dfrac{1}{4}\right)$
7. $\left(\dfrac{1}{8}\right) + \left(-\dfrac{1}{12}\right)$
8. $(-5) - (-2)$
9. $5 - 12$
10. $-4 - 4$
11. $(-4.03) - (-2.1)$
12. $\left(-\dfrac{3}{4}\right) - \left(\dfrac{1}{4}\right)$
13. $\left(\dfrac{1}{3}\right) - \left(-\dfrac{1}{4}\right)$
14. $\left(-\dfrac{4}{7}\right) - \left(-\dfrac{5}{14}\right)$
15. $\dfrac{3}{4} - \dfrac{7}{8}$

2.4 Operations with signed numbers: Multiplication and division

Multiplication of whole numbers emerged as an abbreviated form of addition. So, for example, "6×3" means 3 added to itself 6 times. But to extend this simple understanding to all kinds of numbers, we require that the commutative, associative, and distributive properties continue to hold.

It is not too hard, for example, to extend the definition of multiplication to all integers. First, multiplying a positive number by a negative number, we can reason as in the following example: We can think of $6 \times (-3)$ as -3 added to itself 6 times:

$$(-3) + (-3) + (-3) + (-3) + (-3) + (-3) = -18.$$

If the negative number appears first, this is slightly more difficult: How can we think of $(-2) \times 4$ as 4 added to itself -2 times? However, if we insist that multiplication of signed numbers should still be commutative, then $(-2) \times 4$ must be the same as $4 \times (-2)$, which is

$$(-2) + (-2) + (-2) + (-2) = -8.$$

What should be clear from both of these examples is that *the product of a negative number and a positive number is a negative number.*

What about multiplying a negative number by a negative number? First, you should convince yourself that the opposite of a positive number is negative, and the opposite of a negative number is positive. Second, remember that two numbers are opposites if their sum is 0.

Let's consider the example $(-2) \times (-3)$. We will show that this number is the opposite of $2 \times (-3)$. After all,

$$\begin{aligned}(-2) \times (-3) + 2 \times (-3) &= (2 + (-2)) \times (-3) \quad \text{by the distributive property} \\ &= 0 \times (-3) \\ &= 0.\end{aligned}$$

This calculation shows that $(-2) \times (-3)$ is the opposite of $2 \times (-3)$. But we saw above that $2 \times (-3) = -6$, so $(-2) \times (-3)$ is the opposite of -6. In conclusion, assuming that the distributive property is to hold, $(-2) \times (-3) = +6$.

This example is meant to illustrate a famous but often little-understood property of multiplication of signed numbers: *the product of two negative numbers is a positive number.*

Finally, a word about division of signed numbers. We saw (in the context of fractions) that division is the same as multiplication by the reciprocal. Keeping in mind that two numbers are reciprocals if their product is (positive!) 1, it should not be too hard to see that the reciprocal of a negative number is also negative, and the reciprocal of a positive number is also positive. So for the purpose of signed numbers, division will follow exactly the same rules as multiplication.

This brief discussion leads to the following rules for multiplying signed numbers:

Rules for multiplying and dividing signed numbers

- The product (or quotient) of two numbers with the *same sign* is positive.
- The product (or quotient) of two numbers with *different signs* is negative.

In both case, the magnitude of the product (or quotient) is just the product (or quotient) of the magnitudes of the two numbers.

Notice how nice these rules are compared to the rules for adding signed numbers! We never have to worry about which number has larger magnitude, and we always perform exactly the operation indicated in the problem.

Let's look at a couple of examples. We'll take the opportunity to review operations with decimals also.

Example 2.4.1. *Multiply:* $(-0.004) \times (-2.68)$.

Answer. *Putting aside that the two numbers are written in decimal form, we have the product of two numbers with the same sign—they are both negative. The answer will be positive.*

We will calculate the magnitude separately, by multiplying $(0.004) \times (2.68)$. *Recall that keeping track of the fact that there are a total of five decimal places to the right of the units place (three from the first number, two from the second), we will first multiply* $4 \times 268 = 1072$. *Now we will ensure that the final answer shows five decimal places:* 0.01072.

The answer is $+0.01072$.

Example 2.4.2. *Divide:* $(15.3) \div (-0.03)$.

We are dividing two numbers with different signs. The result will be negative.

We will divide the magnitudes $(15.3) \div (0.03)$. Recall that we will do this as a long division problem, making sure that the divisor (the *second* number in the divison problem) is a whole number. The divisor here has two decimal places; we move the decimal place for both numbers two places to the right (which amounts to multiplying both numbers by 100): $1530 \div 3 = 510$.

In conclusion, $(15.3) \div (-0.03) = -510$.

The answer is -5100.

2.4.1 Exercises

Perform the indicated operations.

1. $(3)(-12)$
2. $(-1)(-25)$
3. $6 \div (-2)$
4. $(-3.2) \div (-0.04)$
5. $\frac{6}{7} \cdot \left(-\frac{2}{3}\right)$
6. $\left(-1\frac{3}{4}\right) \div \left(-1\frac{1}{3}\right)$

2.5 Operations with signed numbers: Exponents and square roots

Whole number exponents indicate repeated multiplication, in the same way that whole number multiplication represent repeated addition.

The *base* of an exponential expression is the quantity immediately to the left of the exponent.

Example 2.5.1. *The base of* $(-4)^{10}$ *is* -4. *(The parentheses immediately to the left of the exponent tells us that the number inside the parentheses is the base.)*

Example 2.5.2. *The base of* -2^5 *is* 2. *Unlike the previous example, the quantity immediately to the left of the exponent is* 2.

Since we will only be concerned (for now!) with whole number exponents, the rules for exponentials with negative base follow immediately from the rules for multiplying signed numbers. In particular, you should notice the following fact:

The sign of exponentials with a negative base

A negative base raised to an even exponent is positive. A negative base raised to an odd exponent is negative

We will also from time to time encounter square roots, written with the symbol $\sqrt{\ }$

.Rememberthatthesquarerootofanumberisthe(non−negative)numberwhich, whenraisedtothese
4 since $(4)^2 = 16$.

For now, we won't go into the different issues that arise from the operation of taking square roots. In fact, for now, we will only consider square roots of *perfect squares*: whole numbers which are the second power of another whole number. The first few perfect squares are

$$1, 4, 9, 16, 25, \ldots$$

If you haven't worked with perfect squares for a while, you should take a moment to make a list of the first 12 or 15 of them (by squaring the numbers from 1 to 15, for example).

There is one important fact that we cannot ignore when we are talking about negative numbers and square roots in the same section. We have already seen that any number, positive or negative, when squared, will result in a positive number. After all, "squaring" a number is raising the number to an even exponent (of 2). For that reason, no real number[1] can be the square root of a negative number. Said differently, **the square root of a negative number cannot be a real number.**

Warning: $-\sqrt{9}$ does not mean the same as $\sqrt{-9}$. $-\sqrt{9}$ means "the opposite of the square root of 9," which is "the opposite of 3," or -3. $\sqrt{-9}$ is the square root of -9, which is not a real number, as we just saw.

2.5.1 Exercises

Perform the indicated operations.

1. $(-3)^4$
2. $(-2)^3$
3. $-(-2)^5$
4. $-\sqrt{81}$
5. $\sqrt{-36}$

2.6 Chapter summary

- Every nonzero number has two "parts:" a magnitude and a sign.
- The number 0 is neither positive nor negative, and has magnitude zero.

[1] For our purposes, the real numbers are all those that can be represented as points on the number line in the manner we have described above. In particular, they can be ordered, and there are points "infinitesimally close" to any other point on the number line. Later, we will see "numbers" that are not real numbers.

- Negative numbers are represented on a number line to the left of 0, while positive numbers are represented to the right of 0. In both cases, the magnitude of the number is represented by the distance from the point representing it to the origin.

- The rules for adding two signed numbers depend on whether the numbers have the same sign or different signs. If the numbers have the same sign, the magnitudes are added and the sign is the same as that of the two numbers. If the numbers have different signs, the magnitudes are subtracted and the sign is the sign of the number with the larger magnitude.

- The rules for multiplying or dividing two signed numbers depend on whether the numbers have the same sign or different signs. If the numbers have the same sign, the result is positive. If the numbers have different signs, the result is negative. The magnitude of the product (or quotient) is the product (or quotient) of the magnitudes.

- A negative base raised to an even power will be positive. A negative base raised to an odd power will be negative.

- The square root of a negative number cannot be a real number.