ARITHMETIC:
A Textbook for Math 01
2nd edition (2012)

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– A.W. January, 2012.

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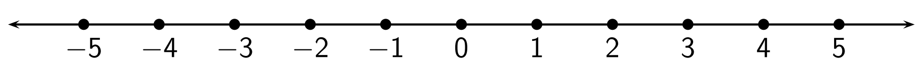
Chapter 1

Whole Numbers

The *natural numbers* are the counting numbers: $1, 2, 3, 4, 5, 6, \ldots$. The dots indicate that the sequence is *infinite* – counting can go on forever, since you can always get the next number by simply adding 1 to the previous number. In order to write numbers efficiently, and for other reasons, we also need the number 0. Later on, we will need the sequence of *negative* numbers $-1, -2, -3, -4, -5, -6, \ldots$. Taken together, all these numbers are called the **integers**.

It helps to visualize the integers laid out on a **number line**, with 0 in the middle, and the natural numbers increasing to the right. There are numbers between any two integers on the number line. In fact, every location on the line represents some number. Some locations represent *fractions* such as one-half (between 0 and 1) or four-thirds (between 1 and 2). Other locations represent numbers which cannot be expressed as fractions, such as π. (π is located between 3 and 4 and expresses the ratio of the circumference to the diameter of any circle.)



For now, we concentrate on the non-negative integers (including 0), which we call **whole numbers**. We need only ten symbols to write any whole number. These symbols are the *digits*

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9.$$

We write larger whole numbers using a **place-value** system. The digit in the right-most place indicates how many *ones* the number contains, the digit in the second-from-right place indicates how many *tens* the number contains, the digit in the third-from-right place indicates how many *hundreds* the number contains, etc.

Example 1.

$$\begin{aligned} 7 \text{ stands for } &\ 7\ \textit{ones} \\ 72 \text{ stands for } &\ 7\ \textit{tens} + 2\ \textit{ones} \\ 349 \text{ stands for } &\ 3\ \textit{hundreds} + 4\ \textit{tens} + 9\ \textit{ones} \\ 6040 \text{ stands for } &\ 6\ \textit{thousands} + 0\ \textit{hundreds} + 4\ \textit{tens} + 0\ \textit{ones} \end{aligned}$$

Notice that when you move left, the place value increases ten-fold. So if a number has five digits, the fifth-from-right place indicates how many ten-thousands the number contains. (Ten-thousand is ten times a thousand.)

1.0.1 Exercises

1. 35 stands for ____
2. 209 stands for ____
3. 9532 stands for ____
4. 21045 stands for ____

1.1 Adding Whole Numbers

When we add two or more integers, the result is called the **sum**. We assume you know the sums of single-digit numbers. Just to make sure, do the following example.

Example 2. Fill in the missing squares in the *digit-addition table* below. For example, the number in the row labelled 3 and the column labelled 4 is the sum $3 + 4 = 7$.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1										
2										
3					7					
4										
5										
6										
7						12				
8										
9										

Table 1.1: The digit addition table

1. Do you notice any patterns or regularities in the digit-addition table? Can you explain them?
2. Why is the second line from the top identical to the top line?
3. What can you say about the left-most column and the second-from-left column?

1.1.1 Commutativity, Associativity, Identity

The first question leads us to an important property of addition, namely, that for any two numbers x and y,

$$x + y = y + x.$$

In other words, the order in which two numbers are added does not effect the sum. This property of addition is called **commutativity**.

The last two questions lead us to an important property of 0, namely, for any number x,

$$x + 0 = x = 0 + x.$$

In other words, when 0 is added to any number, x, you get the identical number, x, again. Because of this property, 0 is called the **additive identity**.

One final property of addition is expressed in the following equation

$$(x + y) + z = x + (y + z),$$

which says that if three numbers are added, it doesn't matter how you "associate" the additions: you can add the first two numbers first, and then add the third to that, or, you could add the second two numbers first, and then add the first to that. This property of addition is called **associativity**.

Example 3. Find the sum of 2, 3, and 5 by associating in two different ways.

Solution. Associating 2 and 3, we calculate

$$(2 + 3) + 5 = 5 + 5 = 10.$$

On the other hand, associating 3 and 5, we calculate

$$2 + (3 + 5) = 2 + 8 = 10.$$

The sum is the same in both cases. □

1.1.2 Multi-digit addition

To add numbers with more than one digit, we line up the numbers vertically so that the *ones* places (right-most) are directly on top of each other, and all other places are similarly arranged. Then we add the digits in each place to obtain the sum.

Example 4. Find the sum of 25 and 134.

Solution. We line up the numbers vertically so that the 5 in the ones place of 25 is over the 4 in the ones place of 134. If we do this carefully, all the other places line up vertically, too. So there is a "ones" column, a "tens" column to the left of it, and a "hundreds" column to the left of that:

$$\begin{array}{r} 2\,5 \\ 1\,3\,4 \end{array}$$

Then we draw a line and add the digits in each column to get the sum:

$$\begin{array}{rr} & 2\,5 \\ + & 1\,3\,4 \\ \hline & 1\,5\,9 \end{array}$$

□

Sometimes we need to "carry" a digit from one place to the next higher place. For example, when adding $38 + 47$, we first add the ones places, $8 + 7 = 15$. But 15 has two digits: it stands for 1 *ten* + 5 *ones*. We "put down" the 5 in the *ones* place, and "carry" the 1 (standing for a *ten*) to the *tens* column. So we have:

Example 5. Find the sum

$$\begin{array}{r} 3\ 8 \\ +\ \ 4\ 7 \\ \hline \end{array}$$

Solution. Put down the 5 in the *ones* place:

$$\begin{array}{r} 3\ 8 \\ +\ \ 4\ 7 \\ \hline 5 \end{array}$$

and carry the 1 to the top of the *tens* column:

$$\begin{array}{r} \{1\}\ \ \\ 3\ 8 \\ +\ \ 4\ 7 \\ \hline 5 \end{array}$$

and finish the job by adding up the *tens* column, including the carried one:

$$\begin{array}{r} \{1\}\ \ \\ 3\ 8 \\ +\ \ 4\ 7 \\ \hline 8\ 5 \end{array}$$

The sum is 85. □

1.1.3 Exercises

Find the sums, carrying where necessary.

1.

$$\begin{array}{r} 2\ 6 \\ +\ \ 5\ 5 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 383 \\ +\quad 47 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 32 \\ 45 \\ +\quad 64 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 129 \\ 377 \\ +\quad 4503 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 909 \\ 777 \\ +\quad 6964 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 320 \\ 984 \\ 902 \\ +\quad 4503 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 9 \\ 999 \\ 902 \\ +\quad 9502 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 56320 \\ 9864 \\ 904 \\ +\quad 6503 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 32000 \\ 9844 \\ 902 \\ +\quad 4503 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 2997 \\ 9844 \\ 205 \\ +\quad 54908 \\ \hline \end{array}$$

1.2 Subtracting Whole Numbers

Another way to say that $5 + 2 = 7$ is to say that $5 = 7 - 2$. In words, "5 is the difference of 7 and 2," or "5 is the result of taking away 2 from 7." The operation of taking away one number from another, or finding their **difference**, is called *subtraction*. For now, we have to be careful that the number we take away is no larger than the number we start with: we cannot have 3 marbles and take 7 of them away! Later on, when we introduce negative numbers, we won't have to worry about this.

We assume you remember the differences of single-digit numbers. Just to make sure, do the following example.

Example 6. Fill in the missing squares in the *digit-subtraction table* below. Here's a start: the number in the row labelled 7 and the column labelled 2 is the difference $7 - 2 = 5$. The digit in the row labelled 3 and the column labelled 3 is the difference $3 - 3 = 0$. (Squares that have an asterisk ($*$) will be filled in later with negative numbers.)

−	0	1	2	3	4	5	6	7	8	9
0		*	*	*	*	*	*	*	*	*
1			*	*	*	*	*	*	*	*
2				*	*	*	*	*	*	*
3				0	*	*	*	*	*	*
4						*	*	*	*	*
5							*	*	*	*
6								*	*	*
7			5						*	*
8										*
9						4				

Table 1.2: The digit subtraction table

1.2.1 Commutativity, Associativity, Identity

When we study negative numbers, we will see that subtraction is **not commutative**. We can see by a simple example that subtraction is also **not associative**.

Example 7. Verify that $(7-4)-2$ is not equal to $7-(4-2)$.

Solution. Associating the 7 and 4, we get

$$(7-4)-2=3-2=1,$$

but associating the 4 and the 2, we get

$$7-(4-2)=7-2=5,$$

a different answer. Until we establish an order of operations, we will avoid examples like this! □

It is true that

$$x-0=x$$

for any number x. However, 0 is not an identity for subtraction, since $0-x$ is *not* equal to 0 (unless $x=0$). To make sense of $0-x$, we will need negative numbers.

1.2.2 Multi-digit subtractions

To perform subtractions of multi-digit numbers, we need to distinguish the number “being diminished” from the number which is “doing the diminishing” (being taken away). The latter number is called the **subtrahend**, and the former, the **minuend**. For now, we take care that the subtrahend is no larger than the minuend.

To set up the subtraction, we line the numbers up vertically, with the minuend over the subtrahend, and the *ones* places lined up on the right.

Example 8. Find the difference of 196 and 43.

Solution. The subtrahend is 43 (the smaller number), so it goes on the bottom. We line up the numbers vertically so that the 6 in the ones place of 196 is over the 3 in the ones place of 43.

$$\begin{array}{r} 1\,9\,6 \\ 4\,3 \end{array}$$

Then we draw a line and subtract the digits in each column, starting with the *ones* column and working right to left, to get the difference:

$$\begin{array}{r} 1\,9\,6 \\ -\;\;4\,3 \\ \hline 1\,5\,3 \end{array}$$

□

1.2.3 Checking Subtractions

Subtraction is the "opposite" of addition, so any subtraction problem can be restated in terms of addition. Using the previous example, and adding the difference to the subtrahend, we obtain

$$\begin{array}{r} 1\,5\,3 \\ +\;\;4\,3 \\ \hline 1\,9\,6 \end{array}$$

which is the original minuend. In general, **if subtraction has been performed correctly, adding the difference to the subtrahend returns the minuend.** This gives us a good way to check subtractions.

Example 9. Check whether the following subtraction is correct:

$$\begin{array}{r} 9\,4 \\ -\;\;5\,1 \\ \hline 3\,3 \end{array}$$

Solution. Adding the (supposed) difference to the subtrahend, we get

$$\begin{array}{r} 3\,3 \\ +\;\;5\,1 \\ \hline 8\,4 \end{array}$$

which is not equal to the minuend (94). Thus the subtraction is incorrect. We leave it to you to fix it! □

1.2.4 Borrowing

Sometimes, when subtracting, we need to "borrow" a digit from a higher place and add its equivalent to a lower place.

Example 10. Find the difference:

$$\begin{array}{r} 8\,5 \\ -\;\;4\,6 \\ \hline \end{array}$$

Solution. The digit subtraction in the *ones* column is not possible (we can't take 6 from 5). Instead we remove or "borrow" 1 *ten* from the *tens* place of the minuend, and convert it into 10 *ones*, which we add to the *ones* in the ones place of the minuend. The minuend is now represented as $\{7\}\{15\}$, standing for 7 *tens* + 15 *ones*. It looks funny (as if 15 were a digit), but it doesn't change the value of the minuend, which is still $7 \times 10 + 15 = 85$. We represent the borrowing operation like this:

$$\begin{array}{rcc} & \{7\} & \{15\} \\ & 8 & 5 \\ - & 4 & 6 \\ \hline \end{array}$$

Ignoring the original minuend, we have $15 - 6 = 9$ for the *ones* place, and $7 - 3 = 4$ for the *tens* place, as follows:

$$\begin{array}{rcc} & \{7\} & \{15\} \\ & 8 & 5 \\ - & 4 & 6 \\ \hline & 3 & 9 \end{array}$$

The difference is 39. We can check this by verifying that the difference + the subtrahend = the original minuend:

$$\begin{array}{rr} & 3\,9 \\ + & 4\,6 \\ \hline & 8\,5 \end{array}$$

□

Sometimes you have to go more than one place to the left to borrow successfully. This happens when the next higher place has a 0 digit – there is nothing to borrow from.

Example 11. Find the difference:

$$\begin{array}{rr} & 2\,0\,7 \\ - & 6\,9 \\ \hline \end{array}$$

Solution. In the *ones* column we can't take 9 from 7, so we need to borrow from a higher place. We can't borrow from the *tens* place, because it has 0 *tens*. But we can borrow from the *hundreds* place. We borrow 1 *hundred*, and convert it into 9 *tens* and 10 *ones*. The minuend is now represented as $\{1\}\{9\}\{17\}$, standing for 1 *hundred* + 9 *tens* + 17 *ones*, (as if 17 were a digit). We represent the borrowing as before:

$$\begin{array}{rccc} & \{1\} & \{9\} & \{17\} \\ & 2 & 0 & 7 \\ - & & 6 & 9 \\ \hline \end{array}$$

Ignoring the original minuend, we have $17 - 9 = 8$ for the *ones* place, $9 - 6 = 3$ for the *tens* place, and $1 - 0 = 1$ for the *hundreds* place:

$$\begin{array}{rccc} & \{1\} & \{9\} & \{17\} \\ & 2 & 0 & 7 \\ - & & 6 & 9 \\ \hline & 1 & 3 & 8 \end{array}$$

The difference is 138. To check, we verify that the difference + the subtrahend = the original minuend:

$$\begin{array}{r} 1\,3\,8 \\ +\quad 6\,9 \\ \hline 2\,0\,7 \end{array}$$

(You may have noticed that the *carrying* you do in the addition check simply reverses the *borrowing* done in the original subtraction!) □

1.2.5 Exercises

Find the differences, borrowing where necessary. Check that the difference + the subtrahend = the minuend.

1.

$$\begin{array}{r} 9\,4 \\ -\quad 3\,7 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 2\,7\,5 \\ -\quad 1\,8\,1 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 3\,5\,0 \\ -\quad 7\,6 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 5\,0\,0 \\ -\quad 1\,9\,1 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 6\,0\,0 \\ -\quad 1\,9\,9 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 1500 \\ -\ 1191 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 5678 \\ -\ 4567 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 50000 \\ -\ 4999 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 801 \\ -\ 790 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 6389 \\ -\ 999 \\ \hline \end{array}$$

11.

$$\begin{array}{r} 500000 \\ -\ 43210 \\ \hline \end{array}$$

12.

$$\begin{array}{r} 9001010 \\ -\ 1111111 \\ \hline \end{array}$$

1.3 Multiplying Whole Numbers

Multiplication is really just repeated addition. When we say "4 times 3 equals 12," we can think of it as starting at 0 and adding 3 four times over:

$$0 + 3 + 3 + 3 + 3 = 12.$$

We can leave out the 0, since 0 is the additive identity ($0+3 = 3$). Using the symbol $\times$ for multiplication, we write

$$3 + 3 + 3 + 3 = 4 \times 3 = 12.$$

The result of multiplying two or more numbers is called the **product** of the numbers. We assume you remember the products of single-digit numbers. Just to make sure, do the following example.

Example 12. Fill in the *digit-multiplication table* below. Here's a start: the number in the row labelled 7 and the column labelled 5 is the product $7 \times 5 = 35$.

$\times$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2										
3					12					
4										
5										
6										
7						35				
8										
9										

Table 1.3: The digit multiplication table

1. Do you notice any patterns or regularities in the digit multiplication table? Can you explain them?

2. Why does the second row from the top contain only 0's?

3. Why is the third row from the top identical to the first row?

4. Is multiplication commutative? How can you tell from the table?

1.3.1 Commutativity, Associativity, Identity, the Zero Property

The second question leads us to an important property of 0, namely, when any number, x, is multiplied by 0, the product is 0:

$$0 \times x = x \times 0 = 0.$$

The third question leads us to an important property of 1, namely, that when any number, x, is multiplied by 1, the product is the identical number, x, again:

$$1 \times x = x \times 1 = x.$$

For this reason, 1 is called the **multiplicative identity**.

The following example should help you to see that **multiplication is commutative**.

Example 13. The figure shows two ways of piling up twelve small squares. On the left, we have piled up 3 rows of 4 squares (3×4); on the right, we have piled up 4 rows of 3 squares (4×3). In both cases, of course, the total number of squares is the product $3 \times 4 = 4 \times 3 = 12$.

Figure 1.1: $3 \times 4 = 4 \times 3 = 12$

Examples like the following help you to see that multiplication is associative.

Example 14. We can find the product $3 \times 4 \times 5$ in two different ways. We could first associate 3 and 4, getting

$$(3 \times 4) \times 5 = 12 \times 5 = 60,$$

or we could first associate 4 and 5, getting

$$3 \times (4 \times 5) = 3 \times 20 = 60.$$

The product is the same in both cases.

Instead of the $\times$ symbol, we often use a **central dot** ($\cdot$) to indicate multiplication. Thus, for example, instead of $2 \times 4 = 8$, we can write

$$2 \cdot 4 = 8.$$

1.3.2 Multi-digit multiplications

To multiply numbers when one of them has more than one digit, we need to distinguish the number "being multiplied" from the number which is "doing the multiplying." The latter number is called the **multiplier**, and the former, the **multiplicand**. It really makes no difference which number is called the multiplier and which the multiplicand (because multiplication is commutative!). But it saves space if we choose the multiplier to be the number with the fewest digits.

To set up the multiplication, we line the numbers up vertically, with the multiplicand over the multiplier, and the *ones* places lined up on the right.

Example 15. To multiply 232 by 3, we write

$$\begin{array}{r} 2\,3\,2 \\ \times\quad 3 \\ \hline \end{array}$$

We multiply place-by-place, putting the products in the appropriate column. 3×2 *ones* is 6 *ones*, so we put 6 in the *ones* place

$$\begin{array}{r} 2\,3\,2 \\ \times\quad 3 \\ \hline 6 \end{array}$$

3×3 *tens* is 9 *tens*, so we put 9 in the *tens* place

$$\begin{array}{r} 2\,3\,2 \\ \times\quad 3 \\ \hline 9\,6 \end{array}$$

Finally, 3×2 *hundreds* is 6 *hundreds*, so that we put down 6 in the hundreds place, and we are done:

$$\begin{array}{r} 2\,3\,2 \\ \times\quad 3 \\ \hline 6\,9\,6 \end{array}$$

Sometimes we need to carry a digit from one place to the next higher place, as in addition. For example, $4 \times 5 = 20$, a number with two digits. So we would "put down" the 0 in the current place, and "carry" the 2 to the column representing the next higher place, as in the next example.

Example 16. Multiply 251 by 4.

Solution. The steps are

$$\begin{array}{r} 2\,5\,1 \\ \times\quad 4 \\ \hline \end{array}$$

For the *ones* place,

$$\begin{array}{r} 2\,5\,1 \\ \times\quad 4 \\ \hline 4 \end{array}$$

For the *tens* place, $4 \times 5 = 20$, so we put down the 0 in the *tens* place and carry the 2 to the *hundreds* column:

$$\begin{array}{r} \{2\} \\ 2\,5\,1 \\ \times\quad 4 \\ \hline 0\,4 \end{array}$$

For the hundreds place, 4×2 *hundreds* is 8 *hundreds*, to which we add the 2 *hundreds* that were carried. This gives us 10 *hundreds*, or 1 *thousand*. We put down 0 in the *hundreds* place, and 1 in the *thousands* place.

$$\begin{array}{r} \{2\} \\ 2\,5\,1 \\ \times \qquad 4 \\ \hline 1\,0\,0\,4 \end{array}$$

The product of 251 and 4 is 1004. □

If the multiplier has more than one digit, the procedure is a little more complicated. We get *partial products* (one for each digit of the multiplier) which are added to yield the total product.

Example 17. Consider the product

$$\begin{array}{r} 2\,4 \\ \times \quad 3\,2 \\ \hline \end{array}$$

Since the multiplier stands for 3 *tens* + 2 *ones*, we can split the product into two partial products

$$\begin{array}{r} 2\,4 \\ \times \quad 2 \\ \hline 4\,8 \end{array}$$

and

$$\begin{array}{r} 2\,4 \\ \times \quad 3 \\ \hline \end{array}$$

Notice that in the second partial product the multiplier is in the *tens* column. This is almost exactly like having a 1-digit multiplier. The second partial product is obtained by simply putting down a 0 in the *ones* place and shifting the digit products one place to the left:

$$\begin{array}{r} 2\,4 \\ \times \quad 3 \\ \hline 7\,2\,0 \end{array}$$

(Notice that we put down 2 and carried 1 when we performed the digit product $3 \times 4 = 12$.) The total product is the sum of the two partial products: $48 + 720 = 768$. We can write the whole procedure compactly by aligning the two partial products vertically

$$\begin{array}{r} 2\,4 \\ \times \quad 3\,2 \\ \hline 4\,8 \\ 7\,2\,0 \end{array}$$

and then performing the addition

$$\begin{array}{rr} & 24 \\ \times & 32 \\ \hline & 48 \\ + & 720 \\ \hline & 768 \end{array}$$

Here's another example.

Example 18. Find the product of 29 and 135.

Solution. We choose 29 as the multiplier since it has the fewest digits.

$$\begin{array}{rr} & 135 \\ \times & 29 \\ \hline \end{array}$$

We use the 1-digit multiplier 9 to obtain the first partial product

$$\begin{array}{rrrrr} & \{3\} & \{4\} & & \\ & & 1 & 3 & 5 \\ \times & & & 2 & 9 \\ \hline & & 12 & 1 & 5 \end{array}$$

Notice that we put down 5 and carried 4 to the *tens* place, and also put down 1 and carried 3 to the *hundreds* place. Next we use the 1-digit multiplier 2 (standing for 2 *tens*) to obtain the second partial product, shifted left by putting a 0 in the *ones* place

$$\begin{array}{rr} & \{1\} \\ & 135 \\ \times & 29 \\ \hline & 1215 \\ & 2700 \end{array}$$

(What carry did we perform?) Finally, we add the partial products to obtain the (total) product

$$\begin{array}{rr} & 135 \\ \times & 29 \\ \hline & 1215 \\ + & 2700 \\ \hline & 3915 \end{array}$$

Note that the whole procedure is compactly recorded in the last step, which is all that you need to write down. ☐

1.3.3 Exercises

Find the products.

1.

$$\begin{array}{r} 1\,2\,2 \\ \times \quad\quad 4 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 8\,3 \\ \times \quad 5 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 1\,0\,4 \\ \times \quad\quad 7 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 3\,0\,0\,8 \\ \times \quad\quad\quad 9 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 2\,1\,2 \\ \times \quad\quad 4\,3 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 8\,3 \\ \times \quad 5\,6 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 1\,3\,6 \\ \times \quad 2\,7 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 3\,0\,8 \\ \times \quad 1\,0\,9 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 2\,1\,0\,3 \\ \times \quad 4\,4 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 8\,3\,7 \\ \times \quad 5\,4 \\ \hline \end{array}$$

1.4 Powers of Whole Numbers

If we start with 1 and repeatedly *multiply* by 3, 4 times over, we get a number that is called the 4*th power of* 3, written

$$3^4 = 1 \times 3 \times 3 \times 3 \times 3.$$

The factor 1 is inserted for a technical reason which we explain in a moment. Ordinarily, we leave it out (since $1 \times 3 = 3$), and simply write

$$3^4 = 3 \times 3 \times 3 \times 3.$$

In the expression 3^4, 3 is called the *base*, and 4 the *exponent* (or power).

Example 19. The 5th power of 2, or 2^5, is the product

$$2 \times 2 \times 2 \times 2 \times 2 = 32.$$

The 3rd power of 4, or 4^3, is the product

$$4 \times 4 \times 4 = 64.$$

It is convenient to make the following definition in the case where the exponent is 0.

For any nonzero signed number N,

$$N^0 = 1.$$

(0^0 is undefined.)

This may seem strange, but it fits into a natural pattern if we recall the "hidden" factor of 1 in every exponential expression. For example,

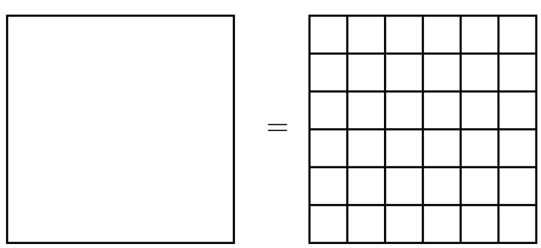
$$\begin{aligned} 3^4 &= 1 \times 3 \times 3 \times 3 \times 3 = 81 \\ 3^3 &= 1 \times 3 \times 3 \times 3 = 27 \\ 3^2 &= 1 \times 3 \times 3 = 9 \\ 3^1 &= 1 \times 3 = 3 \\ 3^0 &= 1. \end{aligned}$$

There is nothing special about the base 3 here – the pattern would hold for any *nonzero* base. It breaks down if the base is 0, which is why 0^0 is left undefined. However, expressions such as 0^2, 0^3, etc., make perfect sense (e.g., $0^3 = 0 \times 0 \times 0 = 0$).

Example 20. $17^0 = 1$. $0^5 = 0$. $0^1 = 0$. 0^0 is undefined.

1.4.1 Squares and Cubes

Certain powers are so familiar that they have special names. For example, the 2nd power is called the **square** and the 3rd power is called the **cube**. Thus 5^2 is read "5 squared," and 7^3 is read as "7 cubed." The source of these special names is geometric (see Section 1.8). The **area** of a square, x units on a side, is x^2 *square units*. This means that the square contains x^2 small squares, each one unit on a side. For example, the figure below shows a square 6 units on a side, with area $6^2 = 36$ square units.



Similarly, the **volume** of a cube, y units on a side, is y^3 *cubic units*. This means that the cube contains y^3 little cubes, each one unit on a side. For example, the volume of an ice cube that measures 2 cm (centimeters) on a side is $2^3 = 8$ cubic centimeters.

1.4.2 Exercises

1. Rewrite using an exponent: $8 \times 8 \times 8 \times 8$

2. Rewrite using an exponent: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

3. Evaluate 2^5
4. Evaluate 9^0
5. Evaluate 0^7
6. Evaluate 5^4
7. Evaluate 10^2
8. Evaluate 10^3
9. Complete the following table of squares:

$0^2 = 0$	$6^2 =$	$12^2 =$	$18^2 =$
$1^2 =$	$7^2 = 49$	$13^2 =$	$19^2 =$
$2^2 =$	$8^2 =$	$14^2 =$	$20^2 =$
$3^2 =$	$9^2 =$	$15^2 =$	$30^2 =$
$4^2 =$	$10^2 =$	$16^2 =$	$40^2 =$
$5^2 =$	$11^2 =$	$17^2 =$	$50^2 =$

10. Complete the following table of cubes:

$0^3 =$	$3^3 =$	$6^3 =$	$9^3 =$
$1^3 =$	$4^3 = 64$	$7^3 =$	$10^3 =$
$2^3 =$	$5^3 =$	$8^3 =$	$100^3 =$

1.4.3 Square Roots

If there is a number, whose square is the number n, we call it the **square root** of *n*, and symbolize it by

$$\sqrt{n}.$$

For example, $2^2 = 4$, so 2 is the square root of 4, and we write

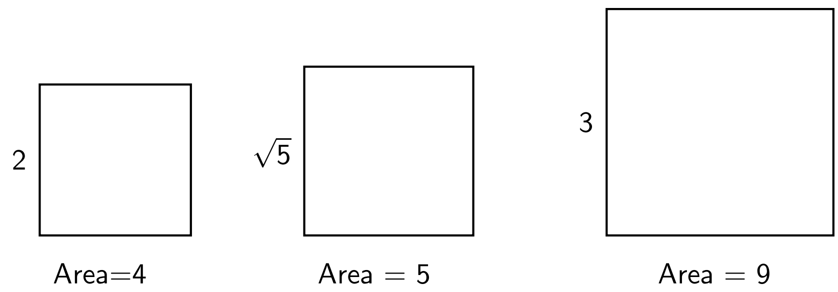
$$\sqrt{4} = 2.$$

Similarly, $3^2 = 9$, so 3 is the square root of 9, and we write

$$\sqrt{9} = 3.$$

Actually, every positive whole number has *two* square roots, one positive and one negative. The positive square root is called the **principal square root**, and, for now, when we say square root, we mean the principal one.

A whole number whose square root is also a whole number is called a **perfect square**. The first few perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121. Clearly, there are lots of whole numbers that are *not* perfect squares. But these numbers must have square roots. Using our geometric intuition, it is easy to believe that there are geometric squares whose areas are not perfect squares. For example, if we want to construct a square whose area is 5 square units, we could start with a square whose area is 4 square units, and steadily expand it on all sides until we get the desired square.



The side length of the middle square is a number which, when squared, yields 5, that is, $\sqrt{5}$. We can see that $\sqrt{5}$ is bigger than $\sqrt{4} = 2$,and less than $\sqrt{9} = 3$.

In general, if a whole number lies between two perfect squares, its square root must lie between the two corresponding square roots.

Example 21. Since 21 lies between the perfect squares 16 and 25, $\sqrt{21}$ must lie between $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

Example 22. Between what two consecutive whole numbers does $\sqrt{53}$ lie?

Solution. Since 53 lies between the perfect squares 49 and 64, the square root $\sqrt{53}$ must lie between $\sqrt{49} = 7$ and $\sqrt{64} = 8$. □

1.4.4 Exercises

Find the square roots:

1. $\sqrt{49}$
2. $\sqrt{81}$
3. $\sqrt{169}$
4. $\sqrt{121}$
5. $\sqrt{64}$

Between what two consecutive whole numbers do the following square roots lie?

6. $\sqrt{19}$
7. $\sqrt{75}$
8. $\sqrt{26}$
9. $\sqrt{32}$
10. Complete the square-root table:

$\sqrt{0} =$	$= 6$	$= 12$	$= 18$
$\sqrt{1} =$	$= 7$	$= 13$	$= 19$
$\sqrt{4} =$	$= 8$	$= 14$	$= 20$
$\sqrt{9} =$	$= 9$	$= 15$	$= 30$
$= 4$	$= 10$	$= 16$	$= 40$
$= 5$	$= 11$	$= 17$	$= 50$

1.5 Division of Whole Numbers

How many times does 23 "go into" 100? Put another way, starting with 100, how many times can we subtract 23 without obtaining a negative number? The answer is easily seen to be 4. Moreover, it is easy to see that there is something "left over," namely, 8. Here are the computations:

$$\begin{array}{r} 1\,0\,0 \\ -\quad 2\,3 \\ \hline 7\,7 \\ -\quad 2\,3 \\ \hline 5\,4 \\ -\quad 2\,3 \\ \hline 3\,1 \\ -\quad 2\,3 \\ \hline 8 \end{array}$$

This operation (repeated subtraction) is called **division**. The number we start with (100 in the example) is called the **dividend**, and the number we repeatedly subtract (23 in the example) is called the **divisor**. We use the symbol $\div$, and note that the dividend is written first:

$$\text{dividend} \div \text{divisor}.$$

1.5.1 Quotient and Remainder

Unlike the other three operations (addition, subtraction, multiplication), the result of a division of whole numbers consists of not one but *two* whole numbers: the number of subtractions performed (4 in the example), and the number left over (8 in the example). These two numbers are called the **quotient** and the **remainder**, respectively.

Whole number divisions with remainder 0 are called **exact**. For example, $48 \div 6$ has quotient 8 and remainder 0, so the division is exact and we can write

$$48 \div 6 = 8,$$

with the understanding that the remainder is 0. Exact divisions can be restated in terms of multiplication. Subtracting 6 (8 times) from 48 yields exactly 0. On the other hand, starting at 0 and adding 6 (8 times) returns 48. Recalling that multiplication is a shorthand for this kind of repeated addition, we see that the two statements

$$48 \div 6 = 8 \qquad \text{and} \qquad 48 = 6 \cdot 8$$

say exactly the same thing. In general,

$$a \div b = c \qquad \text{and} \qquad a = b \cdot c$$

are equivalent statements.

Example 23. Express the statement $72 = 8 \cdot 9$ as an exact division in two ways.

Solution. We can get to 0 by starting at 72 and repeatedly subtracting 8 (9 times), or by repeatedly subtracting 8 (9 times). So, using the division symbol, we can write

$$72 \div 8 = 9$$

or we can write

$$72 \div 9 = 8.$$

□

Example 24. The division $63 \div 7$ is exact. Express this fact using an appropriate multiplication.

Solution. Since the quotient is 9 and the remainder is 0, we can write

$$63 \div 7 = 9 \qquad \text{or} \qquad 63 = 9 \cdot 7.$$

□

1.5.2 Long Division

Divisions with multi-digit divisors and/or dividends can get complicated, so we remind you of a standard way (long division) of organizing the computations. Here is what it looks like:

```
             quotient
         ┌──────────────
 divisor │   dividend
              ***
         ───────────────
                  ***
                  ***
             ───────────
                    ***
                    ***
                ────────
                  remainder
```

The horizontal lines indicate subtractions of intermediate numbers; there is one subtraction for each digit in the quotient. For example, the fact that the division $100 \div 23$ has quotient 4 and remainder 8 is expressed in the long division form as follows:

$$
\begin{array}{r|l}
 & \;\;\;\;\;\;\;\;\,4 \\ \hline
23 & \;\;\;100 \\
 & \;\,-92 \\ \hline
 & \;\;\;\;\;\;\;\;\,8
\end{array}
$$

The 4 repeated subtractions of 23 are summarized as the single subtraction of $4 \cdot 23 = 92$.

In long division, we try to estimate the number of repeated subtractions that will be needed, multiply this estimate by the divisor, and hope for a number that is close to, but not greater than, the dividend. It will be easy to see when our estimate is too large, and to adjust it downward. If it is too small, the result of the subtraction will be too big – there was actually "room" for further subtraction. Going back to our example, $100 \div 23$, it is more or less clear that we will need more than 2 subtractions, since $23 \times 2 = 46$, which leaves a big remainder of 54 (bigger than the divisor, 23). $23 \times 3 = 69$, which also leaves a remainder that is too big. Since $23 \times 5 = 115$, which is bigger than the dividend, 100, we know that the best estimate for the quotient is 4. Now $23 \times 4 = 92$. We subtract 92 from 100, leaving the remainder 8, which is less than the divisor, as it should be. To check our calculations, we verify that $23 \times 4 + 8 = 100$. (In this check, the multiplication is done *before* the addition – this is the standard *order of operations*, which we will say more about later.)

Let's try another simple division problem.

Example 25. Find the quotient and remainder of the division $162 \div 42$.

Solution. Putting the dividend and divisor into the long division form, we have

$$
\begin{array}{r|l}
\hline
42 & \;\;\;162
\end{array}
$$

Let's estimate the number of subtractions that we'll need to perform. 40 goes into 160 four times, so, perhaps four is a good guess. But $4 \times 42 = 168$, which is too big (bigger than the dividend). So we lower our estimate by 1. We get $3 \times 42 = 126$, which is less than the dividend, so this must be the right choice.

$$\begin{array}{r|r} & 3 \\ \cline{2-2} 42 & 162 \\ & -126 \end{array}$$

Subtracting, we obtain the remainder

$$\begin{array}{r|r} & 3 \\ \cline{2-2} 42 & 162 \\ & -126 \\ \cline{2-2} & 36 \end{array}$$

which is less than the divisor, as it should be. Thus, the quotient is 3 and the remainder is 36. As a check, we verify that $3 \times 42 + 36 = 162$. (Multiplication before addition.) □

When the dividend is large, estimating the quotient is not so easy. The next example shows how to break the problem down by considering related, but smaller dividends.

Example 26. Find the quotient and remainder of the division $3060 \div 15$.

Solution. If we read the dividend from the left, one digit at a time, we get, successively, the numbers 3, 30, 306, and 3060. Of course, according to the place-value system, these numbers stand for 3 *thousands*, 30 *hundreds*, etc., but we do not need to think this way. To find the left-most digit of the quotient, observe that the divisor, 15, does not "go into" 3, but it does go into 30 (2 times, exactly). To indicate this we put, for the first digit of the quotient, the digit 2, directly over the right-most digit (0) in 30:

$$\begin{array}{r|l} & \ \ 2 \\ \cline{2-2} 15 & 3060 \end{array}$$

Now we compute the product $15 \times 2 = 30$ and subtract it from the initially selected part of the dividend, i.e., from 30:

$$
\begin{array}{r|l}
 & \;\;\;2 \\
\cline{2-2}
15 & \;3\,0\,6\,0 \\
 & -3\,0 \\
\hline
 & \;\;\;0
\end{array}
$$

In this case, we get a remainder of 0. But we are not done yet. We treat this intermediate remainder as if it were a new dividend. Does 15 “go into” 0? No – we need a larger dividend. To get one, we “bring down” the digit 6:

$$
\begin{array}{r|l}
 & \;\;\;2 \\
\cline{2-2}
15 & \;3060 \\
 & -\;30 \\
\hline
 & \;\;\;06
\end{array}
$$

Now $06 = 6$ (why?), and 15 does not go into 6. More precisely, it goes into it 0 times. We indicate this by putting a 0 directly above the digit 6 in the dividend:

$$
\begin{array}{r|l}
 & \;\;\;20 \\
\cline{2-2}
15 & \;3060 \\
 & -\;30 \\
\hline
 & \;\;\;06
\end{array}
$$

Then we bring down the next (and the last) digit (0) of the dividend:

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & \quad 20 \\ \cline{2-2}
15 & 3060 \\
\multicolumn{1}{r}{} & -\ 30 \\ \cline{2-2}
\multicolumn{1}{r}{} & \quad 060
\end{array}$$

This gives us a new dividend of $060 = 60$, and 15 goes into 60 four times. So we put down 4 above the final digit in the original dividend.

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & \quad 204 \\ \cline{2-2}
15 & 3060 \\
\multicolumn{1}{r}{} & -\ 30 \\ \cline{2-2}
\multicolumn{1}{r}{} & \quad 060
\end{array}$$

Multiplying 15 by 4, we subtract this product from 60. This final difference is the remainder.

$$\begin{array}{r|l}
\multicolumn{1}{r}{} & \quad 204 \\ \cline{2-2}
15 & 3060 \\
\multicolumn{1}{r}{} & -\ 30 \\ \cline{2-2}
\multicolumn{1}{r}{} & \quad 060 \\
\multicolumn{1}{r}{} & \quad -\ 60 \\ \cline{2-2}
\multicolumn{1}{r}{} & \qquad 0
\end{array}$$

We have reached the end of our dividend – there are no more digits to bring down. Hence, our division stops. The quotient is 204, and the remainder is 0. We check by verifying that $15 \times 204 + 0 = 3060$. $\square$

1.5.3 Exercises

Express each exact division as an equivalent multiplication.

1. $88 \div 22 = 4$
2. $42 \div 6 = 7$

3. $51 \div 17 = 3$
4. $168 \div 14 = 12$
5. $96 \div 12 = 8$

Find the quotient and remainder of each division.

6. $37 \div 11$
7. $712 \div 101$
8. $3007 \div 110$
9. $3333 \div 111$
10. $3456 \div 241$
11. $457 \div 41$
12. $578 \div 19$
13. $317 \div 21$
14. $907 \div 201$
15. $712 \div 21$

1.6 Order of operations

We often do calculations that involve more than one operation. For example

$$1 + 2 \times 3$$

involves both addition and multiplication. Which do we do first? If we do the multiplication first, the result is $1 + 6 = 7$, and if we do the addition first, the result is $3 \times 3 = 9$. Obviously, for the expression

$$1 + 2 \times 3$$

to have a definite and unambiguous meaning, we need a convention or agreement about the order of operations. It could have been otherwise, but the convention in this case is:

multiplication before addition.

With this convention, when I write $1 + 2 \times 3$, you know that I mean 7 (not 9). The precedence of multiplication can be made explicit using the **grouping symbols** () (parentheses):

$$1 + (2 \times 3) = 1 + 6 = 7.$$

If one of us insists that the addition be done first, we can do that by re-setting the parentheses:

$$(1 + 2) \times 3 = 3 \times 3 = 9.$$

Thus grouping symbols can be used to force any desired order of operations. Common grouping symbols, besides parentheses, are brackets, [ ], and braces, $\{\}$. The square root symbol $\sqrt{\ }$ is also a grouping symbol. For example

$$\sqrt{4+5} = \sqrt{9} = 3.$$

The $\sqrt{\ }$ symbol acts like a pair of parentheses, telling us to evaluate what is inside (in this case, the sum $4+5$) first, *before* taking the square root.

The **order of operations** is:

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

"In order of appearance" means in order from left to right. Thus in the expression

$$2+5-3,$$

the addition comes first, so it is evaluated first,

$$2+5-3 = 7-3 = 4,$$

while in the expression

$$8-6+11,$$

the subtraction is done first because it comes first,

$$8-6+11 = 2+11 = 13.$$

Example 27. Evaluate $6 \cdot 5 - 4 \div 2 + 2$.

Solution. All four operations appear here. There are no grouping symbols, exponents or roots. Following the order of operations, as well as the order of appearance, we do the computations in the following order: multiplication, division, subtraction, addition. The computations are as follows:

$$\begin{gathered} 6 \cdot 5 - 4 \div 2 + 2 = \\ 30 - 4 \div 2 + 2 = \\ 30 - 2 + 2 = \\ 28 + 2 = \\ 30 \end{gathered}$$

□

In the next two examples, we use the same numbers and the same operations, but we insert grouping symbols to change the order of operations. As expected, this changes the final result.

Example 28. Evaluate $6 \cdot (5-4) \div 2 + 2$.

Solution. The grouping symbols force us to do the subtraction first. After that, the usual order of operations is followed.

$$\begin{gathered} 6 \cdot (5-4) \div 2 + 2 = \\ 6 \cdot 1 \div 2 + 2 = \\ 6 \div 2 + 2 = \\ 3 + 2 = \\ 5 \end{gathered}$$

□

Example 29. Evaluate $6 \cdot 5 - 4 \div [2+2]$.

Solution. Now the grouping symbols (brackets) force us to do the addition first.

$$\begin{gathered} 6 \cdot 5 - 4 \div [2+2] = \\ 6 \cdot 5 - 4 \div 4 = \\ 30 - 4 \div 4 = \\ 30 - 1 = \\ 29 \end{gathered}$$

□

In the next example, there are grouping symbols within grouping symbols. The strategy here is to work *from the inside outward.*

Example 30. Evaluate $6^2 - [3 + (3-1)]^2$.

Solution. We have parentheses within brackets. The parentheses enclose up the innermost group, which is where we start. (That is what "from the inside outward" means.) Thus the first operation to be done is $3 - 1 = 2$, yielding

$$6^2 - [3+2]^2.$$

Next, the bracketed group, $[3+2]$ is evaluated, yielding

$$6^2 - 5^2.$$

At this point, there are no more grouping symbols, and the order of operations tells us to evaluate the expressions with exponents next, yielding

$$36 - 25.$$

All that is left is the remaining subtraction,

$$36 - 25 = 11.$$

□

Example 31. Evaluate $\sqrt{(11-5)^2 + (24 \div 2 - 4)^2}$.

Solution. We evaluate the two inner groups $(11-5)$ and $(24\div 2-4)$ first. In the second group, division precedes subtraction, yielding $12-4=8$. Thus the expression is reduced to

$$\sqrt{6^2+8^2}.$$

Next we evaluate the two expressions with exponents, obtaining

$$\sqrt{36+64}.$$

Then, remembering that $\sqrt{\ }$ is a grouping symbol, we evaluate the group $36+64=100$, and finally, evaluate the square root

$$\sqrt{100}=10.$$

□

1.6.1 Exercises

Evaluate the expressions using the correct order of operations.

1. $6+16\div 4$
2. $16\cdot 4-48$
3. $15-9-4$
4. $2\cdot 6+2(\sqrt{36}-1)$
5. $4\times 3\times 2\div 8-3$
6. $(2\cdot 5)^2$
7. $\sqrt{21-30\div 6}$
8. $[18\div(9\div 3)]^2$
9. $2+2\times 8-(4+4\times 3)$

1.7 Average

The average of 2 numbers is their sum, divided by 2. The average of 3 numbers is their sum, divided by 3. In general, **the average of n numbers is their sum, divided by n.**

Example 32. Find the average of each of the following multi-sets of numbers. (A **multi-set** is a set in which the same number can appear more than once.) (a) $\{10,12\}$; (b) $\{5,6,13\}$; (c) $\{8,12,9,7,14\}$.

Solution. In each case, we take the sum of all the numbers in the multi-set, and then (following the order of operations), divide by number of numbers:

(a) $(10+12)\div 2=11$; (b) $(5+6+13)\div 3=8$; (c) $(8+12+9+7+14)\div 5=10$.

□

The average of a multi-set of numbers is a description of the whole multi-set, in terms of only one number. If all the numbers in the multi-set were the same, the average would be that number. For example, the average of the multi-set $\{5, 5, 5\}$ is

$$(5 + 5 + 5) \div 3 = 5,$$

and the average of the multi-set $\{2, 2, 2, 2, 2, 2\}$ is

$$(2 + 2 + 2 + 2 + 2 + 2) \div 6 = 2.$$

This gives us another way to define the average of a multi-set of n numbers: it is that number which, added to itself n times, gives the same total sum as the sum of all the original numbers in the multi-set.

1.7.1 Exercises

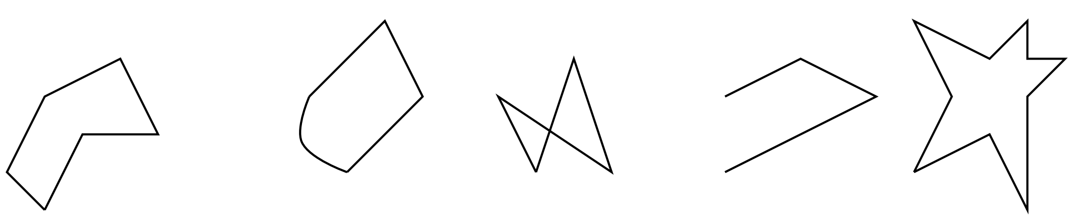
Find the average of each of the following multi-sets of numbers.

1. $\{1, 2, 3, 4, 5, 6, 7\}$
2. $\{13, 13, 19, 15\}$
3. $\{206, 196, 204\}$
4. $\{85, 81, 92\}$
5. A baseball team had 7 games canceled due to rain in the 2010 season. The number of canceled games in the 2002-2009 seasons were 5, 6, 2, 10, 9, 4, 6, 5. What was the average number of canceled games for the 2002-2010 seasons?
6. Suppose the average of the multi-set $\{20, 22, N, 28\}$ is 25, where N stands for an unknown number. Find the value of N.

1.8 Perimeter, Area and the Pythagorean Theorem

Squares and rectangles are examples of **polygons** – closed shapes that can be drawn on a flat surface, using segments of straight lines which do not cross each other. "Closed" means that the line segments form a boundary, with no gaps, which encloses a unique "inside" region, and separates it from the "outside" region.

Example 33. Which of the following figures are polygons?



(a) (b) (c) (d) (e)

Solution. (b) is not a polygon since not all of its sides are straight lines. (c) is not a polygon because it does not have a unique "inside" region. (d) is not a polygon because it is not closed. ☐

There are two useful numerical quantities associated with a polygon: the **perimeter**, which is the length of its boundary, and the **area**, which is (roughly speaking) the "amount of space" it encloses. Perimeters are measured using standard units of length such as feet (ft), inches (in) , meters (m), centimeters (cm). Areas are measured using **square units**, such as square feet (ft^2), square inches (in^2), square meters (m^2), square centimeters (cm^2).

To find the perimeter of a polygon, we simply find the sum of the lengths of its sides.

Example 34. Find the perimeter of each polygon. Assume the lengths are measured in feet.

Solution. Adding the lengths of the sides, we find that the perimeter of polygon (a) is

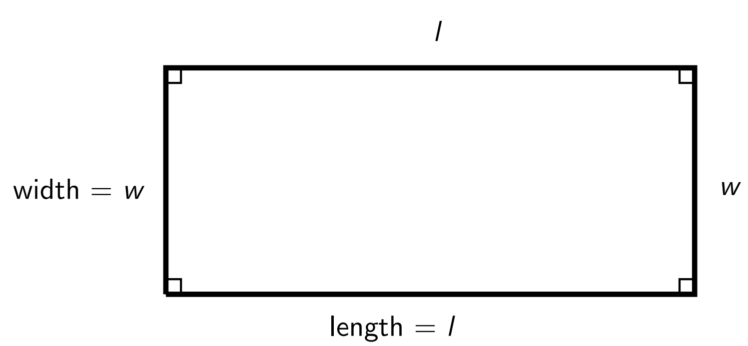
$$3 + 4 + 1 + 5 + 5 = 18 \text{ ft.}$$

Similarly, the perimeter of the polygon (b) is

$$7 + 5 + 6 + 1 + 1 + 1 = 21 \text{ ft.}$$

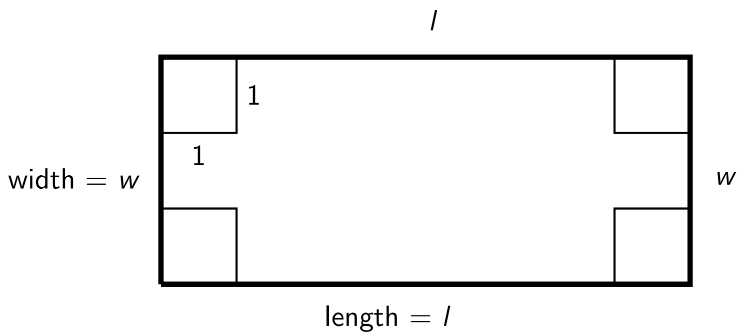
□

Finding the area of a polygon can be complicated, but it is quite simple if the polygon can be divided up into rectangles. A **rectangle** is a four-sided polygon with two pairs of opposite parallel sides, and "square" corners. The square corners – also known as **right angles** – imply that the paired opposite sides to have the same length. The length of the shorter pair of sides is often called the *width* and denoted w, and the length of the longer pair is called the *length* and denoted l.

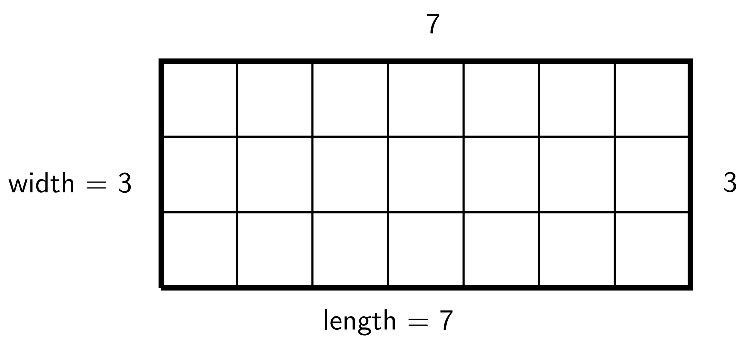


The small squares in the corners are there to indicate that the polygon is a rectangle.

Example 35. Find the area of the rectangle below, assuming the corner squares are 1 unit on a side.



Solution. If the corner squares are 1 unit on a side, the area of the rectangle is the number of square units that fit inside the rectangle. As the next figure shows, exactly 21 such square units fit inside, in 3 rows of 7. The width (w) is evidently 3 units, and the length (l) 7 units. The area is the product $3 \times 7 = 21$ square units.



□

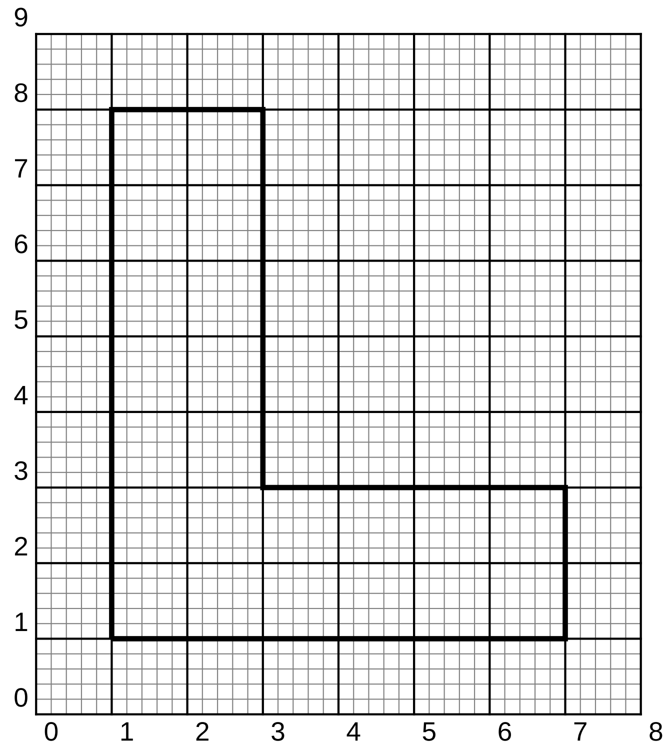
The example illustrates a general fact: **the area of a rectangle is the product of the length and the width.** If A denotes the area, the formula is

$$A = l \cdot w.$$

It is also clear that **the perimeter of a rectangle is the sum of twice the length and twice the width,** since to travel all the way around the boundary of the rectangle, both the length and the width must be traversed twice. If P denotes the perimeter, the formula is

$$P = 2l + 2w.$$

Example 36. In the figure below we have drawn an L-shaped polygon on a grid. Find the area and perimeter of the polygon. Take the unit of area to be one of the large squares of the grid, and assume these squares are 1 cm on a side.



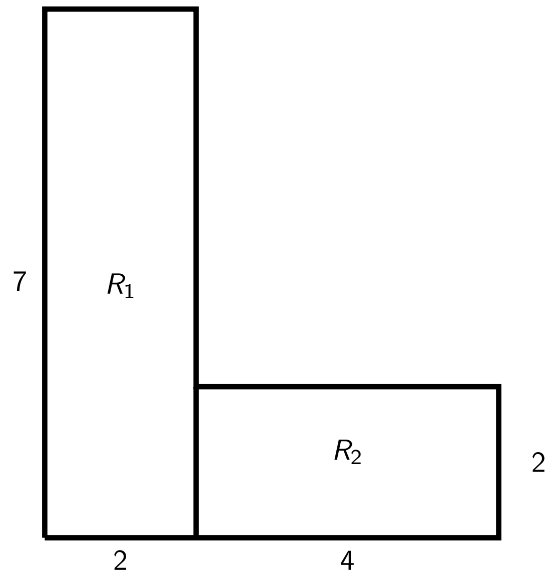
Solution. Using the grid, it is easy to find the lengths of all six sides of the polygon. For example, the left-hand vertical side is 7 cm long (stretching from 1 to 8 on the scale at left), and the top side is 2 cm long. Continuing clockwise around the L-shape, we see that the side lengths are

$$7, 2, 5, 4, 2, 6 \text{ cm},$$

respectively. From this, it follows easily that the perimeter P is given by the sum

$$7 + 2 + 5 + 4 + 2 + 6 = 26 \text{ cm}.$$

To find the area, we draw a line which divides the L-shape into two rectangles (labeled R_1 and R_2) in the figure below. (There is a different line we could have drawn to accomplish the same thing – do you see it?)



It is evident that R_1 has length 7 cm and width 2 cm. So the area of R_1 is

$$A_1 = 2 \cdot 7 = 14 \text{ cm}^2.$$

Similarly, since the length of R_2 is 4 cm and the width is 2 cm, the area of R_2 is

$$A_2 = 2 \cdot 4 = 8 \text{ cm}^2.$$

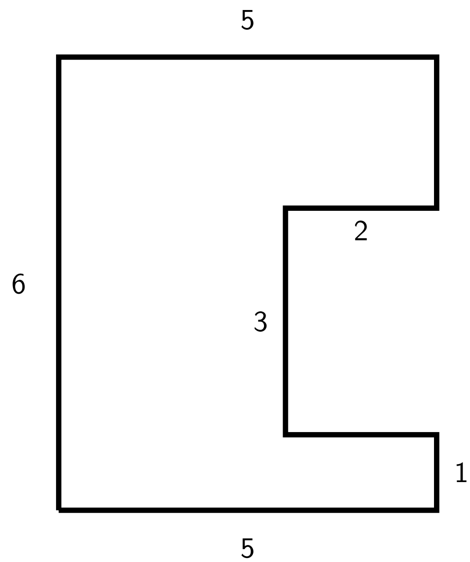
It is visually obvious that the total area A of the L-shaped polygon is the *sum* $A_1 + A_2$ of the areas of R_1 and R_2. Thus

$$A = 14 \text{ cm}^2 + 8 \text{ cm}^2 = 22 \text{ cm}^2.$$

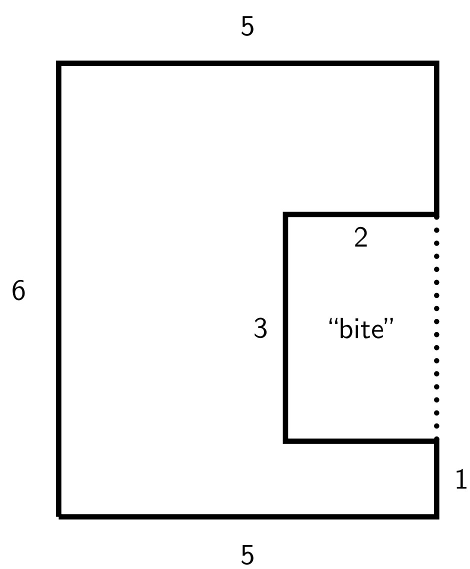
Using the grid, you can easily verify this result by counting the number of large squares inside the polygon. □

The grid is not necessary, as long as we are given enough side-lengths, and assume square corners.

Example 37. Find the area and perimeter of the right-angled (square-cornered) polygon below. Assume the side lengths are measured in feet.



Solution. There are some missing side lengths (for example, the vertical side on the upper right), but the area can be determined without them, using subtraction. We can visualize the polygon as a large rectangle of length 6 ft. and width 5 ft, out of which a rectangular "bite," of length 3 ft and width 2 ft, has been taken.



The area of the large rectangle (bite included) is $6 \cdot 5 = 30$ ft^2, and the area of the bite alone is $3 \cdot 2 = 6$ ft^2. Thus the area of the original polygon is

$$30 - 6 = 24 \text{ ft}^2.$$

To find the perimeter, we need the missing side lengths. For the vertical side on the upper right, we reason as follows. The left edge is 6 ft, and so the total length of all the right vertical edges must also be 6 ft. The two known vertical edges on the right have lengths 1 and 3, and the unkown edge must make up the difference. So its length must be

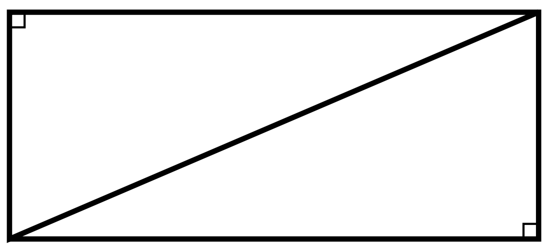
$$6 - (1 + 3) = 2 \text{ ft.}$$

The other missing side (the short horizontal one) is 2 ft. Adding up all the side lengths, starting (arbitrarily) with the bottom edge and proceeding counter-clockwise, yields the perimeter:

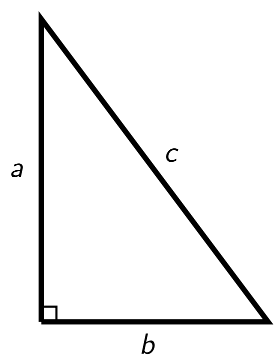
$$5 + 1 + 2 + 3 + 2 + 2 + 5 + 6 = 26 \text{ ft.}$$

□

If we cut a rectangle in two by drawing a **diagonal** from one corner to the opposite corner, we get two **right triangles**, each having exactly the same size and shape, and therefore the same area.



Right triangles are interesting in their own right, and we often consider them in isolation, without reference to the rectangle they came from. The side of a right triangle that is opposite the right angle (the longest side) is known as the **hypotenuse**. The two shorter sides are called the **legs**. In the figure below, the legs are labeled a and b, and the hypotenuse is labeled c.



A famous formula, called the **Pythagorean theorem**, states that, in any right triangle with legs of length a and b, and hypotenuse of length c, the following relation holds:

$$a^2 + b^2 = c^2.$$

We can use this to obtain a formula for the length of the hypotenuse of a right triangle in terms of the lengths of the legs.

$$c = \sqrt{a^2 + b^2}.$$

Since the area of a right triangle is exactly half the area of the rectangle it came from, it follows that **the area of a right triangle is the product of the lengths of the legs, divided by** 2**.** If A denotes the area, and the lengths of the legs are a and b as in the figure, the formula is

$$A = (a \cdot b) \div 2.$$

Example 38. Find the area and perimeter of a triangle whose legs have length 3 feet and 4 feet.

Solution. Putting $a = 4$ and $b = 3$, we find the area using the formula

$$A = (a \cdot b) \div 2 = (4 \cdot 3) \div 2 = 12 \div 2 = 6 \text{ ft}^2.$$

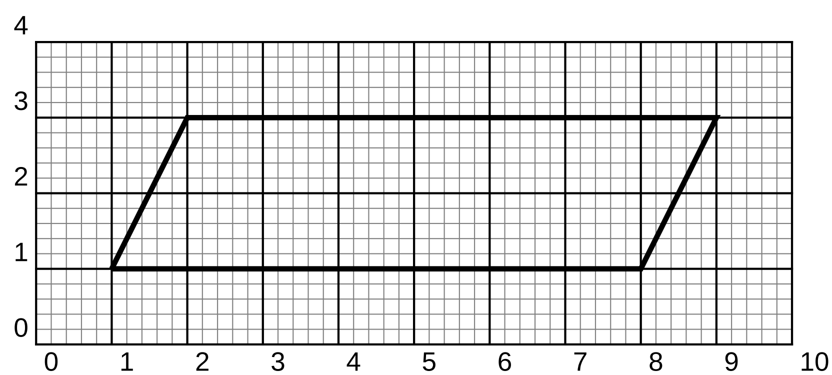
To find the perimeter, we first find the length of the hypotenuse using the formula

$$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \text{ ft}.$$

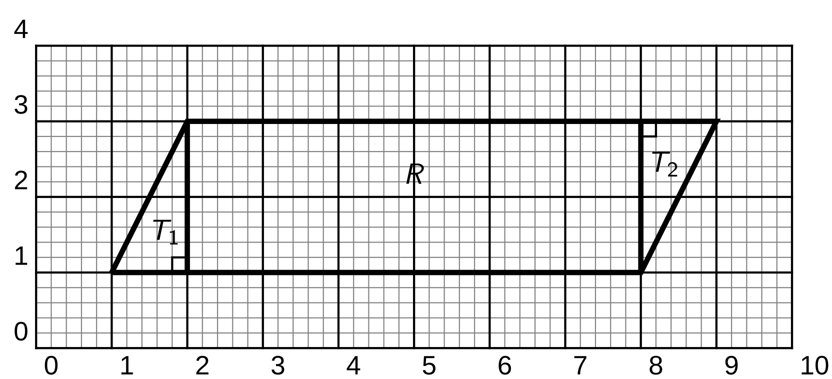
Thus the perimeter is $a + b + c = 4$ ft $+3$ ft $+5$ ft $= 12$ ft. □

Now we can find the area and perimeter of any polygon that can be divided up into rectangles and right triangles.

Example 39. Find the area and perimeter of the polygon below. The large squares of the grid measure 1 centimeter (cm) on a side.



Solution. Drawing two vertical lines, we can divide up the polygon into two right triangles, T_1 and T_2, and a rectangle, R.



The rectangle has length 6 cm and width 2 cm, so it has area $6 \cdot 2 = 12$ cm^2. Both triangles have legs of length 1 and 2, so the area of each is $(2 \cdot 1) \div 2 = 1$ cm^2. Adding the three areas gives the total area of the polygon: $1 + 12 + 1 = 14$ cm^2. To find the perimeter, we use the Pythagorean theorem to find the lengths of the two slanted sides:

$$c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

Thus the perimeter is

$$\sqrt{5} + 6 + \sqrt{5} + 6 = 12 + 2\sqrt{5} \text{ cm}.$$

Since $\sqrt{5}$ is not an integer, we do not simplify the expression further. But we can make the following estimate: Since $\sqrt{5}$ is between 2 and 3, $2\sqrt{5}$ is between 4 and 6, and it follows that the perimeter is between 16 and 18 centimeters. ☐

1.8.1 Exercises

1. Find the area of a right triangle whose legs are 5 in and 12 in.

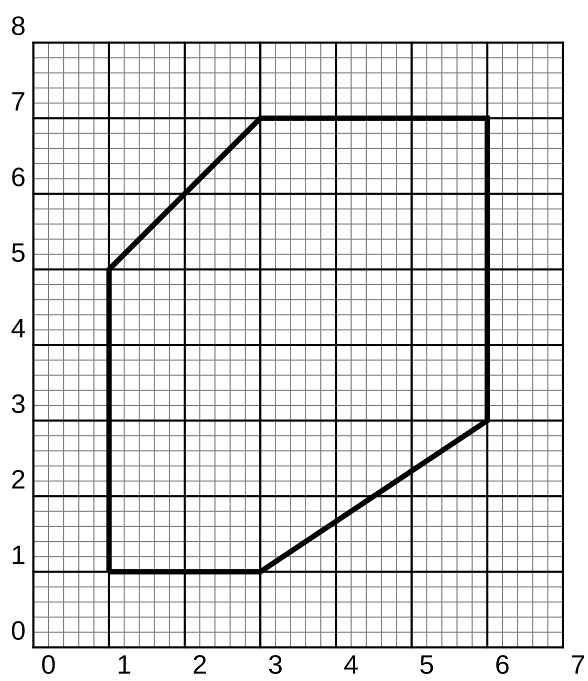
2. Find the perimeter of the right triangle in the previous example.

3. Find the area of a rectangle whose length is 8 ft and whose width is 7 feet.

4. Find the perimeter of the rectangle in the previous example.

5. Find the length of the diagonal of a square which is 2 cm on a side. Between what two whole numbers does the answer lie?

6. Find the area and perimeter of the polygon below.



Chapter 2

Fractions and Mixed Numbers

Fractions are expressions such as

$$\frac{3}{19} \quad \text{or more generally} \quad \frac{a}{b},$$

where a and b are whole numbers, and $b \neq 0$. The numbers a and b are called **terms**. The term on top is called the **numerator**, and the term on the bottom is called the **denominator**. The horizontal line in the middle is called the **fraction bar**. Sometimes to save space, we write fractions in one line, using a slash instead of the fraction bar, putting the numerator on the left and the denominator on the right:

$$\frac{a}{b} = a/b.$$

When reading a fraction out loud, we attach the suffix *th* or *ths* to the denominator, saying "*a bths*" for a/b.

Example 40. The fractions $\frac{1}{8}$ and $\frac{6}{13}$ are spoken "one eighth," and "six thirteenths," respectively.

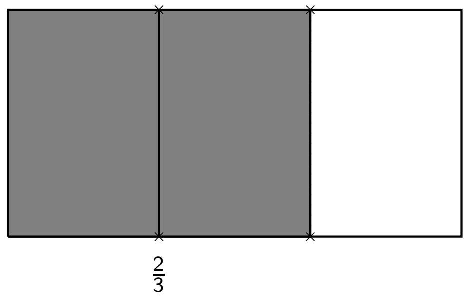
If the denominator is 2 or 3, we say "halves" or "thirds" (not twoths or threeths!), respectively. If the denominator is 4, we sometimes say "fourths," and sometimes, "quarters."

Example 41. 5/2 is spoken "five halves." 4/3 is spoken "four thirds." 3/4 is spoken "three fourths," or "three quarters."

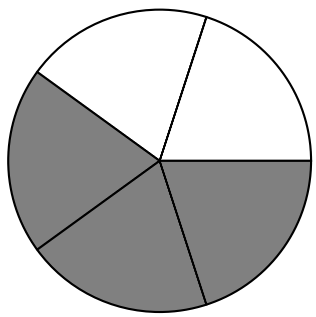
Sometimes, instead of using the *th* suffix, we just say "*a* over *b*." Thus 4/9 can be spoken as "four ninths," or just "four over nine."

2.1 What fractions mean

Fractions represent **parts of a whole** in a precise way. In the fraction a/b, the denominator b represents the *number of equal parts* into which the whole has been divided, and the numerator represents *how many* of those parts are being taken into account. For example, if a rectangle is divided up into 3 equal parts, and 2 of those parts are shaded, then the shaded portion represents $\frac{2}{3}$ (two-thirds) of the whole rectangle.

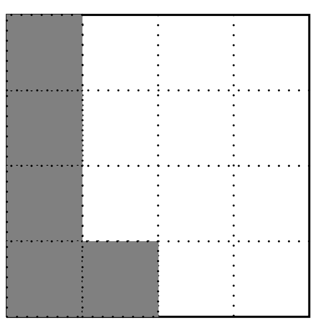


We can divide any convenient figure into equal parts (not just a rectangle), to represent a fraction. For example, the circle below has been divided into 5 equal "wedges," and three of them are shaded. So the picture represents the fraction $\frac{3}{5}$.



Example 42. Use a square to represent the fraction 5/16.

Solution. Since 16 is a perfect square, it's easy to make a square 4 units on a side, and divide that into 16 small squares of equal size. Then we shade 5 of them (any 5 will do) to represent 5/16.

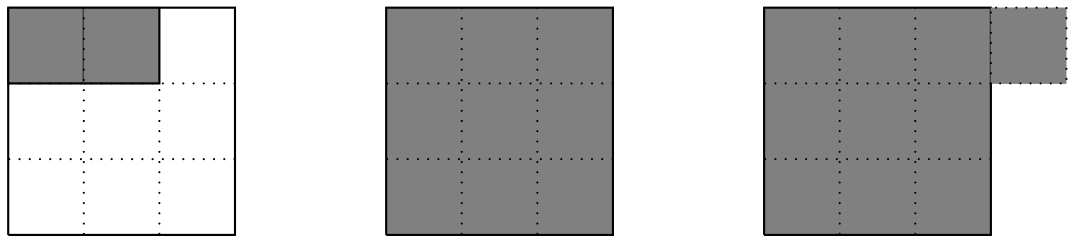


□

2.2 Proper and Improper Fractions

A fraction is called **proper** if its numerator is less than its denominator. A proper fraction represents **less than one whole**. An **improper** fraction has a numerator that is greater than or equal to the denominator and represents a number that is greater than (or equal to) one whole.

Example 43. The figure below represents three fractions with denominator 9: the first one (2/9) is proper, representing a number less than 1. The other two are improper. The one in the middle (9/9) represents the whole number 1. The last (10/9) represents a number that is greater than 1.



$\frac{2}{9}$ $\quad$ $\frac{9}{9} = 1$ $\quad$ $\frac{10}{9} = 1 + \frac{1}{9}$

$9/9 = 1$ illustrates the general fact that the whole number 1 can be represented in infinitely many ways as a fraction – just take any fraction whose numerator and denominator are equal. For example,

$$1 = \frac{2}{2} = \frac{9}{9} = \frac{159}{159}.$$

Intuitively, if an object is divided into equal parts, and *all the parts are taken*, then, in fact, the *whole* (1 whole) has been taken.

For any whole number a (except 0),

$$\frac{a}{a} = 1.$$

Any whole number can be represented as a fraction. For example, to represent the whole number 3, we can think of three whole rectangles, each "divided" into 1 "part," and write

$$3 = \frac{3}{1}.$$

More generally,

For any whole number a (including 0),

$$a = \frac{a}{1}.$$

2.2.1 Zero as Numerator and Denominator

The whole number 0 can be written as a fraction in infinitely many ways:

$$0 = \frac{0}{b}, \quad \text{for any non-zero } b.$$

It is easy to understand why this is true: you can divide something into any number of equal pieces (say, b of them), but if you take *none of them*, you have taken an amount equal to 0 – no matter the value of b. Thus,

$$0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3} = \ldots$$

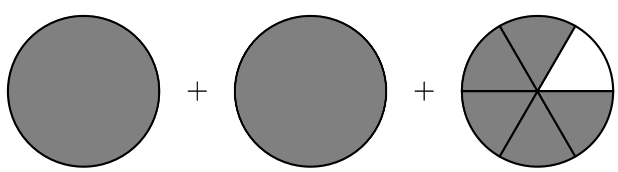
So 0 can certainly be the numerator of a fraction. Can it be the denominator? The answer is no. Here is one way to think about it: does it make sense to divide something into 0 pieces? (1 piece, yes, but 0 pieces?) A related question is: Can 0 be the divisor in a division problem? If you think so, then answer this: how many times does 0 go into 7? Thinking back to our original definition of division as repeated subtraction, the question is "how many times can 0 be subtracted from 7 before we arrive at a negative number?" The answer is "as many times as you like!," since subtracting 0 has no effect on 7, or any other number. For these reasons we attach *no meaning* to a fraction with denominator 0, saying, if it comes up, that it is **undefined.**

For any whole number n, including 0, the fraction $\dfrac{n}{0}$ is undefined.

There is another reason why 0 cannot be the denominator of a fraction. Multiplication and division are mutually inverse operations, meaning that the equation $\frac{a}{b} = c$ is equivalent to the equation $a = b \cdot c$ whenever $b \neq 0$. Suppose we could assign a numerical value to the fraction $1/0$, say, $\frac{1}{0} = 1$. That would mean that $1 = 0 \cdot 1$. But of course, $0 \cdot 1 = 0$, so we arrive at $1 = 0$, an obvious contradiction.

2.2.2 Exercises

1. What improper fraction does the following picture represent?



Use rectangles, circles, or squares to represent the following fractions.

2. $\frac{3}{2}$
3. $\frac{3}{4}$
4. $\frac{5}{8}$
5. $\frac{11}{6}$
6. $\frac{4}{3}$
7. $\frac{6}{2}$
8. Write five fractions which are equal to 1.

9. Write five fractions which are equal to 0.

10. Write the whole number 15 as a fraction.

11. Repeat the reasoning in the end of Subsection 2.1 to convince yourself that the attempt to define $1/0 = -1$ also leads a contradiction.

2.3 Multiplication of Fractions

Fractions are numbers in their own right, and we will learn how to add, subtract, multiply and divide them. We begin with multiplication, since it is one of the easiest operations to perform.

In the next two examples, we will explain multiplication of two fractions in terms of equal parts of rectangles. This will suggest a simple way to multiply any two fractions, without drawing pictures.

It is common practice to use the word "of" for the multiplication operation, especially when the first multiplicand is a proper fraction.

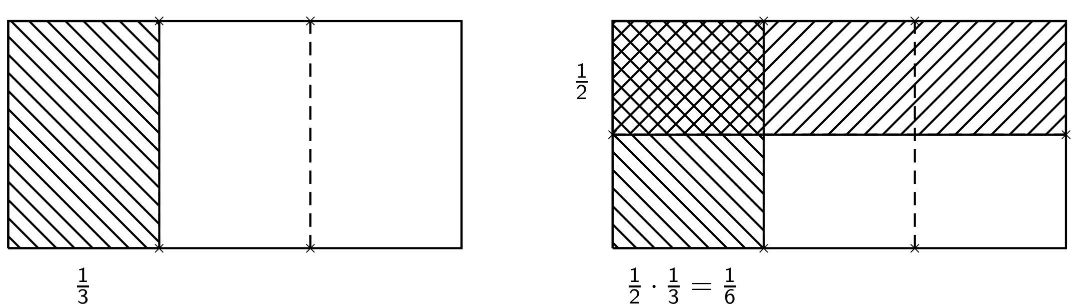
Example 44. What is one half of one third? (Equivalently, what is one third of one half? – multiplication is commutative!)

Solution. Since "of" stands for multiplication, we want to find the product

$$\frac{1}{2} \cdot \frac{1}{3}.$$

Start with a rectangle divided into 3 equal parts, and shade one of the parts, to represent the fraction $1/3$. Then divide the whole rectangle *lengthwise* into 2 equal parts, and shade one of the lengthwise parts (say, the upper part) using a different shading. The rectangle is now cut into $2 \cdot 3 = 6$ equal parts, and the two shadings overlap in precisely 1 of the 6 parts, which is cross-hatched. This part represents one half of one third (or, equivalently, one third of one half). We conclude that

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$



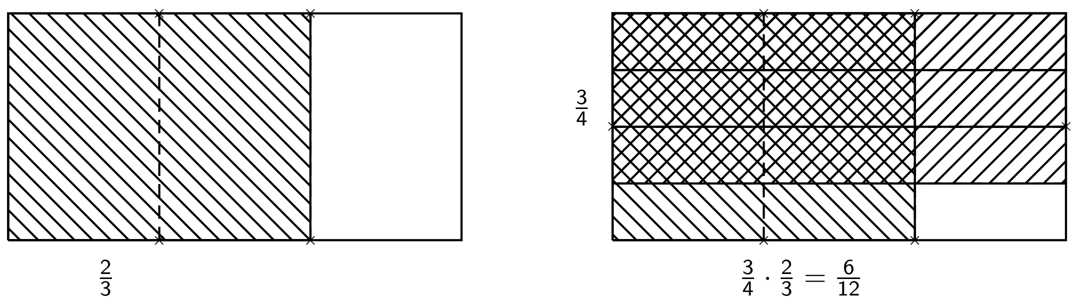
□

Example 45. What is three fourths of two thirds? That is, find the product

$$\frac{3}{4} \cdot \frac{2}{3}.$$

Solution. Start with a rectangle divided into three equal parts, two of them shaded, to represent 2/3. Then, divide the rectangle lengthwise into *four* equal slices, and shade (with a different shading) three of these lengthwise slices, to represent 3/4. The rectangle now has 12 equal parts, and the doubly shaded (cross-hatched) parts (there are 6 of them) represent three fourths of two thirds. Thus

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{6}{12}.$$



□

The two examples suggest the following simple rule for multiplying any two fractions:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \qquad (2.1)}$$

In words: *the product of two fractions is the product of the numerators over the product of the denominators.* Note that this rule gives the correct results for the two previous examples we did using rectangles:

$$\frac{1}{2} \cdot \frac{1}{3} = \frac{1 \cdot 1}{2 \cdot 3} = \frac{1}{6},$$

$$\text{and} \quad \frac{3}{4} \cdot \frac{2}{3} = \frac{3 \cdot 2}{4 \cdot 3} = \frac{6}{12}.$$

Recall that every whole number can be written as a fraction with denominator 1. So the rule can also be applied when one (or both) of the multiplicands is a whole number.

Example 46. What is two thirds of five? That is, find the product $\frac{2}{3} \cdot 5$.

Solution. Writing 5 as the fraction 5/1 and using the fraction multiplication rule, we get

$$\frac{2}{3} \cdot \frac{5}{1} = \frac{2 \cdot 5}{3 \cdot 1} = \frac{10}{3}.$$

In words: "two thirds of five is ten thirds." □

2.3.1 Exercises

Find the following products. (Just use the fraction product rule – no need to draw rectangles.)

1. $\frac{1}{3} \cdot \frac{5}{7}$
2. One half of one half
3. $\frac{3}{4} \cdot \frac{3}{4}$
4. $3 \cdot \frac{5}{8}$
5. $\left(\frac{2}{3}\right)^2$
6. $\frac{1}{2} \cdot \frac{7}{8} \cdot 3$

2.4 Mixed Numbers

A **mixed number** is the sum of a whole number and a proper fraction. Even though it is a sum, the $+$ sign is omitted. Thus

$$3\frac{1}{2}$$

represents the sum $3 + \frac{1}{2}$. A proper fraction is a mixed number with whole number part 0. We can think of the proper fraction $\frac{2}{3}$ (for example) as

$$0\frac{2}{3} = 0 + \frac{2}{3},$$

but we never write it this way. The important fact is: *every improper fraction can be written as a mixed or whole number.*

2.4.1 Converting an improper fraction into a mixed or whole number

Suppose we have an improper fraction

$$\frac{\text{numerator}}{\text{denominator}}.$$

To convert this to a mixed number, we simply perform the division

$$\text{numerator} \div \text{denominator},$$

obtaining a quotient and a remainder. Then we use these to build the mixed number as follows: the quotient becomes the whole number part; the remainder over the divisor becomes the fractional part. Note that the fractional part obtained in this way is *always proper*, because the remainder is always smaller than the divisor. Summarizing:

$$\frac{\text{numerator}}{\text{denominator}} = \text{quotient}\frac{\text{remainder}}{\text{divisor}}$$

Example 47. Write the improper fraction 13/3 as a mixed number.

Solution. Dividing 13 by 3 yields a quotient of 4 and a remainder of 1. The corresponding mixed number has whole number part 4 and fractional part 1/3. Thus

$$\frac{13}{3} = 4\frac{1}{3}.$$

□

If the remainder is 0, the mixed number is actually just a whole number.

Example 48. Write the improper fraction 84/7 as a mixed number.

Solution. Dividing 84 by 7 yields a quotient of 12 and a remainder of 0. Thus

$$\frac{84}{7} = 12\frac{0}{7} = 12,$$

a whole number. □

Here are some everyday uses of mixed numbers.

Example 49. Five hikers want to share seven chocolate bars fairly. How many bars does each hiker get?

Solution. For fairness, each of the five hikers should get the same amount: exactly one fifth of the chocolate. There are seven bars, so the amount (in chocolate bars) that each hiker should get is

$$\frac{7}{5} = 1\frac{2}{5}.$$

Each hiker gets $1\frac{2}{5}$ chocolate bars. □

Example 50. One of the five hikers in the last example is very big. To support him, his teammates decide to double his share. How many bars does each hiker get now?

Solution. We simply count the big guy as two regular hikers. Now we divide seven bars into six equal parts. Each "regular" hiker gets $\frac{7}{6} = 1\frac{1}{6}$ chocolate bars, while the big one gets

$$2 \times \frac{7}{6} = \frac{2}{1} \cdot \frac{7}{6} = \frac{14}{6} = 2\frac{2}{6} = 2\frac{1}{3}$$

bars. □

Example 51. Julissa waited 5 minutes for the train on Monday, 2 minutes on Tuesday, and then 4, 8, and 3 minutes on Wednesday, Thursday, and Friday, respectively. What was her average wait time for the week?

Solution. Recall that the average of a set of numbers is their sum, divided by the number of numbers. In this case, we want the average of the five numbers $\{5, 2, 4, 8, 3\}$, which is

$$\frac{5+2+4+8+3}{5} = \frac{22}{5} = 4\frac{2}{5}.$$

Her average wait time was $4\frac{2}{5}$ minutes. (Extra credit: how long is two fifths of a minute, in seconds?) ☐

Example 52. One third of a twelve-member jury are women. How many women are on the jury?

Solution. We need to find one third of twelve. $\frac{1}{3} \cdot \frac{12}{1} = \frac{12}{3}$. We divide 12 by 3 to get a mixed number. The quotient is 4 and the remainder is 0, so

$$\frac{12}{3} = 4\frac{0}{3} = 4,$$

a whole number. There are four women on the jury. ☐

2.4.2 Exercises

Convert the following improper fractions into mixed numbers:

1. $\frac{19}{3}$
2. $\frac{11}{2}$
3. $\frac{135}{5}$
4. $\frac{99}{98}$
5. $\frac{77}{5}$

Use mixed numbers to answer the following questions:

6. Three boys share four sandwiches fairly. How many sandwiches does each boy get?
7. What is two thirds of nine?
8. What is the average of the set of numbers $\{11, 14, 9, 12\}$?
9. During the first seven days of January, Bill travels $31, 37, 46, 31, 77, 50$ and 40 miles respectively. What is his average daily travel?

2.4.3 Converting a mixed or whole number to an improper fraction

Sometimes we need to change a mixed number back into an improper fraction. The key fact, again, is that the fraction

$$\frac{a}{a}$$

is *always equal to* 1, for any number a except 0. For example,

$$\frac{3}{3} = 1.$$

Example 53. Convert the mixed number $2\frac{1}{3}$ to an improper fraction, using the fact that $\frac{3}{3} = 1$.

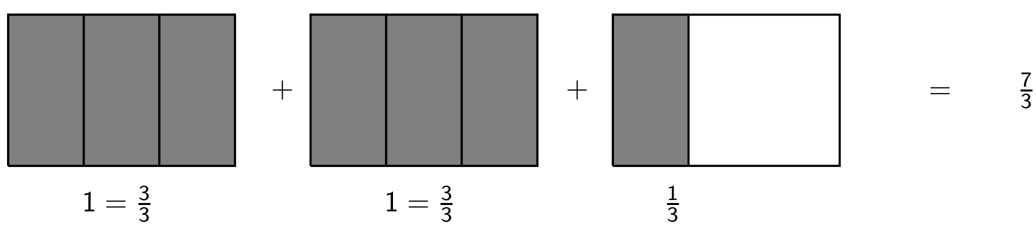
Solution. Think of the whole number 2 in the following way:

$$2 = 1 + 1 = \frac{3}{3} + \frac{3}{3}.$$

It follows that the mixed number $2\frac{1}{3}$ can be written

$$1 + 1 + \frac{1}{3} = \frac{3}{3} + \frac{3}{3} + \frac{1}{3}.$$

The figure below should make it clear that this sum is equal to the improper fraction $\frac{7}{3}$.



(The figure also demonstrates that *fractions with the same denominator add up to a new fraction, with the same denominator, and a numerator which is the sum of all the old numerators.* We'll say more about adding fractions later.) ☐

The procedure in the previous example is easily turned into a general formula.

The mixed number $N\frac{p}{q}$ is equal to the improper fraction $\frac{N \cdot q + p}{q}$.

Example 54. Convert the mixed number $8\frac{2}{3}$ into an improper fraction, using the boxed rule.

Solution.

$$8\frac{2}{3} = \frac{8 \cdot 3 + 2}{3} = \frac{24 + 2}{3} = \frac{26}{3}.$$

Note that we follow the order of operations (multiplication before addition) when evaluating $8 \cdot 3 + 2$. ☐

Example 55. Write the whole number 8 as an improper fraction in three different ways.

Solution. We have

$$8 = 8 \cdot \frac{q}{q} = \frac{8 \cdot q}{q},$$

for any nonzero q. Picking three nonzero numbers for q, say, $2, 5$ and 10, we get

$$8 = \frac{8 \cdot 2}{2} = \frac{16}{2}, \qquad 8 = \frac{8 \cdot 5}{5} = \frac{40}{5}, \qquad \text{and} \quad 8 = \frac{8 \cdot 10}{10} = \frac{80}{10}.$$

□

2.4.4 Exercises

Convert the mixed numbers into improper fractions:

1. $1\frac{1}{2}$
2. $8\frac{1}{3}$
3. $15\frac{3}{8}$
4. $5\frac{3}{4}$
5. $11\frac{5}{6}$

Using $N = N \cdot \frac{q}{q}$ for any nonzero q, write three fractions equal to each given whole number:

6. 5
7. 11
8. 10

2.5 Equivalent Fractions

Fractions which look very different can represent the same number. For example, the fractions

$$\frac{2}{4}, \ \frac{5}{10}, \ \frac{6}{12}, \ \text{and} \ \frac{50}{100}$$

all represent the number $\frac{1}{2}$. What property do all these fractions share? Each has a denominator that is exactly *twice* its numerator; the simplest fraction with this property is $\frac{1}{2}$.

Fractions which represent the same number are called **equivalent**, and we use the equal sign to indicate this. Thus, for example,

$$\frac{1}{2} = \frac{50}{100}.$$

There is an easy way to tell when two fractions are equivalent. We give it here because it is so simple and pleasing, but we postpone the explanation until we discuss *proportions*.

$$\frac{a}{b} = \frac{c}{d} \quad \text{if (and only if)} \quad a \cdot d = b \cdot c.$$

The box above clearly shows why the products $a \cdot d$ and $b \cdot c$ are called *cross-products*. Thus,

Two fractions are equivalent if (and only if) their corresponding *cross-products* are equal.

Starting with a given fraction, we can generate equivalent fractions easily, using the fact that 1 is the multiplicative identity, and that

$$1 = \frac{c}{c}$$

for any non-zero c. Then

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{c}{c},$$

and so we have

The Fundamental Property of Fractions:

If we multiply both the numerator and denominator of a fraction by any nonzero c, then the new fraction is equivalent to the original one:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}.$$

Example 56. Write two fractions equivalent to $\frac{2}{3}$.

Solution. We use the fact that

$$\frac{2}{3} = \frac{2 \cdot c}{3 \cdot c}$$

for any nonzero c. Picking two values for c, say, 6 and 7, we get two fractions equivalent to 2/3:

$$\frac{2}{3} = \frac{2 \cdot 6}{3 \cdot 6} = \frac{12}{18} \qquad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{14}{21}.$$

Of course, other choices of c would have produced other fractions equivalent to 2/3. □

2.5.1 Cancellation and Lowest Terms

The boxed rule above produces equivalent fractions with higher (larger) terms. It is sometimes possible to go the other way, producing lower (smaller) terms. If both numerator and denominator have a **common factor** – a whole number greater than 1 which divides them both with zero remainder – we can "cancel" it by division. The two quotients become the terms of an equivalent fraction with lower terms. For example, the numerator and denominator of $\frac{18}{24}$ have the common factor 6. Thus

$$\frac{18}{24} = \frac{18 \div 6}{24 \div 6} = \frac{3}{4}.$$

In general,

For any non-zero c,

$$\frac{a}{b} = \frac{a \div c}{b \div c}.$$

This method of obtaining lower terms is called **cancellation** or **cancelling out**. It is often indicated as follows:

$$\frac{18}{24} = \frac{\cancel{18}^{\,3}}{\cancel{24}^{\,4}} = \frac{3}{4}.$$

This is useful short-hand, but it has one disadvantage: the common factor (6, in this case) is not made visible. Therefore, sometimes it is convenient, *before cancelling*, to write explicitly the prime factorizations of the numerator and denominator of the fraction as follows:

$$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6} = \frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}} = \frac{3}{4}.$$

Of course, some steps here can be omitted.

Example 57. Find lower terms for the fraction $\frac{28}{70}$.

Solution. Since 7 is a common factor of the numerator and denominator, it can be cancelled, yielding

$$\frac{\cancel{28}^{\,4}}{\cancel{70}^{\,10}} = \frac{4}{10} \quad \text{(cancelling 7)}.$$

Still lower terms are possible, since 4 and 10 have the common factor 2:

$$\frac{\cancel{4}^{\,2}}{\cancel{10}^{\,5}} = \frac{2}{5} \quad \text{(cancelling 2)}.$$

□

If the only common factor of the numerator and denominator is 1, then no further reduction is possible, and the fraction is said to be in **lowest terms**. For example, 2 and 5 have no common factor other than 1, so the fraction $\frac{2}{5}$, and also the improper $\frac{5}{2}$, are in lowest terms.

Example 58. Reduce the fraction $\frac{24}{36}$ to lowest terms.

Solution. A common factor of the numerator and denominator is 4, so we can reduce the terms as follows:

$$\frac{24}{36} = \frac{\cancel{24}^{\,6}}{\cancel{36}^{\,9}} = \frac{6}{9} \qquad \text{(cancelling 4)};$$

6 and 9 have the common factor 3, so we can further reduce

$$\frac{6}{9} = \frac{\cancel{6}^{\,2}}{\cancel{9}^{\,3}} = \frac{2}{3} \qquad \text{(cancelling 3)}.$$

It is clear that $\frac{2}{3}$ is in lowest terms, since 2 and 3 have no common factor other than 1.

Note that this reduction to lowest terms could have been accomplished in one step, had we realized, at the outset, that both 24 and 36 are divisible by 12:

$$\frac{24}{36} = \frac{\cancelto{2}{24}}{\cancelto{3}{36}} = \frac{2}{3} \qquad \text{(cancelling 12).}$$

□

Example 59. Reduce the fraction $\frac{13}{39}$ to lowest terms.

Solution. A common factor of the numerator and denominator is 13, so that

$$\frac{13}{39} = \frac{1 \cdot 13}{3 \cdot 13} = \frac{1}{3}.$$

□

Example 60. Reduce the fraction $\frac{15}{3}$ to lowest terms.

Solution. $\frac{15}{3} = \frac{5}{1} = 5.$ □

2.5.2 Exercises

Using the rule $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ for any nonzero c, write four fractions equivalent to each given fraction:

1. $\dfrac{1}{4}$
2. $\dfrac{3}{4}$
3. $\dfrac{1}{5}$
4. $\dfrac{2}{5}$
5. $\dfrac{1}{8}$
6. $\dfrac{5}{8}$

Using cancellation, reduce each fraction to lowest terms. Convert improper fractions to mixed numbers.

7. $\dfrac{12}{8}$
8. $\dfrac{12}{18}$
9. $\dfrac{20}{45}$
10. $\dfrac{84}{60}$

11. $\dfrac{54}{108}$

12. $\dfrac{360}{120}$

2.6 Prime Factorization and the GCF

Reducing a fraction to lowest terms requires recognizing a common factor (greater than 1) of the numerator and denominator. Doing it *in one step* requires finding and using the *largest* or *greatest* common factor. In this section, we develop a systematic way of finding the **greatest common factor** (GCF) of a set of numbers.

A whole number greater than 1 is **prime** if it has no factors other than itself and 1. The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots .$$

The dots indicate that there are infinitely many larger primes. (This fact is not obvious but was proved more than two thousand years ago by Euclid.) The whole numbers greater than 1 which are *not* prime are called **composite**. (Note that 1 is a special case according to these definitions: it is neither prime nor composite!) The first few composite numbers are

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, \ldots$$

How do we know that these numbers are composite? Because each has at least one factor, other than 1 and itself. For example, 15 has factors 3 and 5, that is, $15 = 3 \cdot 5$.

Every composite whole number has a unique **prime factorization**, which is its expression as the product of its prime factors, listed in order of increasing size. For example:

$$\begin{aligned}
4 &= 2 \cdot 2 & &= 2^2 \\
6 &= 2 \cdot 3 & & \\
8 &= 2 \cdot 2 \cdot 2 & &= 2^3 \\
9 &= 3 \cdot 3 & &= 3^2 \\
10 &= 2 \cdot 5 & & \\
12 &= 2 \cdot 2 \cdot 3 & &= 2^2 \cdot 3 \\
14 &= 2 \cdot 7 & & \\
15 &= 3 \cdot 5 & & \\
16 &= 2 \cdot 2 \cdot 2 \cdot 2 & &= 2^4
\end{aligned} \tag{2.2}$$

To find prime factorizations, we use repeated division with prime divisors. Start by testing the number for divisibility by the smallest prime, 2 (a number is divisible by 2 if its final digit is *even*: $0, 2, 4, 6$ or 8). If it is divisible by 2, we divide by 2 as many times as possible, until we arrive at a quotient which is no longer divisible by 2. We then repeat the procedure, starting with the last quotient obtained, and using the next larger prime, 3 (a number is divisible by 3 if the *sum of its digits* is divisible by 3 – did you know that?). Repeating again (if necessary) with the next larger prime, we eventually arrive at a quotient which is itself a prime number. At that point, we are almost finished. We collect all the primes that were used as divisors (if the same prime has been used more than once, it should be collected as many times), together with the final (prime) quotient. The product of all these numbers is the prime factorization.

Example 61. Find the prime factorization of 300.

Solution. 300, being even, is divisible by 2, so we start by dividing by 2. The steps are as follows:

$$\begin{aligned} 300 \div 2 &= 150; & & \\ 150 \div 2 &= 75; & & \text{75 is not divisible by 2; move on to 3} \\ 75 \div 3 &= 25; & & \text{25 is not divisible by 3; move on to 5} \\ 25 \div 5 &= 5; & & \text{5 is a prime number; stop.} \end{aligned}$$

The primes that were used as divisors were 2, 2, 3, 5. Note that 2 was used twice, so it is listed twice. The final prime quotient is 5. The prime factorization is the product of all those prime divisors and the final prime quotient:

$$300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \quad \text{or} \quad 2^2 \cdot 3 \cdot 5^2.$$

We can easily check our work by multiplying the prime factors and verifying that the product obtained is the original number. ☐

2.6.1 Exercises

Find the prime factorizations of the following numbers. Check the results by multiplication.

1. 60
2. 48
3. 81
4. 360
5. 85
6. 154
7. Which of the numbers above is divisible by 3?

2.6.2 Finding the GCF

Using prime factorizations, it is easy to find the greatest common factor of a set of numbers.

Example 62. Find the greatest common factor (GCF) of the two-number set $\{a, b\}$, where a and b have the following prime factorizations.

$$a = 2^4 \cdot 3^2 \cdot 7 \qquad \text{and} \quad b = 2 \cdot 3^4 \cdot 11.$$

Solution. First, look for the *common* prime factors of a and b: they are 2 and 3. Note that 7 and 11 are not common, and therefore cannot be factors of the GCF. Now look at the *powers* (exponents) on 2 and 3. A small power of any prime is a factor of any larger power of the same prime. The smallest power of 2 that appears is $2 = 2^1$ (in the factorization of b), and the smallest power of 3 that appears is 3^2 (in the factorization of a). It follows that the GCF is the product of the two smallest powers of 2 and 3. Thus

$$\text{GCF}\{a, b\} = 2^1 \cdot 3^2 = 18.$$

Notice that the actual values of a and b (which we could have determined by multiplication) were not used – only their prime factorizations. □

Here is a summary of the procedure:

> To find the GCF of a set of numbers:
>
> 1. find the prime factorization of each number;
> 2. determine the prime factors *common* to all the numbers;
> 3. if there are no common prime factors, the GCF is 1; otherwise
> 4. for each common prime factor, find the smallest exponent that appears on it;
> 5. the GCF is the product of the common prime factors with the exponents found in step 4.

Example 63. (a) Find the GCF of the set $\{60, 135, 150\}$. (b) Find the GCF of the subset $\{60, 150\}$.

Solution. (a) Following the boxed procedure:

1. the prime factorizations are

$$\begin{aligned} 60 &= 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5 \\ 135 &= 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5 \\ 150 &= 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2 \end{aligned}$$

2. the common prime factors are 3 and 5;
3. (does not apply to this example);
4. the smallest exponent on 3 is 1 (in the factorizations of 60 and 150); the smallest exponent on 5 is also 1 (in the factorizations of 60 and 135);
5. the GCF is $3^1 \cdot 5^1 = 15$.

(b) For the two-number subset $\{60, 150\}$, the common prime factors are 2, 3 and 5. The smallest exponent on all three factors is 1. So the GCF is $2^1 \cdot 3^1 \cdot 5^1 = 30$.

Can you explain why the GCF in part (b) is bigger than in part (a)? □

2.6.3 Exercises

Find the GCF of each of the following sets of numbers:

1. $\{72, 48\}$
2. $\{72, 48, 36\}$
3. $\{72, 36\}$
4. $\{48, 36\}$
5. $\{36, 15\}$
6. $\{36, 14\}$
7. $\{15, 14\}$

2.6.4 Cancelling the GCF for lowest terms

Knowing that the GCF of $\{60, 150\} = 30$ allows us to reduce the fraction $\frac{60}{150}$ to lowest terms in one step: we simply cancel it out. Thus,

$$\frac{60}{150} = \frac{\cancel{60}^{\,2}}{\cancel{150}^{\,5}} = \frac{2}{5} \qquad \text{(cancelling the GCF, 30).}$$

Recall that this is short-hand for

$$\frac{60 \div 30}{150 \div 30} = \frac{2}{5}.$$

Example 64. Reduce $\frac{168}{252}$ to lowest terms by finding and cancelling the GCF.

Solution. The prime factorizations of the numerator and denominator are

$$168 = 2^3 \cdot 3 \cdot 7$$
$$252 = 2^2 \cdot 3^2 \cdot 7.$$

The smallest exponent on 2 is 2, and the smallest exponents on 3 and 7 are 1; it follows that the GCF is $2^2 \cdot 3 \cdot 7 = 84$. Cancelling the GCF yields lowest terms:

$$\frac{168}{252} = \frac{\cancel{168}^{\,2}}{\cancel{252}^{\,3}} = \frac{2}{3} \qquad \text{(cancelling the GCF, 84).}$$

□

2.6.5 Exercises

Reduce each fraction to lowest terms by finding and cancelling the GCF of the numerator and denominator. Convert improper fractions to mixed numbers.

1. $\frac{36}{72}$

2. $\frac{48}{36}$

3. $\frac{14}{48}$

4. $\frac{36}{14}$

5. $\frac{14}{15}$

6. $\frac{72}{84}$

7. $\frac{48}{180}$

8. $\frac{96}{56}$

9. $\frac{105}{147}$

10. $\frac{300}{360}$

2.7 Pre-cancelling when Multiplying Fractions

In the fraction product

$$\frac{21}{5} \cdot \frac{15}{14},$$

we could simply follow the rule that the product of fractions is the product of the numerators over the product of the denominators,

$$\frac{21 \cdot 15}{5 \cdot 14},$$

calculate the products in the numerator and denominator and, finally, reduce the resulting fraction to lowest terms by cancelling out the GCF. But there's a short-cut: Since multiplication is commutative, we can reverse the order of multiplication in the numerator, obtaining

$$\frac{15 \cdot 21}{5 \cdot 14} = \frac{15}{5} \cdot \frac{21}{14}.$$

The fractions on the right-hand side are easily reduced to lowest terms by cancelling the obvious GCFs (5 and 7, respectively)

$$\frac{\cancelto{3}{15}}{\cancelto{1}{5}} \cdot \frac{\cancelto{3}{21}}{\cancelto{2}{14}},$$

leaving a simpler product

$$\frac{3}{1} \cdot \frac{3}{2} \quad = \frac{9}{2},$$

which has the further advantage that the solution, $\frac{9}{2}$, is already in lowest terms. The trick of cancelling *before* multiplying – pre-cancellation – saves us from bigger numbers,

$$\frac{21}{5} \cdot \frac{15}{14} = \frac{315}{70},$$

and the extra work of finding the $\mathrm{GCF}\{315, 70\} = 35$ for cancellation:

$$\frac{\cancel{315}^{\,9}}{\cancel{70}^{\,2}} = \frac{9}{2} \quad \text{(cancelling 35)},$$

The general rule is this:

> In a product of fractions, a factor which is common to one of the numerators and one of the denominators can be cancelled before multiplying. The numerator and denominator need not belong to the same fraction.

Notice that pre-cancellation works with any number of factors.

Example 65. Find the product

$$\frac{3}{4} \cdot \frac{8}{5} \cdot \frac{10}{9}.$$

Solution. The numerator of the first fraction (3) has a common factor with the denominator of the third fraction (9), so the product is equal to

$$\frac{\cancel{3}^{\,1}}{4} \cdot \frac{8}{5} \cdot \frac{10}{\cancel{9}^{\,3}} = \frac{1}{4} \cdot \frac{8}{5} \cdot \frac{10}{3}.$$

Continuing on, the numerator of the second fraction (8) has a common factor with the denominator of the first fraction (4), so the product is equal to

$$\frac{1}{\cancel{4}^{\,1}} \cdot \frac{\cancel{8}^{\,2}}{5} \cdot \frac{10}{3} = 1 \cdot \frac{2}{5} \cdot \frac{10}{3}.$$

Finally, the numerator of the third fraction (10) has a common factor with the denominator of the second fraction (5), so (omitting the factor 1) the product is equal to

$$\frac{2}{\cancel{5}^{\,1}} \cdot \frac{\cancel{10}^{\,2}}{3} = \frac{2}{1} \cdot \frac{2}{3}.$$

No further cancellation is possible. The final answer is now a simple product

$$\frac{2}{1} \cdot \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3},$$

which is already in lowest terms. These cancellations could have been done in a different order, or (carefully) all at once. □

Mixed numbers are multiplied by simply converting them into improper fractions.

Example 66. Find the product $2\frac{3}{8} \cdot 1\frac{1}{4} \cdot 2\frac{2}{3}$. Express the result as a mixed number.

Solution. Rewriting each mixed number as an improper fraction, we have the product

$$\frac{19}{8}\cdot\frac{5}{4}\cdot\frac{8}{3}.$$

Cancelling 8's, we have

$$\frac{19\cdot 5}{4\cdot 3}=\frac{85}{12}=7\frac{1}{12}.$$

□

Example 67. A gas tank with a $13\frac{1}{2}$-gallon capacity is only one third full. How much gas is in the tank?

Solution. We need to find the product $\frac{1}{3}\cdot 13\frac{1}{2}$. Converting $13\frac{1}{2}$ to the improper fraction $\frac{27}{2}$, we have

$$\frac{1}{3}\cdot 13\frac{1}{2}=\frac{1}{3}\cdot\frac{27}{2}.$$

Cancelling the common factor 3,

$$\frac{1}{\cancel{3}_{1}}\cdot\frac{\cancel{27}^{9}}{2}=\frac{9}{2}.$$

Converting $\frac{9}{2}$ to a mixed number, we conclude that the tank contains $4\frac{1}{2}$ gallons of gas. □

2.7.1 Exercises

Find the products, using pre-cancellation where possible. Check that the answers are in lowest terms. Express any improper fractions as mixed numbers.

1. $\frac{4}{5}\cdot\frac{7}{12}$
2. $\frac{32}{45}\cdot\frac{9}{16}\cdot\frac{15}{6}$
3. $12\cdot\frac{5}{8}\cdot\frac{2}{3}$
4. $\frac{9}{22}\cdot\frac{21}{12}\cdot 2\frac{4}{7}$
5. $\frac{2}{3}$ of 24.
6. $\frac{3}{4}$ of $\frac{2}{3}$ of 50.
7. $2\frac{2}{3}\cdot 1\frac{3}{4}$.

Use multiplication to answer the following questions.

8. What is the area of a rectangle with length $2\frac{1}{2}$ feet and width $2\frac{3}{10}$ feet?
9. A $12\frac{1}{2}$-gallon fish tank is only three-fifths full. How many gallons of water must be added to fill it up?
10. $2\frac{2}{3}$ cups of beans are needed to make 4 bowls of chili. How many cups are needed to make 8 bowls? 1 bowl? 3 bowls?

2.8 Division of Fractions

Our intuition fails when we think of dividing two fractions. How many times does $\frac{2}{3}$ "go into" $\frac{3}{4}$, for example? It is not at all obvious. But the following notion will help us.

2.8.1 Reciprocals

Two non-zero numbers are **reciprocal** if their product is 1. Thus, if x and y are nonzero numbers, and if

$$x \cdot y = 1,$$

then x *is the reciprocal of* y, and, also, y *is the reciprocal of* x.

The rule for multiplying fractions, together with obvious cancellations, shows that

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{\cancel{a}^{1} \cdot b}{b \cdot \cancel{a}^{1}} = \frac{1 \cdot \cancel{b}^{1}}{\cancel{b}^{1} \cdot 1} = \frac{1}{1} = 1.$$

This means that

> The reciprocal of the fraction $\dfrac{a}{b}$ is the fraction $\dfrac{b}{a}$.
> (both a and b nonzero)

Since every whole number n can be written as the fraction $\frac{n}{1}$, we have the following special case:

> The reciprocal of the whole number n (nonzero)
> is the fraction $\dfrac{1}{n}$.

Example 68. The reciprocal of 5 is $\frac{1}{5}$. The reciprocal of $\frac{1}{9}$ is 9. The reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$, or, expressed as a mixed number, $2\frac{2}{3}$.

We note some important facts and special cases:

- the *reciprocal of the reciprocal* of a number is the number itself. For example, the reciprocal of 2/3 is 3/2, and in turn, the reciprocal of 3/2 is 2/3.
- 1 is the only (positive) number that is its own reciprocal (since $1 = \frac{1}{1}$.)
- the reciprocal of a number less than 1 is a number greater than 1, and vice versa (since the reciprocal of a *proper* fraction is an *improper* fraction).
- 0 has no reciprocal (since division by 0 is undefined).

2.8.2 Exercises

Find the reciprocals of the following numbers. Change improper fractions to mixed or whole numbers.

1. $\frac{3}{7}$
2. 11
3. $\frac{14}{5}$
4. $\frac{1}{25}$
5. $\frac{5}{6}$

2.8.3 Division is Multiplication by the Reciprocal of the Divisor

The concept of reciprocal allows us to convert fraction division into fraction multiplication. The division problem

$$\frac{3}{4} \div \frac{2}{3}$$

can be expressed as a complicated fraction

$$\frac{\frac{3}{4}}{\frac{2}{3}},$$

where the numerator and the denominator are themselves fractions. Recall, once again, that

$$\frac{n}{n} = 1$$

for any nonzero number n. This is true even if n itself is a (nonzero) fraction! Thus, putting $\frac{a}{b}$ in for n,

$$\frac{\frac{a}{b}}{\frac{a}{b}} = 1.$$

For example,

$$\frac{\frac{3}{2}}{\frac{3}{2}} = 1.$$

Going back to our example, and using the fact that 1 (in any form) is the multiplicative identity,

$$\frac{3}{4} \div \frac{2}{3} = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{\frac{3}{4}}{\frac{2}{3}} \cdot 1 = \frac{\frac{3}{4}}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}}.$$

Things look complicated, but they will soon get simpler! The denominator on the right is the product of a fraction and its reciprocal ($\frac{2}{3} \cdot \frac{3}{2}$), and so is equal to 1. It follows that

$$\frac{\frac{3}{4}}{\frac{2}{3}} = \frac{\frac{3}{4} \cdot \frac{3}{2}}{1} = \frac{3}{4} \cdot \frac{3}{2}.$$

The division has been converted into multiplication by the reciprocal. So,

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8}.$$

A complicated argument has led us to a very simple rule:

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}}$$

In words: the quotient of two fractions is the product of the dividend and the *reciprocal of the divisor*.

Example 69. Perform the division $\frac{5}{12} \div \frac{5}{6}$.

Solution. We convert the division into multiplication by the reciprocal of $\frac{5}{6}$.

$$\begin{aligned} \frac{5}{12} \div \frac{5}{6} &= \frac{5}{12} \cdot \frac{6}{5} \\ &= \frac{30}{60} \\ &= \frac{1}{2} \qquad \text{(cancelling 30).} \end{aligned}$$

□

Example 70. What is $\frac{3}{7}$ divided by 6?

$$\begin{aligned} \frac{3}{7} \div 6 &= \frac{3}{7} \div \frac{6}{1} \\ &= \frac{3}{7} \cdot \frac{1}{6} \\ &= \frac{3}{42} \\ &= \frac{1}{14} \qquad \text{(cancelling 3).} \end{aligned}$$

We can divide mixed numbers, too, by simply converting them into improper fractions first.

Example 71. Perform the division $2\frac{8}{9} \div 1\frac{1}{2}$. Express the result as a mixed number.

Solution. Converting mixed numbers to improper fractions, we have

$$2\frac{8}{9} \div 1\frac{1}{2} = \frac{26}{9} \div \frac{3}{2}.$$

Now we convert the division into multiplication by the reciprocal of $\frac{3}{2}$:

$$= \frac{26}{9} \cdot \frac{2}{3}.$$

No pre-cancellation is possible, so

$$\frac{26 \cdot 2}{9 \cdot 3} = \frac{52}{27} = 1\frac{25}{27},$$

where, at the last step, we converted the improper fraction into a mixed number. □

Example 72. How many books, each $1\frac{3}{4}$ inch thick, can fit on a 35 inch bookshelf?

Solution. The question calls for division, since we need to know how many times $1\frac{3}{4}$ "goes into" 35.

$$35 \div 1\frac{3}{4} = \frac{35}{1} \div \frac{7}{4} = \frac{35}{1} \cdot \frac{4}{7}.$$

Cancelling 7, we get

$$\frac{5}{1} \cdot \frac{4}{1} = \frac{20}{1} = 20.$$

So, exactly 20 books will fit on the shelf. □

2.8.4 Exercises

Perform the divisions. Reduce to lowest terms, and change improper fractions to mixed numbers.

1. $\dfrac{1}{5} \div \dfrac{3}{2}$
2. $15 \div \dfrac{2}{3}$
3. $\dfrac{8}{15} \div \dfrac{16}{21}$
4. $10\dfrac{1}{2} \div 5\dfrac{5}{6}$
5. $\dfrac{7}{8} \div \dfrac{21}{4}$
6. $\dfrac{100}{34} \div \dfrac{50}{17}$
7. $100 \div 33$
8. $\dfrac{100}{3} \div 5$

Use division to answer the following questions.

9. How many $1\frac{1}{2}$ inch pieces can be cut from a board that is 36 inches long?
10. Seeds are sold in $2\frac{1}{2}$-ounce packets. A gardener needs 32 ounces of seed. How many packets should he buy to be sure he has enough?

2.9 Adding and Subtracting Fractions

If I eat a third of a pizza for lunch, and another third for dinner, then I have eaten two thirds in total. That is,

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Similar logic applies whenever we add two (or more) fractions *with the same denominator* – we simply add the numerators, while keeping the denominator fixed:

$$\boxed{\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.}$$

A very similar rule holds for subtraction of fractions with the same denominator:

$$\boxed{\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.}$$

Fractions with the same denominator are called **like** fractions.

Example 73. Here are three examples involving addition or subtraction of like fractions:

$$\frac{1}{5}+\frac{2}{5}=\frac{1+2}{5}=\frac{3}{5}$$

$$\frac{13}{7}-\frac{2}{7}=\frac{13-2}{7}=\frac{11}{7}=1\frac{4}{7}$$

$$\frac{7}{8}-\frac{5}{8}=\frac{7-5}{8}=\frac{2}{8}=\frac{1}{4}.$$

Notice that we reduced to lowest terms where necessary, and changed improper fractions to mixed numbers.

2.9.1 Exercises

Add or subtract the like fractions as indicated. Reduce the final answers to lowest terms if necessary. Change improper fractions to mixed numbers.

1. $\frac{1}{5}+\frac{3}{5}=$
2. $\frac{7}{5}+\frac{13}{5}=$
3. $\frac{11}{15}+\frac{13}{15}+\frac{8}{15}=$
4. $\frac{20}{51}-\frac{3}{51}=$
5. $\frac{5}{13}-\frac{4}{13}=$
6. $\frac{11}{25}+\frac{9}{25}-\frac{3}{25}=$
7. $\frac{109}{7}-\frac{11}{7}=$

8. $\frac{13}{2} - \frac{1}{2} =$

9. $\frac{10}{7} + \frac{6}{7} - \frac{11}{7} =$

10. $\frac{11}{25} - \frac{2}{25} + \frac{8}{25} =$

2.9.2 Adding and Subtracting Unlike Fractions

How do we add **unlike** fractions? The trick is to replace each fraction by an equivalent fraction, so that the new fractions have the same denominator.

Example 74. Find the sum:

$$\frac{2}{3} + \frac{1}{5}.$$

Solution. Observe that

$$\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15} \quad \text{and} \quad \frac{1}{5} = \frac{1 \cdot 3}{5 \cdot 3} = \frac{3}{15}.$$

It follows that

$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15}.$$

Using the rule for adding like fractions, we obtain

$$\frac{10}{15} + \frac{3}{15} = \frac{10 + 3}{15} = \frac{13}{15}.$$

Since equivalent fractions represent the same number, we conclude that

$$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}.$$

□

Why (and how) did we choose 15 as the common denominator? We could have chosen any number that is a *multiple* of both 3 and 5 (the two original denominators in the example). But to make our computations as simple as possible, we chose the *smallest* (least) such number. The multiples of 3 are

$$3, 6, 9, 12, 15, 18, 21, \text{etc.},$$

and the multiples of 5 are

$$5, 10, 15, 20, 25, 30, \text{etc.},$$

and it is easy to see that the *least* number that is a multiple of both – a *common* multiple – is 15.

2.9.3 The LCM

The method of listing small multiples is often the simplest way to find the **least common multiple**, or LCM, of a set of whole numbers.

Example 75. Find the LCM$\{6, 10, 15\}$.

Solution. The multiples of 6 are

$$6, 12, 18, 24, 30, 36, \dots ,$$

the multiples of 10 are

$$10, 20, 30, 40, \dots ,$$

and the multiples of 15 are

$$15, 30, 45, \dots .$$

It is evident that the smallest number which is a multiple of *all three* numbers is 30. ☐

The LCM can also be found using prime factorizations. This is useful when the numbers are rather large.

Example 76. Find the LCM of the two-number set $\{a, b\}$, where a and b have the following prime factorizations.

$$a = 2^4 \cdot 3^2 \cdot 7 \qquad \text{and} \quad b = 2 \cdot 3^4 \cdot 11.$$

Solution. The LCM must be a multiple of both numbers, so that it must be divisible by the highest power of every prime factor that appears in any one of the factorizations. The prime factors of a and b are 2, 3, 7 and 11. The highest powers that appear are

$$2^4, \quad 3^4, \quad 7^1, \quad 11^1.$$

The LCM is the product of these powers:

$$\text{LCM}\{a, b\} = 2^4 \cdot 3^4 \cdot 7 \cdot 11.$$

This is a rather large number, but it is the smallest which is a multiple of both a and b. Notice that the actual values of a and b, and their LCM (which we could calculate by multiplication) were not needed – only their prime factorizations. ☐

Here is a summary of the procedure:

To find the LCM of a set of numbers:

1. find the prime factorization of each number;
2. for each prime factor, find the largest exponent that appears on it in any of the factorizations;
3. the LCM is the product of the prime factors with the exponents found in step 2.

Compare this procedure with the procedure for finding the GCF of a set of numbers. There are similarities and significant differences.

2.9.4 Exercises

Find the LCM of the following sets of numbers. Use the prime factorization method for larger numbers.

1. LCM$\{25, 10\}$
2. LCM$\{48, 60\}$
3. LCM$\{10, 15, 25\}$
4. LCM$\{8, 12\}$
5. LCM$\{9, 6, 12\}$
6. LCM$\{60, 168\}$
7. LCM$\{51, 34, 17\}$
8. LCM$\{15, 12\}$
9. LCM$\{18, 8\}$
10. LCM$\{3, 4, 5\}$
11. LCM$\{4, 14\}$
12. LCM$\{2, 5, 9\}$
13. LCM$\{3^2 \cdot 5^2 \cdot 11, \quad 3^4 \cdot 7^2\}$

2.9.5 The LCD

To add unlike fractions, we need the LCM of their denominators. This is such a frequent operation with fractions that it has its own name: the LCD or **Least Common Denominator**.

Example 77. Find the LCD of the fractions

$$\frac{1}{8}, \quad \frac{3}{10}, \quad \text{and} \quad \frac{1}{18}.$$

Solution. The LCD of the fractions is the LCM of their denominators,

$$\text{LCM}\{8, 10, 18\}.$$

Looking at the prime factorizations

$$8 = 2^3, \qquad 10 = 2 \cdot 5, \qquad 18 = 2 \cdot 3^2,$$

and taking the highest power of each prime that occurs, we see that the LCM is

$$2^3 \cdot 3^2 \cdot 5 = 360.$$

This is LCD of the fractions. □

Example 78. Find the sum of the unlike fractions $\frac{1}{8} + \frac{3}{10} + \frac{1}{18}$.

Solution. The LCD is the LCM from the previous example: 360. Now we observe that

$$360 \quad = \quad 8 \cdot 45 \quad = \quad 10 \cdot 36 \quad = \quad 18 \cdot 20.$$

Thus

$$\frac{1}{8} = \frac{1 \cdot 45}{8 \cdot 45}, \qquad \frac{3}{10} = \frac{3 \cdot 36}{10 \cdot 36} \qquad \text{and} \quad \frac{1}{18} = \frac{1 \cdot 20}{18 \cdot 20}.$$

It follows that

$$\begin{aligned} \frac{1}{8} + \frac{3}{10} + \frac{1}{18} &= \frac{1 \cdot 45}{8 \cdot 45} + \frac{3 \cdot 36}{10 \cdot 36} + \frac{1 \cdot 20}{18 \cdot 20} \\ &= \frac{45 + 108 + 20}{360} \\ &= \frac{173}{360}. \end{aligned} \tag{2.3}$$

Since the 173 is not divisible by 2, 3, or 5, the fraction is in lowest terms. □

Practically the same method works for subtracting unlike fractions.

Example 79. Find the difference $\frac{14}{25} - \frac{2}{10}$, using the LCD. Reduce to lowest terms, if necessary.

Solution. The LCD is $\mathrm{LCM}\{25, 10\} = 50$. Now $50 = 25 \cdot 2 = 10 \cdot 5$. So

$$\frac{14}{25} = \frac{14 \cdot 2}{25 \cdot 2} = \frac{28}{50}, \qquad \text{and} \qquad \frac{2}{10} = \frac{2 \cdot 5}{10 \cdot 5} = \frac{10}{50}.$$

Thus, the difference of the two fractions is

$$\frac{28}{50} - \frac{10}{50} = \frac{28 - 10}{50} = \frac{18}{50}.$$

The latter fraction is not in lowest terms, since the GCF of 18 and 50 is 2. Cancelling the GCF, we get

$$\frac{18}{50} = \frac{\cancel{18}^{\,9}}{\cancel{50}^{\,25}} = \frac{9}{25}.$$

□

Example 80. Subtract $\frac{1}{3} - \frac{1}{4}$, and reduce to lowest terms if necessary.

Solution. The LCD is 12. Changing both fractions to equivalent fractions with denominator 12, we get

$$\frac{1}{3} = \frac{1 \cdot 4}{3 \cdot 4} = \frac{4}{12} \quad \text{and} \quad \frac{1}{4} = \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12}.$$

Thus,

$$\frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{4 - 3}{12} = \frac{1}{12}.$$

The latter fraction is already in lowest terms, so we are done. □

Example 81. Find the sum $\frac{4}{5} + \frac{3}{4}$, reduce to lowest terms, and express the answer as a mixed number.

Solution. The $\mathrm{LCD}\{4, 5\} = 20$, so that

$$\frac{4}{5} + \frac{3}{4} = \frac{4 \cdot 4}{5 \cdot 4} + \frac{3 \cdot 5}{4 \cdot 5} = \frac{16}{20} + \frac{15}{20} = \frac{31}{20} = 1\frac{11}{20}.$$

□

2.9.6 Exercises

Add or subtract the following fractions as indicated, reduce to lowest terms if necessary, and change improper fractions to mixed numbers.

1. $\frac{1}{5} + \frac{3}{6}$
2. $\frac{7}{5} + \frac{13}{3}$
3. $\frac{15}{1} + \frac{2}{3}$
4. $\frac{1}{5} + \frac{0}{2}$
5. $\frac{2}{3} + \frac{3}{4} + \frac{4}{5}$
6. $\frac{11}{15} + \frac{13}{25}$
7. $\frac{17}{51} - \frac{3}{50}$
8. $\frac{5}{3} - \frac{4}{13}$
9. $\frac{11}{5} - \frac{2}{25}$
10. $\frac{11}{7} - \frac{11}{17}$
11. $\frac{3}{2} - \frac{1}{12}$
12. $\frac{6}{7} - \frac{11}{17}$
13. $\frac{11}{20} - \frac{2}{25}$

2.10 Mixed Numbers and Mixed Units

Mixed numbers can be added and subtracted by first converting them into improper fractions, and very often this is what we do. But a problem such as

$$75\frac{1}{2} + 59\frac{3}{4}$$

is not well-suited to this method. If we convert to improper fractions, the numerators of both fractions would be quite large and unwieldy.

2.10.1 Vertical Addition and Subtraction

We can align mixed numbers vertically so that the fractional and whole number parts line-up, and then separately add the fraction and whole number columns. We must ensure that the resulting mixed number is in proper form (the fractional part must be a *proper* fraction), and this involves a procedure analogous to "carrying" in whole number addition.

Example 82. Add $2\frac{1}{2} + 6\frac{3}{4}$.

Solution. We first align the whole number and fractional places vertically:

$$\begin{array}{rrr} & 2 & \frac{1}{2} \\ + & 6 & \frac{3}{4} \\ \hline \end{array}$$

To add the fractional parts, we write them both with their LCD, which is 4.

$$\begin{array}{rrr} & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline \end{array}$$

Then we add the columns separately.

$$\begin{array}{rrr} & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline & 8 & \frac{5}{4} \end{array}$$

The resulting mixed number, $8\frac{5}{4}$, is not in proper form, because $\frac{5}{4} = 1\frac{1}{4}$. So we leave the proper fractional part, $\frac{1}{4}$, in the fractions column, and carry the whole number part, 1, over to the whole number column. (We indicate the carrying by writing $\{1\}$ above the whole number column.) Then we recompute the whole number sum, obtaining the final answer in proper form:

$$\begin{array}{rrr} & \{1\} & \\ & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline & 9 & \frac{1}{4} \end{array}$$

□

To subtract mixed numbers, we sometimes need a "borrowing" procedure, analogous to the procedure we use when subtracting whole numbers.

Example 83. Subtract $7 - 3\frac{2}{5}$.

Solution. Note that the whole number 7 is a mixed number with 0 fractional part, and we can represent 0 as a fraction using any convenient denominator. Here the convenient choice is 5. So we write $7 = 7\frac{0}{5}$. Aligning whole number parts vertically, we have

$$\begin{array}{rrc} & 7 & \dfrac{0}{5} \\ - & 3 & \dfrac{2}{5} \\ \hline \end{array}$$

Now we see that the subtraction in the fractions place is not possible (we can't take 2 fifths from 0 fifths), so we need to borrow 1 from the whole numbers column. $1 = \frac{n}{n}$ for any convenient n (except 0), and in this example, it is convenient to write $1 = \frac{5}{5}$. Borrowing $1 = \frac{5}{5}$ from 7 (reducing it to 6), and adding $1 = \frac{5}{5}$ to $\frac{0}{5}$ at the top of the fractions column, we have

$$\begin{array}{rrc} & 6 & \dfrac{5}{5} \\ - & 3 & \dfrac{2}{5} \\ \hline & 3 & \dfrac{3}{5} \end{array}$$

□

Both of the previous examples could have been done by first converting the mixed numbers into improper fractions. For example,

$$7 - 3\frac{2}{5} = \frac{35}{5} - \frac{17}{5} = \frac{18}{5} = 3\frac{3}{5}.$$

Perhaps you find this easier. But here is an example that would involve much more work if we were to use that method.

Example 84. Subtract $153\frac{2}{15} - 67\frac{4}{9}$.

Solution. The LCD$\{15, 9\} = 45$, and since $\frac{2}{15} = \frac{6}{45}$ and $\frac{4}{9} = \frac{20}{45}$, we write, in vertical form,

$$\begin{array}{rrc} & 153 & \dfrac{6}{45} \\ - & 67 & \dfrac{20}{45} \\ \hline \end{array}$$

The subtraction in the fractions column is not possible, so we borrow 1 from 153 (in the form $\frac{45}{45}$) and add it to $\frac{6}{45}$, obtaining

$$\begin{array}{rrc} & 152 & \dfrac{51}{45} \\ - & 67 & \dfrac{20}{45} \\ \hline \end{array}$$

Now the subtraction in the fractions column is possible, and we obtain

$$\begin{array}{rrr} & 152 & \dfrac{51}{45} \\ - & 67 & \dfrac{20}{45} \\ \hline & 85 & \dfrac{31}{45} \end{array}$$

The fractional part is in lowest terms, and we are done. □

If we had done the previous example by first converting the mixed numbers into improper fractions, we would have had to work with rather large numerators:

$$\frac{6891}{45} - \frac{3035}{45}$$

Not impossible, but the other method is easier!

2.10.2 Exercises

Add or subtract the mixed numbers as indicated. Express the final answers as mixed numbers in proper form.

1. $1\frac{1}{3} + \frac{1}{2}$
2. $3\frac{2}{5} + 7\frac{1}{3} + 1\frac{3}{5}$
3. $1\frac{1}{5} + 10$
4. $75\frac{3}{4} + 91\frac{2}{3}$
5. $12\frac{11}{15} - 2\frac{13}{20}$
6. $3\frac{7}{8} - 2\frac{1}{4}$
7. $1\frac{2}{3} - 1\frac{4}{13}$
8. $3\frac{3}{5} + 1\frac{1}{2} + \frac{3}{10}$
9. $1\frac{1}{7} - \frac{3}{7}$
10. $21\frac{3}{5} - 19\frac{3}{4}$
11. $3\frac{6}{7} - \frac{11}{12}$
12. $5\frac{11}{20} - 4\frac{4}{5}$

2.10.3 Measurements in Mixed Units

When a number results from a measurement, it is given in terms of a *unit of measure*, or *unit* for short. For example, if you use a standard ruler to measure the length of a table, the length is given in units of feet (ft). All 1-foot rulers are divided into 12 smaller units called inches (in), so that

$$1 \text{ ft} = 12 \text{ in}.$$

More precise measurements can be given in terms of the **mixed units** feet-and-inches. For example, the length of your table might be 3 feet 6 inches or 3 ft 6 in.

Other familiar examples of mixed units are hours-and-minutes, or hours-minutes-and-seconds. The abbreviations are hr, min and sec. Of course,

$$1 \text{ hr} = 60 \text{ min} \quad \text{and} \quad 1 \text{ min} = 60 \text{ sec}.$$

Mixed units are just like mixed numbers, because the smaller units are simple fractions of the larger ones:

$$1 \text{ in} = \frac{1}{12} \text{ ft} \qquad 1 \text{ min} = \frac{1}{60} \text{ hr} \qquad 1 \text{ sec} = \frac{1}{60} \text{ min}.$$

It follows that measurements in mixed units can be added and subtracted just like mixed numbers. The notion of proper form remains since, for example, a measurement of

$$4 \text{ ft } 17 \text{ in}$$

makes sense, but is *properly* written as

$$5 \text{ ft } 5 \text{ in},$$

since any measurement of over 12 inches is at least 1 foot. Borrowing and carrying are sometimes needed, and are easily accomplished, as the next example shows.

Example 85. Hugo and Hector go to separate movies that start at the same time. Hugo's movie is 2 hours and 5 minutes long, while Hector's is 1 hour and 43 minutes. How long will Hector have to wait, after his movie is over, for Hugo to come out?

Solution. Hector's waiting time is the difference in the lengths of the movies,

$$\begin{array}{rrr} & 2 \text{ hr} & 5 \text{ min} \\ - & 1 \text{ hr} & 43 \text{ min} \\ \hline \end{array}$$

The subtraction in the minutes column can't be done, so we borrow 1 hr = 60 min from the hours column, reducing the number of hours at the top of the hours column from 2 to 1, and increasing the number of minutes at the top of the minutes column from 5 min to 5 min + 60 min = 65 min.

$$\begin{array}{rrr} & 1 \text{ hr} & 65 \text{ min} \\ - & 1 \text{ hr} & 43 \text{ min} \\ \hline & 0 \text{ hr} & 22 \text{ min} \end{array}$$

Hector will have to wait 22 minutes. □

Here's an example involving length measurements.

Example 86. Two boards are laid end-to-end. One of the boards measures 6 ft 8 in, and the other measures 11 ft 10 in. What is the total length of the boards? Give your answer in proper form.

Solution. We need to calculate the mixed-unit sum

$$\begin{array}{rrr} & 6\ \text{ft} & 8\ \text{in} \\ + & 11\ \text{ft} & 10\ \text{in} \\ \hline \end{array}$$

Adding the inches column, we obtain 18 in = 1 ft 6 in. So we put down 6 in in the inches column, and carry 1 ft to the top of the feet column:

$$\begin{array}{rrr} & \{1\ \text{ft}\} & \\ & 6\ \text{ft} & 8\ \text{in} \\ + & 11\ \text{ft} & 10\ \text{in} \\ \hline & & 6\ \text{in} \end{array}$$

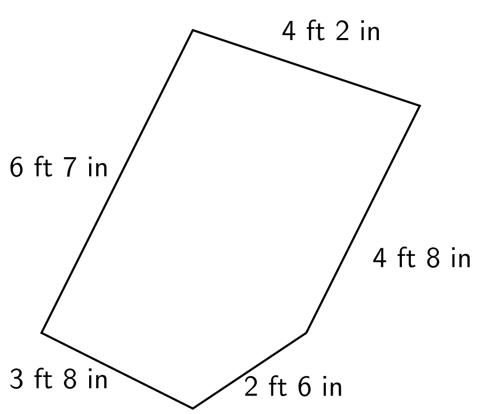
We add up the feet column to obtain the final answer:

$$\begin{array}{rrr} & \{1\ \text{ft}\} & \\ & 6\ \text{ft} & 8\ \text{in} \\ + & 11\ \text{ft} & 10\ \text{in} \\ \hline & 18\ \text{ft} & 6\ \text{in} \end{array}$$

The total length, in proper form, is 18 ft 6 in. □

2.10.4 Exercises

1. A carpenter cuts a 6 ft 7 in piece from a 9 ft 4 in board. What is the length of the leftover piece?
2. Find the perimeter of the polygon:



3. A film critic watched three movies. The first was $2\frac{1}{2}$ hrs, the second 2 hrs 15 min, and the last 1 hr 45 min. (a) What was the total length of the movies? (b) What was the average length of the movies?

2.11 Comparison of Fractions

When two fractions have the same denominator, it is clear that the larger fraction is the one with the larger numerator. This suggests a simple way to compare any set of fractions: replace each fraction by an equivalent fraction having the LCD. Then the largest fraction is the one with the largest numerator, the second-largest is the one with the second-largest numerator, etc.

Example 87. Which is larger, $\dfrac{2}{3}$ or $\dfrac{3}{4}$?

Solution. The LCD is 12,

$$\frac{2}{3} = \frac{8}{12}, \qquad \text{and} \qquad \frac{3}{4} = \frac{9}{12}.$$

The larger of the two fractions with denominator 12 is the one with the larger numerator, $\frac{9}{12}$. It follows that $\frac{3}{4}$ is larger than $\frac{2}{3}$. $\square$

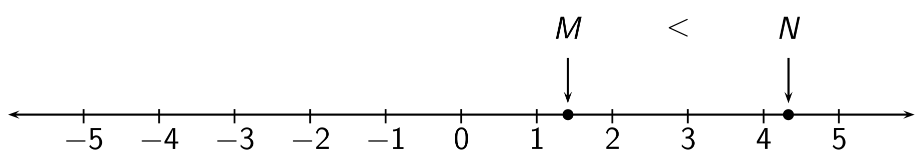
The **inequality symbols** $<$ (less than) and $>$ (greater than) are used to state size comparisons between numbers. For example, the inequalities

$$\frac{1}{2} < 1, \qquad 2\frac{1}{3} > 2$$

state that $\frac{1}{2}$ is less than 1, and that $2\frac{1}{3}$ is greater than 2. Visualizing the number line yields a good rule of thumb:

"Left is Less"

$M < N$ if M is *left of* N on the number line:



It is a happy coincidence that the symbol $<$ for "Less than" looks rather like the letter "L."

Numbers arranged in **increasing order** are written from left to right, separated by the $<$ symbol. (This is the left-to-right ordering on the number line.) We can also arrange numbers in **decreasing order**, from left to right, separated by the $>$ ("greater than") symbol.

Example 88. Arrange the fractions $\dfrac{3}{4}, \dfrac{2}{3},$ and $\dfrac{7}{9}$ in increasing order.

Solution. $\mathrm{LCD}\{3, 4, 9\} = 36$, and

$$\frac{3}{4} = \frac{27}{36} \qquad \frac{2}{3} = \frac{24}{36} \qquad \frac{7}{9} = \frac{28}{36}.$$

We are asked to arrange the fractions in *increasing* order, so we list them in increasing order of their numerators, separated by the inequality symbol $<$. Thus,

$$\frac{24}{36} < \frac{27}{36} < \frac{28}{36}.$$

Now, going back to the equivalent fractions in lowest terms, we have

$$\frac{2}{3} < \frac{3}{4} < \frac{7}{9}.$$

$\square$

Example 89. Arrange the numbers $1\frac{3}{5}, 1\frac{3}{4}, 1\frac{6}{7}$, and $\frac{11}{12}$ in decreasing order.

Solution. Recall that a proper fraction is less than 1, so $\frac{11}{12}$ is obviously the smallest of the four numbers (the others being mixed numbers greater than 1). To compare the other three numbers, it is enough to compare their fractional parts, because each has the same whole number part (1). To compare $\frac{3}{5}$, $\frac{3}{4}$, and $\frac{6}{7}$, we find the LCM$\{5, 4, 7\} = 140$, and write equivalent fractions with denominator 140:

$$\frac{3}{5} = \frac{84}{140} \quad \frac{3}{4} = \frac{105}{140} \quad \frac{6}{7} = \frac{120}{140}.$$

In decreasing order, the numerators are $120 > 105 > 84$. The corresponding ordering of the fractions is

$$\frac{6}{7} > \frac{3}{4} > \frac{3}{5},$$

and it follows that the corresponding order of the mixed numbers (each with whole number part 1) is

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5}.$$

Adjoining the smallest (proper) fraction $\frac{11}{12}$ at the end, we have, finally, the decreasing order

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5} > \frac{11}{12}.$$

$\square$

2.11.1 Exercises

1. Arrange in decreasing order: $\frac{4}{9}, \frac{3}{8}$, and $\frac{1}{3}$.

2. Arrange in increasing order: $\frac{3}{5}, \frac{3}{4}$, and $\frac{5}{7}$.

3. Is $\frac{5}{12}$ of an inch more or less than $\frac{7}{16}$ of an inch?

4. Is $9\frac{5}{8}$ inches closer to 9 or 10 inches?

5. A stock price changed from $5\frac{5}{8}$ to $5\frac{3}{4}$. Did the price go up or down?

2.12 Combined operations with fractions and mixed numbers

Recall that when there is more than one operation to be performed in a calculation, we agree to follow the **order of operations**:

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

We remind you that "*in order of appearance*" means in order *from left to right*, and that "grouping symbols" include parentheses, brackets, braces (curly brackets), the square root symbol, and the horizontal fraction line as in $\frac{a+b}{c-d}$.

Example 90. Calculate $2 - \dfrac{3}{5} + \dfrac{1}{2}$.

Solution. Subtraction comes first in the left-to-right order. Rewriting 2 as $\frac{10}{5}$, we have

$$\begin{aligned}\left(\frac{10}{5} - \frac{3}{5}\right) + \frac{1}{2} &= \frac{7}{5} + \frac{1}{2} \\ &= \frac{14}{10} + \frac{5}{10} \qquad\qquad (\text{LCD} = 10) \\ &= \frac{19}{10} \\ &= 1\frac{9}{10}.\end{aligned}$$

□

Example 91. Calculate $2 - \left(\dfrac{3}{5} + \dfrac{1}{2}\right)$.

Solution. This differs from the previous example only in the parentheses, which force us to do the addition first. Rewriting $\frac{3}{5}$ and $\frac{1}{2}$ with the LCD 10, we have

$$\begin{aligned}2 - \left(\frac{6}{10} + \frac{5}{10}\right) &= 2 - \frac{11}{10} \\ &= \frac{20}{10} - \frac{11}{10} \\ &= \frac{9}{10}.\end{aligned}$$

□

Example 92. Calculate $\left(\dfrac{2}{3}\right)^2 + 1\dfrac{2}{3} \cdot \dfrac{1}{10}$.

Solution. We evaluate the expression with an exponent first ($(\frac{2}{3})^2 = \frac{4}{9}$), followed by the mixed number multiplication, followed by the addition.

$$\begin{aligned}
\left(\frac{2}{3}\right)^2 + 1\frac{2}{3} \cdot \frac{1}{10} &= \frac{4}{9} + 1\frac{2}{3} \cdot \frac{1}{10} \\
&= \frac{4}{9} + \frac{5}{3} \cdot \frac{1}{10} && \text{(converting the mixed number to an improper fraction)} \\
&= \frac{4}{9} + \frac{5}{30} \\
&= \frac{4}{9} + \frac{1}{6} && \text{(cancelling 5)} \\
&= \frac{8}{18} + \frac{3}{18} && \text{(LCD = 18)} \\
&= \frac{11}{18}.
\end{aligned}$$

□

Example 93. Calculate $1\frac{1}{4} \cdot \frac{2}{3} - 1\frac{1}{6} \div 1\frac{1}{2}$.

Solution. The order of operations is multiplication, division, subtraction. We first convert mixed numbers into improper fractions, and change the division into multiplication by the reciprocal.

$$\begin{aligned}
\frac{5}{4} \cdot \frac{2}{3} - \frac{7}{6} \div \frac{3}{2} &= \frac{10}{12} - \frac{7}{6} \cdot \frac{2}{3} \\
&= \frac{10}{12} - \frac{14}{18} \\
&= \frac{5}{6} - \frac{7}{9} && \text{(lowest terms)} \\
&= \frac{15}{18} - \frac{14}{18} && \text{(LCD = 18)} \\
&= \frac{1}{18}.
\end{aligned}$$

□

Example 94. Calculate $\{1 + [2 + \left(1\frac{1}{4} \cdot \frac{2}{3} - 1\frac{1}{6} \div 1\frac{1}{2}\right)]\}$.

Solution. In this example we have so-called *nested* grouping symbols, that is, grouping symbols inside other grouping symbols. In this case, we must work "from the inside out." This means we work within the innermost grouping symbols first. From the previous example we know that

$$1\frac{1}{4} \cdot \frac{2}{3} - 1\frac{1}{6} \div 1\frac{1}{2} = \frac{1}{18},$$

thus we get

$$\{1 + [2 + \left(1\frac{1}{4} \cdot \frac{2}{3} - 1\frac{1}{6} \div 1\frac{1}{2}\right)]\}$$

$$= \{1 + [2 + \left(\frac{1}{18}\right)]\} = \{1 + [\frac{37}{18}]\} = \frac{55}{18} = 3\frac{1}{18}.$$

□

2.12.1 Exercises

Calculate, using the correct order of operations. Express improper fractions as mixed numbers.

1. $\dfrac{3}{4} + \dfrac{2}{3} \div \dfrac{4}{9}$
2. $2 - \dfrac{3}{4} + 3\dfrac{1}{2}$
3. $\left(\dfrac{3}{4} + 3\dfrac{1}{2}\right) - 2$
4. $\left(8\dfrac{1}{4} - 1\dfrac{2}{3}\right) \div \dfrac{1}{3}$
5. $7\dfrac{1}{2} \div \dfrac{3}{5} + 1\dfrac{7}{8} \cdot 2\dfrac{2}{5}$
6. $\left(\dfrac{3}{4}\right)^2 + 2\dfrac{4}{5} \cdot 1\dfrac{1}{4}$
7. Find the perimeter of a rectangle with length $2\frac{1}{2}$ ft and width $1\frac{3}{4}$ ft.
8. Julia has two closets. The floor of one is a square $3\frac{1}{2}$ ft on a side. The floor of the other is a rectangle of width $3\frac{1}{2}$ ft and length $4\frac{1}{2}$ ft. She wants to put square tiles on the floors of the closets. How much will it cost if the tiles are \$5 per square foot?

Chapter 3

Decimals and Percents

Decimals are a convenient type of mixed number, in which the proper fractional part has a denominator which is 10, or 100, or 1000, or, more generally, a *power* of 10. Here are some powers of 10:

$$\begin{aligned}
10^0 &= 1 \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1000 \\
10^4 &= 10000 \\
10^n &= 1 \text{ followed by } n \text{ 0's.}
\end{aligned}$$

The number above and to the right of 10 is the *exponent*, or *power*, and indicates the number of times that 1 is to be multiplied by 10. Thus

$$10^3 = 1 \times 10 \times 10 \times 10 = 1000,$$

and so forth. (This explains why $10^0 = 1$.)

Here are some examples of mixed numbers in which the denominator of the fractional part is a power of ten, together with their representations as decimals:

$$\begin{aligned}
1\frac{3}{10} &= 1.3 \\
21\frac{37}{100} &= 21.37 \\
13\frac{21}{1000} &= 13.021
\end{aligned}$$

The *decimal point* (which looks like the period at the end of a sentence) separates the whole number part from the proper fractional part. The digits to the right of the decimal point represent the fractional part according to the following rule:

- the digits to the right of the decimal point constitute the numerator;
- the *number* of digits to the right of the decimal point is the power of 10 which constitutes the denominator.

There is no need to show the fraction bar. As we shall see, this makes computation very easy. There are some subtleties to watch out for, however. In the example

$$13\frac{21}{1000} \quad = \quad 13.021,$$

we had to use 021 to represent the numerator 21 because we need *three* digits to show that the denominator is 10^3.

3.1 Decimal place values

In Chapter 1 we showed that any whole number can be written using just the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, by using the *place value* system. Thus, you will recall that

$$4267 \text{ stands for } \quad 4 \textit{ thousands} \ + \ 2 \textit{ hundreds} \ + \ 6 \textit{ tens} \ + \ 7 \textit{ ones}.$$

Using powers of 10, we can write this more compactly:

$$4267 = 4 \times 10^3 \ + \ 2 \times 10^2 \ + \ 6 \times 10^1 \ + \ 7 \times 10^0.$$

The decimal point allows us to adjoin more places (to the right of the decimal point), with place values less than 1. Thus

$$23.59 \text{ stands for } \quad 2 \textit{ tens} \ + \ 3 \textit{ ones} \ + \ 5 \textit{ tenths} \ + \ 9 \textit{ hundredths}.$$

The first place to the right of the decimal point has the place value $\frac{1}{10}$, or one *tenth*; the second place to the right of the decimal point has the place value $\frac{1}{100}$, or one *hundredth*, and so forth. More generally,

> The nth place to the right of the decimal point has the place value $\dfrac{1}{10^n}$.

This is a continuation of the pattern we established in Chapter 1: as we move to the right, place values decrease ten-fold (by a factor of *one tenth*). If we adopt the convention that

$$10^{-n} = \frac{1}{10^n},$$

then the sequence of place values, from left to right, is just the sequence of decreasing powers of ten:

$$\ldots, 10^3, \quad 10^2, \quad 10^1, \quad 10^0 \quad \text{(decimal point)} \quad 10^{-1}, \quad 10^{-2}, \quad 10^{-3}, \quad \ldots$$

$$\ldots, \textit{thousands, hundreds, tens, ones} \quad \text{(decimal point)} \quad \textit{tenths, hundredths, thousandths,} \ \ldots$$

Example 95. Name the digit in tens place, and the digit in the hundredths place, in the decimal 304.671.

Solution. The tens (10^1) place is the second place to the left of the decimal point. The digit in that place is 0. The hundredths (10^{-2}) place is the second place to the right of the decimal point. The digit in that place is 7. □

Example 96. What is the digit in the ten-thousandths place of the decimal 0.00286?

Solution. One ten-thousandth is

$$\frac{1}{10000} = \frac{1}{10^4},$$

so the ten-thousandths place is the fourth place to the right of the decimal point. The digit in that place is 8. □

3.1.1 Exercises

1. 19.01 stands for
2. 0.946 stands for
3. 98.6 stands for
4. Name the digit in the hundred-thousandths place in the decimal 1.870457.
5. Express the following mixed numbers as decimals:
   (a) $2\frac{23}{100}$
   (b) $\frac{237}{1000}$
   (c) $19\frac{7}{10}$
   (d) $\frac{12}{10000}$.
6. Express the following decimals as mixed numbers or proper fractions:
   (a) 14.3
   (b) 28.61
   (c) 100.1
   (d) 4.003

3.2 Significant and Insignificant 0's

Decimals are read aloud in the following way:

- read the whole number part;
- read "and" for the decimal point;
- read the digits to the right of the decimal point as if they represented a whole number;
- append the place-name of the last decimal place.

Example 97. (a) 1.13 is read aloud as "one and thirteen hundredths."
(b) 31.045 is read aloud as "thirty-one and forty-five thousandths".
(c) 48.006 is read aloud as "forty-eight and six thousandths."

In the decimal 48.006, the fractional part .006 stands for 0 tenths + 0 hundredths + 6 thousandths. Reading aloud, we simply say "six thousandths." Nonetheless, the two 0's in .006 are absolutely necessary. If we had written .6 instead of .006, we would have named a different fraction, namely, $\frac{6}{10}$, rather than $\frac{6}{1000}$. On the other hand, in the decmial 006.23, the first two 0's are unnecessary, since the whole number 006 is just 6. In one instance the 0's are crucial, and in the other, they are unnecessary.

How do we know which 0's are significant, and which are insignificant? We can precede any whole number or whole number part of a decimal by as many 0's as we want, because these 0's do not change the value of the number (as in $006 = 6$). Similarly, we can *follow* the fractional part of a decimal with as many 0's as we want, because these 0's do not change the value of the number. For example, $.98000 = .98$ because $.98000 = 9 \times 10^{-1} + 8 \times 10^{-2} + 0 \times 10^{-3} + 0 \times 10^{-4} + 0 \times 10^{-5}$. In these cases, the 0's are *insignificant*. In contrast, *internal* 0's hold a place and cannot be omitted.

Example 98. 2.3004 is not equal to 2.304, or to 2.34, because the fractional parts, $\frac{3004}{10000}$, $\frac{304}{1000}$, and $\frac{34}{100}$ are not equal. The 0's in the hundredths and thousandths places of 42.3004 are *significant*, and therefore cannot be omitted.

> In a decimal, a 0 digit is *insignificant*, and may be omitted, if (and only if)
>
> - it precedes the left-most non-zero digit of the whole number part; or
> - it follows the right-most non-zero digit of the fractional part.
>
> All other 0's are *significant*, and cannot be omitted.

Example 99. In the decimal 0304.90800, the 0 between 3 and 4, and the 0 between 9 and 8 are significant; the others are insignificant. Thus

$$0304.90800 = 304.908.$$

Example 100. Does 23400 contain any insignificant 0's?

Solution. 23400 is a whole number. The decimal point is not shown, but is "understood" to be at the rightmost end. It follows that the two 0's in 23400 are internal, and therefore, significant. □

It is sometimes convenient to preserve or adjoin non-significant 0's. For example, when a decimal represents a *proper* fraction, the whole number part is 0, but that 0, to the left of the decimal point, is often retained for emphasis or clarity, even though it is insignificant. Thus the proper fraction $\frac{67}{100} = .67$ is often written as 0.67.

3.3 Comparing Decimals

Insignificant 0's are also useful in comparing decimals as to size. Recall that when fractions have the same (common) denominator, the one with the largest numerator represents the largest number. It is very easy to find a common denominator for two or more decimals, because the number of decimal *places* to the right of the decimal point is all that is needed to determine the denominator: if there are n places to the right of the decimal point, the denominator is 10^n. Given a set of decimals, we just find the one with the largest number of decimal *places* to the right of the decimal point, and use that number to determine the common denominator. There is no computation involved: we simply "pad out" the shorter decimals with insignificant 0's to the right of the decimal place until all have the same number of decimal places. Then it is easy to tell which decimal has the largest numerator, and hence, which is largest.

Example 101. Arrange the decimals in descending order, from largest to smallest: 0.102, 0.09876, 0.2.

Solution. Of the three given decimals, 0.09876 is the longest, with 5 places to the right of the decimal point. So we pad out all three to 5 places, using insignificant 0's: Thus

$$\begin{aligned} 0.102 &= 0.10200 \qquad (3.1) \\ 0.09876 &= 0.09876 \\ 0.2 &= 0.20000 \end{aligned}$$

In this form, the decimals have a common denominator of $10^5 = 100000$:

$$0.102 = \frac{10200}{100000} \quad 0.09876 = \frac{9876}{100000} \quad 0.2 = \frac{20000}{100000}.$$

Now, it is easy to see which has the largest numerator, which the second-largest, and which the smallest. Hence we have

$$0.2 > 0.102 > 0.09876.$$

□

3.3.1 Exercises

Eliminate the insignificant 0's in the following decimals.

1. 210304.0900

2. 00206.006070

3. 210.00

4. 21030900

Arrange each group of decimals in descending order, from largest to smallest:

5. 0.2, .009, .121.

6. 1.31, 1.9, 1.224.

7. 0.106, 0.5, 0.61.

8. 9.104, 9.14, 9.137, 9.099.

3.4 Rounding-off

The numbers

$$0.1, \quad 0.11, \quad 0.111, \quad 0.1111, \quad 0.11111$$

get closer and closer together (on the number line) as we move from left to right. After a few steps, they are almost too close together to visualize. The second number was obtained from the first by adding $\frac{1}{100}$, and the last from the second-to-last by adding just $\frac{1}{100000}$, a very small quantity indeed. Such small quantities can be important in the sciences, but even in circumstances where precision is important, there is always a limit beyond which small differences become negligible – not worth worrying about. (For example, when your bank calculates the interest on your savings account, it calculates, but then ignores, amounts that are less than one half of one cent.) Being precise, but not overly precise, is the purpose of *rounding-off* numbers.

To round off a number, we must first decide how precise to be. That means choosing the place whose value we consider the smallest worth worrying about. For example, in negotiating an annual salary, we would probably not argue about amounts less than $100, but in negotiating an hourly wage, we would be willing to argue about pennies. In the first instance, we would round off our dollar amounts to the nearest hundred, and in the second, to the nearest hundredth (cent).

Example 102. \$175 is closer to \$200 than to \$100, so, rounding to the *nearest hundred*, we say that an annual salary of \$36,175 is approximately \$36,200. We write

$$36175 \approx 36200 \quad \text{(to the nearest hundred).}$$

The symbol $\approx$ means "approximately equal to."

Example 103. 16.84 is closer to 16.80 than to 16.90, so, *to the nearest tenth*,

$$16.84 \approx 16.8.$$

Notice that we dropped the insignificant 0 in 16.80.

Example 104. 0.135 is exactly halfway between 0.130 and 0.140, so, if we want to round *to the nearest hundredth*, it is not clear whether we should round "up" to 0.140, or "down" to 0.130. Here, we have to make up a rule. The convention is that "halfway is almost home," so we round up.

$$0.135 \approx 0.14 \quad \text{(to the nearest hundredth),}$$

where, again, we have dropped the insignificant 0 in 0.140.

Below is a summary of the procedure for rounding off a number expressed as a decimal to a given place, called the *round-off place*.

To round off a decimal to a given place (the *round-off place*):

1. Preserve the digit in the round-off place if the right-neighboring digit is less than 5; otherwise, increase the digit in the round-off place by 1;
2. Replace all digits to the right of the round-off place by 0's;
3. Eliminate insignificant 0's (except a 0 in the round-off place).

Example 105. Round 26.03 to the nearest tenth.

Solution. The round-off place is the *tenths* place, and the right-neighboring digit (3) is less than 5, so we preserve the digit in the round-off place (0) and replace all digits to the right (there is only one) with 0. We get 26.00. Here the exception in step 3 of the procedure applies: both the 0's are insignificant, but we keep the 0 in the round-off place (otherwise it might seem as if we had rounded to the nearest unit). Thus

$$26.03 \approx 26.0 \quad \text{(to the nearest tenth)}$$

□

Example 106. Round 51 to the nearest hundred.

Solution. The round off place (hundreds) doesn't appear, but we can show it by adjoining an insignificant 0 on the left: $51 = 051$. The right neighbor of the round-off digit is 5, so, by the "halfway" convention, we round the 0 in the round-off place up to 1, and replace all digits to the right with 0. We get

$$51 \approx 100 \quad \text{(to the nearest hundred).}$$

Notice that the two 0's in 100 are significant, and cannot be eliminated! □

One final remark about step 2 of the procedure. If the digit in the round-off place is 9, and we need to round up (to 10), we must carry the 1 to the left-neighboring place.

Example 107. Round 6.597 to the nearest hundredth.

Solution. The 9 in the round-off place has right-neighbor 7, which is greater than 5. So we increase 9 to 10, and carry the 1 to the (left-neighboring) tenths place:

$$6.597 \approx 6.60 \quad \text{(to the nearest hundredth)}$$

Notice we preserved the insignificant 0, since it is in the round-off place. ☐

3.4.1 Exercises

Round off each number twice: (a) to the nearest ten; (b) to the nearest hundredth.

1. 304.0900
2. 96.075
3. 115.497
4. 100.0055
5. 7.009

3.5 Adding and Subtracting Decimals

One of the advantages of decimals over ordinary fractions is ease of computation. The four operations (addition, subtraction, multiplication and division) are done almost exactly as with whole numbers. The only question is where to put the decimal point.

Addition and subtraction are the easiest: we line up the numbers vertically, with the ones places aligned on top of eachother. This puts the decimal points in alignment as well. The decimal point in the sum is placed directly below the decimal points in the numbers being added. Carrying and borrowing are done as with whole numbers.

Example 108. Find the sum of 96.8 and 342.03.

Solution. Line up the numbers vertically so that the ones places (and hence the neighboring decimal points) are directly on top of eachother.

$$\begin{array}{ccccccc} & & 9 & 8 & . & 8 & \\ + & 3 & 4 & 2 & . & 0 & 3 \\ \hline \end{array}$$

For clarity, you can "pad out" the top number with an insignificant 0 so that both numbers have the same number of decimal places. (This is not strictly necessary: just remember that an empty place is occupied by a 0.) Place a decimal point, vertically aligned with the others, where the sum will go.

$$\begin{array}{ccccccc} & & 9 & 8 & . & 8 & 0 \\ + & 3 & 4 & 2 & . & 0 & 3 \\ \hline & & & & . & & \end{array}$$

As before, add the digits in each column starting from the right-most one, carrying when necessary:

$$\begin{array}{rrrcrr} & 9 & 8 & . & 8 & 0 \\ + \; 3 & 4 & 2 & . & 0 & 3 \\ \hline \\ 4 & 4 & 0 & . & 8 & 3 \end{array}$$

The sum is 440.83. □

Example 109. Find the sum of 804.09, 56.384, 107, and 0.205.

Solution. Line up the numbers vertically at the decimal points (equivalently, at the ones place). In the whole number 107, the decimal point is understood to be directly to the right of the ones place. Add the columns from right to left, imagining 0's in the empty places, and carrying when necessary.

$$\begin{array}{crrrcrrr} & 8 & 0 & 4 & . & 0 & 9 & \\ & & 5 & 6 & . & 3 & 8 & 4 \\ & 1 & 0 & 7 & . & & & \\ + & & & 0 & . & 2 & 0 & 5 \\ \hline \\ & 9 & 6 & 7 & . & 6 & 7 & 9 \end{array}$$

□

Subtraction follows the same vertical alignment rule.

Example 110. Find the difference $50 - 4.706$.

Solution. The larger number, 50, goes on top. We fill it out with insignificant 0's so that both the minuend and the subtrahend have the same number of decimal places.

$$\begin{array}{crrcrrr} & 5 & 0 & . & 0 & 0 & 0 \\ - & & 4 & . & 7 & 0 & 6 \\ \hline \\ & & & . & & & \end{array}$$

Borrowing from the *tens* place of the subtrahend, we can write

$$\begin{aligned} 50.000 &= 5 \textit{ tens} + 0 \textit{ ones} + 0 \textit{ tenths} + \text{ etc.} \\ &= 4 \textit{ tens} + 9 \textit{ ones} + 9 \textit{ tenths} + 9 \textit{ hundredths} + \{10\} \textit{ thousandths} \\ &= 49.99\{10\} \end{aligned}$$

Then our subtraction becomes

$$\begin{array}{crrcrrr} & 4 & 9 & . & 9 & 9 & \{10\} \\ - & & 4 & . & 7 & 0 & 6 \\ \hline \\ & 4 & 5 & . & 2 & 9 & 4 \end{array}$$

□

3.5.1 Exercises

Find the sums or differences:

1. $680.48 + 56.09$
2. $804.09 + 5.8409$
3. $58.09 - 32.1$
4. $4.09 + 0.38409$
5. $14.093 - 6.39$
6. $100 - 23.441$
7. $830 - 16.61$
8. $80 - 56.384$
9. Verify the subtraction problems above by doing an appropriate addition.
10. Find the sum of 64.09, 15.3, 4, and 9.09.
11. The difference of 20 and the sum of 4.6 and 0.07

3.6 Multiplying and Dividing Decimals by Powers of 10

What happens when we multiply a whole number by 10? *Ones* become *tens*, *tens* become *hundreds*, *hundreds* become *thousands*, etc. For example, in the multiplication

$$234 \times 10 = 2340,$$

the 4 *ones* in 234 turn into the 4 *tens* in 2340, the 3 *tens* turn into 3 *hundreds* and so forth. (In the product, there are no longer any *ones*, and that fact must be recorded with a significant 0 in the *ones* place.)

Remember that a whole number is a decimal, with the decimal point understood to be immediately to the right of the ones place. Making the decimal point explicit in our example,

$$234.0 \times 10 = 2340.$$

We can describe what happens this way: when we multiply by 10, all the digits shift one place to the left, including the insignificant 0 which was understood to be in the *tenths* place of the whole number 234. (In this description, we imagine the decimal point remaining fixed.) Similarly, if we multiply a whole number by 100, all the digits shift *two* places to the left, including the *two* insignificant 0's which are understood to be in the *tenths* and *hundredths* places. Thus, $597 \times 100 = 59\ 700$, or, more explicitly,

$$597.00 \times 100 = 59\ 700.$$

In general, if we multiply a whole number by any positive power of 10, say, by 10^n, all the digits shift n places to the left, including the n insignificant 0's understood to be in the n places to the right of the decimal point (while the decimal point remains fixed). Thus, for example,

$$281 \times 10^4 = 2\ 810\ 000.$$

If there are non-zero digits to the right of the decimal place, they shift in exactly the same way. Here are some examples:

$$
\begin{aligned}
38.623 \times 100 &= 3862.3 \\
0.6 \times 10 &= 6 \\
0.0031 \times 1000 &= 3.1 \\
12.09 \times 10^4 &= 120\ 900 \\
100 \times 10^2 &= 10000.
\end{aligned}
$$

Note that insignificant 0's have been omitted from the products.

It is often convenient to imagine that, when multiplying by a power of 10, the digits remain fixed while the decimal point moves (to the *right*). (Electronic calculators work this way, using a "floating" decimal point.) This description leads to a very easy rule for multiplying a decimal by a positive power of 10:

To multiply a decimal by 10^n, move the decimal point n places to the right.

If we *divide* a decimal by 10, *hundreds* becomes *tens*, *tens* become *ones*, *ones* become *tenths*, *tenths* become *hundredths*, etc. The whole discussion above can be repeated, except that, in this case of division, digits shift *to the right*, or, equivalently, the decimal point moves *to the left*. The easy rule is

To divide a decimal by 10^n, move the decimal point n places to the left.

Here are some examples:

$$
\begin{aligned}
623 \div 10 &= 62.3 \\
0.023 \div 100 &= 0.00023 \\
480 \div 10 &= 48 \\
37.5 \div 10^3 &= 0.0375
\end{aligned}
$$

With the convention that

$$10^{-n} = \frac{1}{10^n},$$

division by 10^n can be thought of as multiplication by 10^{-n}. For example,

$$
\begin{aligned}
5 \div 100 &= 5 \times 10^{-2} = 0.05 \\
6.5 \div 10 &= 6.5 \times 10^{-1} = 0.65 \\
86.37 \div 10000 &= 86.37 \times 10^{-4} = 0.008637 \\
8 \div 100000 &= 8 \times 10^{-5} = 0.00008
\end{aligned}
$$

Notice that the left movement of the decimal point is indicated by the negative exponent.

3.6.1 Exercises

Multiply or divide each decimal by the indicated power of 10.

1. 6080.48×10
2. $6080.48 \div 10$
3. 19×10^{-1}
4. 804.09×10^{2}
5. $804.09 \div 10^{2}$
6. 0.0908×10^{3}
7. $0.0908 \div 10^{3}$
8. 38×10^{-3}
9. 260804.09×10^{4}
10. $260804.09 \div 10^{5}$
11. 48.3×10^{-3}
12. What is the value, in dollars, of 5 million pennies?

3.7 Multiplication of general decimals

We already know how to multiply and divide decimals when one of them is a power of 10. If neither factor is a power of ten, there is more to the process than just moving the decimal point. Consider the following example.

Example 111. Multiply 0.3×0.07.

Solution. Writing this as a product of ordinary fractions, we get

$$0.3 \times 0.07 = \frac{3}{10} \times \frac{7}{100} = \frac{3 \times 7}{10 \times 100} = \frac{21}{1000}.$$

The last fraction can be written $21 \div 10^3$, or 21×10^{-3} which, from the results of the last section, is equal to 0.021. Thus,

$$0.3 \times 0.07 = 0.021.$$

□

Multiplication of two decimals always involves a whole number multiplication for the numerator ($3 \times 7 = 21$ in the example) and a multiplication of powers of 10 for the denominator ($10 \times 100 = 10^1 \times 10^2 = 1000 = 10^3$ in the example). By looking back at the previous section, it is easy to see the product of two powers of 10 is itself a power of 10. Which power of 10? The rule is simple:

$$\boxed{10^n \times 10^m = 10^{(n+m)}}$$

This implies that the number of decimal places in the product of two (or more) decimals is the sum of the numbers of decimal places in the factors. Thus any set of decimals can be multiplied by following a two step procedure:

> To multiply two or more decimals:
>
> - Multiply the decimals as if they were whole numbers, ignoring the decimal points (this gives the numerator of the decimal fraction);
> - Add the number of decimal places in each factor (this gives the denominator of the decimal fraction by specifying the number of decimal places).

Example 112. Find the product 21.02×0.004.

Solution. Temporarily ignoring the decimal points, we multiply $2102 \times 4 = 8408$. 21.02 has two decimal places, and 0.004 has three. So the product will have $2 + 3 = 5$ decimal places. In other words,

$$21.02 \times 0.004 = 8408 \div 10^5 = 0.08408.$$

□

Example 113. Find the product of 12, 0.3, and 0.004.

Solution. Ignoring the decimal points, the product is $12 \times 3 \times 4 = 144$. The numbers of decimal places in the three decimals are, left to right, 0, 1 and 3, which add up to 4. Thus,

$$12 \times 0.3 \times 0.004 = 144 \div 10^4 = 0.0144.$$

□

3.7.1 Exercises

Find the following products.

1. 68.4×23
2. 804×6.2
3. 26.09×0.004
4. 4.09×93
5. 100×9.9
6. 14.093×6.39
7. 64.9×0.345
8. 0.0001×0.001
9. 1000×0.053
10. $6 \times 0.9 \times 0.02 \times 0.001$

3.8 Division of a decimal by a whole number

If the divisor is a whole number, the division procedure is almost exactly like the long division procedure for whole numbers. The only difference is this: a decimal point is placed where the quotient will go, vertically aligned with the decimal point in the dividend. For example, to set up the division $12.0342 \div 31$, we write, as before

$$\begin{array}{r|l} 3\,1 & \overline{1\,2\,.\,0\,3\,4\,2\quad} \end{array}$$

Then we put a decimal point vertically aligned with the decimal point in the divisor, where the quotient will go:

$$\begin{array}{rl} & \,. \\ 3\,1 & \big)\overline{1\,2\,.\,0\,3\,4\,2\quad} \end{array}$$

Now the long division is performed with no further attention paid to the decimal point. In this example, since 31 is bigger than 12, we put 0 in the units place of the quotient. Continuing, we estimate that 31 goes into 120 3 times, so 3 is placed in the next (tenths) place in the quotient, and 3×31 is subtracted from 120 to yield 27. (Note that the decimal point between 2 and 0 is ignored at this point.)

$$\begin{array}{rl} & 0\,.\,3 \\ 3\,1 & \big)\overline{1\,2\,.\,0\,3\,4\,2\quad} \\ & \underline{-\,9\ \ 3} \\ & \ \ 2\ \ 7 \end{array}$$

We bring down the next digit in the dividend, which is 3, and start over with a new dividend of 273. We estimate that 31 goes into 273 8 times. $8 \times 31 = 248$, which is subtracted from 273 to yield 25.

$$
\begin{array}{r|l}
 & 0.38 \\ \hline
31 & 12.0342 \\
 & -93 \\ \hline
 & 273 \\
 & -248 \\ \hline
 & 25
\end{array}
$$

We bring down the next digit in the dividend, which is 4, and start over with a new dividend of 254, etc. The remaining steps are as follows.

$$
\begin{array}{r|l}
 & 0.3882 \\ \hline
31 & 12.0342 \\
 & -93 \\ \hline
 & 273 \\
 & -248 \\ \hline
 & 254 \\
 & -248 \\ \hline
 & 62 \\
 & -62 \\ \hline
 & 0
\end{array}
$$

Thus, $12.0342 \div 31 = 0.3882$, with 0 remainder. We can check the result by multiplication, of course: $31 \times 0.3882 = 12.0342$, as it should.

In this example, the remainder was 0. In fact, because of the decimal point, we never need to write remainders again. The reason is that we can always adjoin insignificant 0's to to the end of the dividend, bring them down, and so continue the division process.

Example 114. Perform the division $12.4 \div 5$.

Solution. We set up the long division process as before, with a decimal point where the quotient will go, directly above the decimal point in the dividend.

$$
\begin{array}{r|l}
 & . \\ \hline
5 & 13.4
\end{array}
$$

After two steps, we arrive at

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & 2.6 \\ \cline{2-2}
5 & 13.4 \\
\multicolumn{1}{r}{} & -10 \\ \cline{2-2}
\multicolumn{1}{r}{} & 3\;4 \\
\multicolumn{1}{r}{} & -3\;0 \\ \cline{2-2}
\multicolumn{1}{r}{} & 4
\end{array}
$$

Now, instead of stopping with a remainder of 4, we adjoin an insignificant 0 to the dividend, bring it down, and continue the process:

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & 2.6 \\ \cline{2-2}
5 & 13.4\;0 \\
\multicolumn{1}{r}{} & -10 \\ \cline{2-2}
\multicolumn{1}{r}{} & 3\;4 \\
\multicolumn{1}{r}{} & -3\;0 \\ \cline{2-2}
\multicolumn{1}{r}{} & 4\;0
\end{array}
$$

5 goes into 40 exactly 8 times:

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & 2.6\;8 \\ \cline{2-2}
5 & 13.4\;0 \\
\multicolumn{1}{r}{} & -10 \\ \cline{2-2}
\multicolumn{1}{r}{} & 3\;4 \\
\multicolumn{1}{r}{} & -3\;0 \\ \cline{2-2}
\multicolumn{1}{r}{} & 4\;0 \\
\multicolumn{1}{r}{} & -\;\;4\;0 \\ \cline{2-2}
\multicolumn{1}{r}{} & 0
\end{array}
$$

The 0 remainder signals the end of the process. Thus $13.4 \div 5 = 2.68$. □

Since we can adjoin as many insignificant 0's as we need, it might seem that we can always continue until we get a 0 remainder. But this doesn't always happen. Sometimes, we never get a 0 remainder, but instead, we get a sequence of non-zero remainders that repeats, forever. In this case, we get what is know as a *repeating* decimal.

Example 115. Perform the division $3.2 \div 11$.

Solution. We set up the long division process as usual.

$$\begin{array}{rl} & \quad\;\; . \\ 11 & \overline{\big)\quad 3\,.\,2 \qquad\qquad} \end{array}$$

11 goes into 32 twice with a remainder of 10:

$$\begin{array}{rl} & \quad\;\; .\,2 \\ 11 & \overline{\big)\quad 3\,.\,2 \qquad\qquad} \\ & \underline{\;-\;\; 2\;\;\; 2 \qquad\qquad\quad} \\ & \qquad 1\;\;\; 0 \end{array}$$

Adjoin an insignificant 0 to the dividend, bring it down, and perform another step: 11 goes into 100 9 times with a remainder of 1.

$$\begin{array}{rl} & \quad\;\;\; .\,2\,9 \\ 11 & \overline{\big)\quad 3\,.\,2\,0 \qquad\quad} \\ & \underline{\;-\;\; 2\;\;\; 2 \qquad\qquad\quad} \\ & \qquad 1\;\;\; 0\,0 \\ & \underline{\quad -\qquad 9\,9 \qquad\quad} \\ & \qquad\qquad\;\; 1 \end{array}$$

Now if we adjoin another insignificant 0 to the dividend, and bring it down, we see that 11 goes into 10 0 times. We record that with a 0 digit in the quotient, adjoin another insignificant 0 to the dividend, and bring it down. Now we see that (as above) 11 goes into 100 9 times with a remainder of 1.

$$
\begin{array}{r|l}
 & \ .\,2\,9\,0\,9 \\
\hline
11 & \ 3\,.\,2\,0 \\
 & -\,2\ \ 2 \\
\hline
 & \ 1\ \ 0\,0 \\
 & -\ \ \ \ \ 9\,9 \\
\hline
 & \ \ \ \ \ \ 1\,0\,0 \\
 & -\ \ \ \ \ \ \ \ \ 9\,9 \\
\hline
 & \ \ \ \ \ \ \ \ \ \ \ 1
\end{array}
$$

It is evident that these last two steps will now repeat again and again, forever. The decimal will never terminate, since a 0 remainder will never occur. Still, it is easy to describe the quotient: it will consist of 29 after the decimal point, followed by an endless string of 09's. To indicate this, we put a bar over the repeated string:

$$0.2909090909090909\cdots = 0.29\overline{09}.$$

Thus, $3.2 \div 11 = 0.29\overline{09}$. □

We emphasize that it may take a while for the digits to start repeating, and the bar is only placed over the repeating part. For example, the repeating decimal

$$0.796812341234123412341234\ldots$$

is written, using the bar notation, as

$$0.7968\overline{1234}.$$

The repeated string can be quite long, or just a single digit. Here are some examples, which you should verify by carrying out the division process:

$$
\begin{aligned}
3 \div 7 &= 0.\overline{428571} \\
2 \div 11 &= 0.\overline{09} \\
5 \div 3 &= 1.\overline{6}
\end{aligned}
$$

Sometimes, we don't care whether a decimal terminates or repeats. For example, if we know in advance that we can round off our answer to a given decimal place, we carry out the division only as far as the right neighboring place and stop (this gives us all we need to know to round off).

Example 116. Suppose you bought 900 buttons for \$421. To the nearest cent, how much did you spend for each button?

Solution. We need to perform the division $421 \div 900$. Since we are rounding to the nearest *hundredth* (cent), we need only carry out the division as far as the *thousandths* place. Since the thousandths place is the the third place to the right of the decimal point, we adjoin three insignificant 0's to the dividend, setting up the division as $421.000 \div 900$. We omit the details, but you can verify that the quotient, carried out to three decimal places, is 0.467, which, rounded to the nearest hundredth, is 0.47. Therefore, the buttons cost approximately 47 cents each.

If you carried out the division process a little further, you noticed that the quotient is actually a repeating decimal:

$$421 \div 900 = 0.46777777\cdots = 0.46\overline{7}.$$

□

3.8.1 Exercises

Perform the indicated divisions. Use the bar notation for repeating decimals. If you round off, indicate to what place.

1. $91 \div 20$
2. $14.68 \div 5$
3. $23 \div 90$
4. $86 \div 71$
5. $6.02 \div 9$
6. $4.19 \div 13$
7. $804.09 \div 215$
8. $353 \div 37$
9. $17 \div 19$
10. $38.8 \div 40$

3.9 Division of a decimal by a decimal

What do we do when the divisor is a decimal? For example, how do we perform the division $8.61 \div 2.5$? The answer is simple. Just multiply both the dividend and divisor by a power of 10 sufficiently large to make the divisor into a whole number, and then proceed as before. In this example, if we multiply both numbers by 10^1, the divisor 2.5 turns into the whole number 25 and the dividend turns into 86.1. Recall that two fractions are *equivalent*, that is, represent the same number, if one is obtained from the other by multiplying both numerator and denominator by the same nonzero number. In this case, the convenient choice of nonzero multiplier is $10^1 = 10$, because it turns the denominator (divisor) into a whole number. Thus,

$$\frac{8.61}{2.5} = \frac{8.61 \times 10}{2.5 \times 10} = \frac{86.1}{25}.$$

In other words, the division problem $8.61 \div 2.5$ has the same quotient as the division problem $86.1 \div 25$. We do the latter problem exactly as in the previous section.

$$\begin{array}{r|rrrrrr}
 & & 3 & . & 4 & 4 & 4 \\ \cline{2-7}
25 & 8 & 6 & . & 1 & 0 & 0 \\
 & -7 & 5 & & & & \\ \cline{2-7}
 & 1 & 1 & & 1 & & \\
 & -1 & 0 & & 0 & & \\ \cline{2-7}
 & & 1 & & 1 & 0 & \\
 & & -1 & & 0 & 0 & \\ \cline{2-7}
 & & & & 1 & 0 & 0 \\
 & & & & -1 & 0 & 0 \\ \cline{2-7}
 & & & & & & 0
\end{array}$$

Since the remainder is 0, the quotient is a terminating decimal. $8.61 \div 2.5 = 3.444$. (Question: why didn't we write $3.\overline{4}$?)

Example 117. Perform the division $32.067 \div 6.41$ and round off the quotient to the nearest hundredth.

Solution. There are two decimal places in the divisor, 6.41, so multiplying it by $10^2 = 100$ will give us a whole number: $6.41 \times 10^2 = 641$. Doing the same to the dividend yields the equivalent division problem,

$$3206.7 \div 641.$$

Since we are rounding off to the nearest hundredth, we will need to carry out the division to the thousandths place. We therefore add two insignificant 0's to the dividend, and perform the division as follows:

$$\begin{array}{rr} & 5.002 \\ 641 & \overline{)\,3206.700} \\ - & 3205 \\ \hline & 1700 \\ - & 1282 \\ \hline & 418 \end{array}$$

Since the digit in the thousandths place of the quotient is less than 5, we preserve the digit in the hundredths place. The quotient is ≈ 5.00 (to the nearest hundredth). □

3.9.1 Exercises

Write each division as an equivalent division with a whole number divisor.

1. $680.4 \div 1.01$
2. $0.48 \div 11.4$
3. $804.09 \div 9.18$
4. $9.8 \div 0.0215$

Perform the divisions.

5. $4.19 \div 0.5$
6. $4.09 \div 0.21$
7. $353 \div 2.5$
8. $0.004 \div 0.002$
9. $29.997 \div 0.01$

3.10 Percents, Conversions

A *percent* is a fraction in which the denominator is 100. The word "percent" comes from the Latin phrase *per centum* meaning "out of 100," and is symbolized by %. For example,

$$97\% \quad \text{means} \quad \frac{97}{100} \quad \text{or} \quad 0.97.$$

Percents need not be whole numbers, and they need not represent proper fractions. For example,

$$150\% = \frac{150}{100} = 1.5$$
$$0.5\% = \frac{0.5}{100} = 0.005$$

Fractions, decimals, and percents represent the same quantities in different ways, and we need to know how to convert one to another.

A decimal can be converted to a percent by moving the decimal point two places to the right and adjoining the % symbol. This is the same as multiplying the decimal fraction by 100, which exactly cancels the denominator, leaving just the numerator (the percent). Thus

$$\begin{aligned} 0.68 &= 68\% \\ 2.05 &= 205\% \\ 0.708 &= 70.8\% \\ 1.4 &= 140\% \\ 0.0067 &= 0.67\% \end{aligned}$$

To convert a percent back into a decimal, we simply divide by 100, which, as we know, is equivalent to moving the decimal point two places to the left. Thus

$$\begin{aligned} 92\% &= 0.92 \\ 0.2\% &= 0.002 \\ 138\% &= 1.38 \\ 71.02\% &= 0.7102 \end{aligned}$$

To convert a percent to a fraction (or mixed number), first convert the percent to a decimal, as above, then express the decimal as a fraction (with a visible denominator), and finally, reduce the fraction to lowest terms, if needed.

Example 118. Convert 10.8% into a fraction in lowest terms.

Solution. We first write 10.8% as a decimal.

$$10.8\% = 0.108.$$

Then, we write the decimal as a fraction with a visible denominator. In this case, because there are 3 decimal places, the denominator is 1000.

$$0.108 = \frac{108}{1000}.$$

Finally, the GCF of the numerator and denominator is 4, so we cancel 4 from both of these numbers, obtaining a fraction in lowest terms:

$$\frac{108}{1000} = \frac{\cancel{108}^{\,27}}{\cancel{1000}_{\,250}} = \frac{27}{250}.$$

□

Here are some common percents with their decimal and fractional equivalents (in lowest terms):

$$10\% = 0.1 = \frac{1}{10}, \quad 20\% = 0.2 = \frac{1}{5}, \quad 25\% = 0.25 = \frac{1}{4}, \quad 50\% = 0.5 = \frac{1}{2}.$$

To convert a fraction to a decimal, we just perform long division.

Example 119. Convert $\frac{1}{4}$ to a decimal.

Solution. Performing the division $1 \div 4$, we obtain

$$\begin{array}{r|l} & 0.25 \\ \hline 4 & 1.00 \\ & -\ 8 \\ \hline & \ \ 20 \\ & -\ 20 \\ \hline & \ \ \ 0 \end{array}$$

Thus $\frac{1}{4} = 0.25$. □

To convert a fraction to a percent, first convert it to a decimal, and then convert the decimal to a percent.

Example 120. Convert $\frac{1}{40}$ to a percent.

Solution. Performing the division $1 \div 40$, we obtain 0.025 (verify this – it is quite similar to the last example). Then, we convert 0.025 to a percent by multiplying by 100 (equivalently, by moving the decimal point two places to the right). We obtain

$$\frac{1}{40} = 2.5\%.$$

□

3.10.1 Exercises

Convert the following percents to decimals.

1. 43%
2. 608%

3. 56.04%
4. 4.09%

Convert the following decimals to percents.

5. 14.09
6. 0.00679
7. 1.384
8. 0.384

Convert the following decimals or percents to fractions (or mixed numbers) in lowest terms.

9. 44%
10. 2%
11. 0.15
12. 0.25
13. 40%
14. 5%
15. 98%
16. 7.2%
17. 18%

Convert the following fractions, first to decimals, then to percents. Round percents to the nearest tenth of a percent.

18. $\frac{1}{8}$
19. $\frac{1}{6}$
20. $\frac{2}{5}$
21. $\frac{3}{8}$
22. $\frac{3}{4}$
23. $\frac{1}{13}$
24. $\frac{1}{12}$
25. $\frac{5}{12}$

3.11 Fractional parts of numbers

We now have several ways of indicating a fractional part of a number. For example, the phrases

$$\begin{array}{ccc} \frac{1}{4} & \text{of} & 90 \\ 0.25 & \text{of} & 90 \\ 25\% & \text{of} & 90 \end{array}$$

are all different ways of describing the number $\frac{1}{4} \times 90 = 22.5$. The word "of" indicates multiplication. In the case of percent, the multiplication is done after first converting the percent to a fraction or a decimal.

The fractional part taken need not be a *proper* fractional part. That is, we could end up with more than we started with.

Example 121. Find 125% of 500.

Solution. Converting the percent to the decimal 1.25, and then multiplying by 500, we get

$$1.25 \times 500 = 625.$$

□

Example 122. Sales tax in New York State is $8\frac{1}{4}\%$. What is the sales tax on a shirt priced at \$25?

Solution. $8\frac{1}{4} = 8.25$, and 8.25%, as a decimal, is .0825. So the sales tax on a \$25 shirt is

$$.0825 \times 25 = 2.0625 \approx \$2.06.$$

Note that we rounded off to the nearest cent. □

Example 123. Josh spends $\frac{2}{5}$ of his income on rent. If he earns \$1250 per month, how much does he have left over, after paying his rent?

Solution. The left over part is $\frac{3}{5}$, since $1 - \frac{2}{5} = \frac{5}{5} - \frac{2}{5} = \frac{3}{5}$. So he has

$$\frac{3}{5} \times 1250 = \frac{3}{\cancelto{1}{5}} \times \frac{\cancelto{250}{1250}}{1} = \$750$$

left over after paying rent. □

3.11.1 Exercises

1. Find 16% of 75
2. Find $\frac{3}{8}$ of 60
3. Find .05 of 280
4. Find 150% of 105
5. Find $\frac{1}{2}\%$ of 248

6. A \$500 television is being sold at a 15% discount. What is the sale price?
7. Angela gets a 5% raise. Her original salary was \$36,000 per year. What is her new salary?
8. A car loses $\frac{2}{5}$ of its value over a period of years. If the car originally sold for \$12,500, what it would it sell for now?
9. On a test, Jose answers $\frac{7}{8}$ of the problems correctly. If there were 24 problems on the test, how many did he get wrong?
10. Medical expenses can be deducted from a person's income tax if they exceed 2% of total income. If Maribel's medical expenses were \$550, and her total income was \$28,000, can she deduct her medical expenses?

Chapter 4

Ratio and Proportion

In this chapter we develop another interpretation of fractions, as comparisons between two quantities.

4.1 Ratio

If a team wins ten games and loses five, we say that the **ratio** of wins to losses is 2 : 1, or "2 to 1." Where did the numbers 2 : 1 come from? We simply made a fraction whose numerator is the number of games the team won, and whose denominator is the number of games they lost, and reduced it to lowest terms: $\frac{\text{wins}}{\text{losses}} = \frac{10}{5} = \frac{2}{1}$.

We can do the same for any two quantities, a and b, as long as $b \neq 0$.

> The ratio of a to b $(b \neq 0)$ is the fraction $\dfrac{a}{b}$, reduced to lowest terms.

Example 124. Find (a) the ratio of 9 to 18, (b) the ratio of 21 to 12, (c) the ratio of 64 to 4.

Solution. (a) The ratio of 9 to 18 is

$$\frac{9}{18} = \frac{1}{2} \quad \text{or} \quad 1:2.$$

(b) The ratio of 21 to 12 is

$$\frac{21}{12} = \frac{7}{4} \quad \text{or} \quad 7:4.$$

(c) The ratio of 64 to 4 is

$$\frac{64}{4} = \frac{16}{1} \quad \text{or} \quad 16:1.$$

□

Notice that in (c) we left the denominator 1 (rather than just writing 16) to maintain the idea of a *comparison* between two numbers.

We can form ratios of non-whole numbers. When the ratio is expressed in lowest terms, however, it is always the ratio of two whole numbers, as small as possible. This is the main point of a ratio comparison: if the given two numbers were *small whole numbers*, how would they compare?

Example 125. What is the ratio of $3\frac{3}{4}$ to $1\frac{1}{2}$?

Solution. Converting the mixed numbers to improper fractions, and rewriting the division as multiplication by the reciprocal, we obtain

$$3\frac{3}{4} \div 1\frac{1}{2} = \frac{15}{4} \times \frac{2}{3} = \frac{\cancelto{5}{15}}{\cancelto{2}{4}} \times \frac{\cancelto{1}{2}}{\cancelto{1}{3}} = \frac{5}{2}.$$

The ratio is 5 : 2. □

Example 126. What is the ratio of 22.5 to 15?

Solution. We could perform the decimal division $22.5 \div 15$, but it is easier to simplify the equivalent fraction

$$\frac{225}{150} = \frac{3}{2}.$$

The ratio is 3 : 2. □

When finding the ratio of two measurable quantities, we must be sure the quantities are expressed in the *same units*. Otherwise the ratio will be a skewed or false comparison.

Example 127. Find the ratio of 15 dollars to 30 cents.

Solution. If we form the ratio $\frac{15}{30} = \frac{1}{2}$, we imply that a person walking around with \$15 in his pocket has half as much money as a person walking around with only 30 cents! The proper comparison is obtained by converting both numbers to the same units. In this case, we convert dollars to cents. The correct ratio is

$$\frac{1500 \text{ cents}}{30 \text{ cents}} = \frac{50}{1}.$$

□

Example 128. Manuel gets a five minute break for every hour he works. What is the ratio of work time to break time?

Solution. Since 1 hour = 60 minutes, the ratio of work time to break time is

$$\frac{55 \text{ minutes}}{5 \text{ minutes}} = \frac{11}{1}.$$

□

Recall that percent (%) means "per hundred." So a percent can be viewed as a ratio whose second term (denominator) is 100.

Example 129. At a certain community college, 55% of the students are female. Find (a) the ratio of female students to the total number of students, and (b) the ratio of female students to male students.

Solution. 55% indicates the fraction

$$\frac{55}{100} = \frac{11}{20},$$

so the ratio of female students to the total number of students is 11 : 20. For (b), we first deduce that 45% of the students are male (since 45% = 100% − 55%). So the ratio of female students to male students is

$$\frac{55}{45} = \frac{11}{9} \quad \text{or} \quad 11 : 9.$$

□

Example 130. A printed page $8\frac{1}{2}$ inches in width has a margin $\frac{5}{8}$ inches wide on either side. Text is printed between the margins. What is the ratio of the width of the printed text to the total margin width?

Solution. The total margin width is $\frac{5}{8} + \frac{5}{8} = \frac{5}{4} = 1\frac{1}{4}$ inches (taking into account both left and right margins.) The width of the printed text is the difference

$$\text{total page width} - \text{total margin width} = 8\frac{1}{2} - 1\frac{1}{4} = 7\frac{1}{4} \text{ inches.}$$

The ratio of the width of the printed text to total margin width is

$$7\frac{1}{4} \div 1\frac{1}{4} = \frac{29}{4} \div \frac{5}{4} = \frac{29}{4} \cdot \frac{4}{5} = \frac{29}{5},$$

or 29 : 5. ☐

4.1.1 Exercises

Find the ratios.

1. 14 to 4
2. 30 to 32
3. 56 to 21
4. $1\frac{5}{8}$ to $3\frac{1}{4}$
5. $2\frac{1}{12}$ to $1\frac{1}{4}$
6. 14.4 to 5.4
7. 1.69 to 2.6
8. 3 hours to 40 minutes
9. 8 inches to $5\frac{1}{2}$ feet
10. In the late afternoon, a 35 foot tree casts an 84 foot shadow. What is the ratio of the tree's height to the shadow's length?

4.2 Proportions

A **proportion** is a statement that two ratios are equal. Thus

$$\frac{40}{20} = \frac{10}{5}$$

is a proportion, because both ratios are equivalent to the ratio 2 : 1 (= the fraction $\frac{2}{1}$).

A proportion is a statement of the form

$$\frac{a}{b} = \frac{c}{d}$$

where $b, d \neq 0$.

4.2.1 The cross-product property

There is a very useful fact about proportions: If the proportion $\frac{a}{b} = \frac{c}{d}$ is true, then the "cross-products," ad and bc, are equal, and, conversely, if the cross-products are equal, then the proportion must be true.

Example 131. Verify that the cross-products are equal in the (true) proportion

$$\frac{40}{20} = \frac{10}{5}.$$

Solution. $40 \times 5 = 10 \times 20 = 200$. □

The general cross-product property is stated below for reference:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

To prove the cross-product property, we need to say a little bit about **equations**. An equation is a mathematical statement of the form $X = Y$. If the equation $X = Y$ is true, and N is any nonzero number, then the following equations are also true, and have exactly the same solution(s):

$$N \times X = N \times Y \qquad \text{and} \qquad \frac{X}{N} = \frac{Y}{N}.$$

Now go back to the proportion $\dfrac{a}{b} = \dfrac{c}{d}$, and multiply both sides by the nonzero number bd. We get

$$bd\frac{a}{b} = bd\frac{c}{d}.$$

Cancelling b on the left and d on the right yields the cross-product property, $ad = bc$. Conversely, if we have four numbers a, b, c, d, $(b,\ d \neq 0)$, and we know that $ad = bc$, then, dividing both sides by the nonzero number bd gives us the proportion $\frac{a}{b} = \frac{c}{d}$.

Example 132. Decide if the given proportions are true or false, using the cross-product property. (a) $\dfrac{3}{11} = \dfrac{2}{7}$; (b) $\dfrac{4}{10} = \dfrac{6}{15}$.

Solution. (a) The cross-products are $3 \cdot 7 = 21$ and $2 \cdot 11 = 22$. They are not equal, so the proportion is false. (b) The cross-products are $4 \cdot 15$ and $10 \cdot 6$, both equal to 60. So the proportion is true. □

4.2.2 Solving a proportion

If one of the four terms of a proportion is missing or unknown, it can be found using the cross-product property. This procedure is called **solving** the proportion. In the proportion below, x represents an unknown term (any other letter would do).

$$\frac{x}{3} = \frac{34}{51}.$$

There is a unique x which makes the proportion true, namely, the one which makes the cross-products, $51x$ and $34(3)$, equal. The equation

$$51x = 34(3) = 102$$

can be divided by 51 (the number which multiplies x) on both sides, giving

$$\frac{51x}{51} = \frac{102}{51}.$$

Cancellation yields

$$\frac{\cancel{51}^{1}x}{\cancel{51}^{1}} = \frac{\cancel{102}^{2}}{\cancel{51}^{1}} = 2.$$

It follows that

$$x = 2$$

which is the solution of the proportion.

It doesn't matter which of the four terms is missing; the proportion can always be solved by a similar procedure.

Example 133. Solve the proportion

$$\frac{9}{100} = \frac{36}{y}$$

for the unknown term y.

Solution. The cross-products must be equal, so

$$9y = 3600.$$

Dividing both sides of the equation by 9 (the number which multiplies y), and cancelling, yields

$$\frac{\cancel{9}^{1}y}{\cancel{9}^{1}} = \frac{\cancel{3600}^{400}}{\cancel{9}^{1}}$$

$$y = 400.$$

The unknown term is 400. We check our solution by verifying that the cross-products

$$9(400) = 36(100) = 3600$$

are equal. □

Neither the terms nor the solution of a proportion are necessarily whole numbers.

Example 134. Solve the proportion

$$\frac{10}{B} = \frac{15}{4}$$

for B and check the solution.

Solution. Setting the cross products equal,

$$40 = 15B.$$

Dividing both sides of the equation by 15 (the number multiplying B), and simplifying, yields

$$\begin{aligned} \frac{\cancelto{8}{40}}{\cancelto{3}{15}} &= \frac{15B}{15} \\ \frac{8}{3} &= B. \end{aligned}$$

The unknown term is $\frac{8}{3}$ or $2\frac{2}{3}$.

Substituting $2\frac{2}{3}$ for B in the original proportion, we check that the cross-products are equal.

$$\begin{aligned} 10(4) &\stackrel{?}{=} \left(2\frac{2}{3}\right)(15) \\ 40 &\stackrel{?}{=} \frac{8}{\cancelto{1}{3}}\,\frac{\cancelto{5}{15}}{1} \\ 40 &= 40. \end{aligned}$$

$\square$

Example 135. Solve the proportion

$$\frac{\left(\frac{2}{3}\right)}{5} = \frac{x}{15}.$$

Solution. Set the cross-products equal, and solve for x:

$$\begin{aligned} \frac{2}{3} \cdot 15 &= 5x \\ \frac{2}{\cancelto{1}{3}} \cdot \cancelto{5}{15} &= 5x \\ 10 &= 5x \\ 2 &= x. \end{aligned}$$

$\square$

Example 136. Solve the proportion

$$\frac{42}{70} = \frac{x}{1.5}.$$

Solution. We could immediately set the cross-products equal, but it is simpler to first reduce the fraction $\frac{42}{70}$ to its lowest terms, using – by the way– a proportion: $\frac{42}{70} = \frac{3}{5}$. Stringing two proportions together

$$\frac{3}{5} = \frac{42}{70} = \frac{x}{1.5}$$

lets us skip over the middle fraction. It is evident that the solution to our original proportion is the solution to the simpler proportion

$$\frac{3}{5} = \frac{x}{1.5}.$$

Setting the cross-products equal

$$4.5 = 5x,$$

we obtain the solution $x = \frac{4.5}{5} = 0.9$.

□

We summarize the procedure for solving a proportion.

To solve a proportion,

1. Reduce the numerical ratio (not containing the unknown) to lowest terms, if necessary;
2. Set the cross-products equal;
3. Divide both sides of the resulting equation by the number multiplying the unknown term.

To check the solution,

1. In the original proportion, replace the unknown term with the solution you obtained;
2. Verify that the cross-products are equal.

Equivalence of fractions, and hence, most of fraction arithmetic, is based on proportion. To add

$$\frac{1}{2} + \frac{1}{3}$$

for example, we first solve the two proportions

$$\frac{1}{2} = \frac{x}{6} \qquad \text{and} \qquad \frac{1}{3} = \frac{y}{6}$$

(a task we have performed up to now without much comment) so that the fractions can be written with the LCD (6). Then

$$\frac{1}{2} + \frac{1}{3} = \frac{x+y}{6}.$$

Practice: solve the two proportions for x and y.

4.2.3 Exercises

Solve the following proportions.

1. $\frac{1}{5} = \frac{3}{x}$
2. $\frac{15}{y} = \frac{2}{3}$
3. $\frac{100}{5} = \frac{20}{y}$
4. $\frac{A}{9} = \frac{5}{3}$
5. $\frac{11}{B} = \frac{1}{2}$
6. $\frac{5}{3} = \frac{c}{6}$
7. $\frac{s}{3} = \frac{4}{13}$
8. $\frac{1.2}{7} = \frac{A}{0.84}$
9. $\frac{P}{100} = \frac{75}{125}$
10. $\frac{3\frac{1}{5}}{x} = \frac{4}{2\frac{1}{2}}$

4.3 Percent problems

Any problem involving percent can be stated (or restated) in the form

"A is P percent of B"

where one of the numbers A, B or P is unknown. We can make this into a mathematical equation by making the following "translations:"

$$\begin{array}{ccc} \text{"is"} & \longleftrightarrow & = \\ \text{"}P\text{ percent"} & \longleftrightarrow & \dfrac{P}{100} \\ \text{"of"} & \longleftrightarrow & \times \end{array}$$

This gives us the equation

$$A = \frac{P}{100} \times B.$$

If we divide both sides of the equation by B, we obtain the proportion in the box below.

The statement "A is P percent of B" is equivalent to the proportion

$$\frac{A}{B} = \frac{P}{100}.$$

The letter B is used to suggest "base amount" – that is, B is the amount *from which* (or *of which*) a percentage is taken. Notice that B follows "of" in the verbal statement. It could be that A (the percentage taken) is greater than B, but only if the percentage P is greater than 100.

In the following examples, the phrase involving "what," as in "what number?" or "what percent?" helps determine which of A, B of P is the unknown.

Example 137. 6 is what percent of 300?

Solution. The percent is unknown, and the base amount B is 300 (since it follows the word "of"). We can restate the question as: "6 is P percent of 300." The corresponding proportion is

$$\frac{6}{300} = \frac{P}{100}$$

and the solution (verify it!) is $P = 2$. Therefore, 6 is 2% of 300. □

Example 138. What number is 8% of 150?

Solution. The base amount B is 150, since it follows the word "of," the percentage is $P = 8$, and therefore A is the unknown amount. The question can be restated as: "A is 8% of 150." The corresponding proportion is

$$\frac{A}{150} = \frac{8}{100}$$

and the solution is $A = 12$ (verify!). It follows that 12 is 8% of 150.

(Note that this result could have been obtained without using a proportion, since 8% of 150 simply means $.08 \times 150 = 12$.) □

Example 139. 58% of what number is 290?

Solution. The base amount B is unknown, and A is 290. The question can be restated as "58% of B is 290." The corresponding proportion is

$$\frac{290}{B} = \frac{58}{100}$$

and the solution is $\frac{29000}{58} = 500$. So 58% of 500 is 290. □

Everyday questions involving percent are not always as straightforward as in the previous examples. But with a little thought, they can be converted into such statements.

Example 140. Approximately 55% of the students enrolled at BCC are female. If there are 2970 female students, what is the total enrollment at BCC?

Solution. According to the given information, 2970 is 55% of the total enrollment, which is the base amount, B. The question can be restated as "2970 is 55% of B" and the corresponding proportion is therefore

$$\frac{2970}{B} = \frac{55}{100}.$$

Using lowest terms of the fraction $\frac{55}{100} = \frac{11}{20}$, we solve

$$\begin{aligned}\frac{2970}{B} &= \frac{11}{20}\\ 59400 &= 11B\\ B &= \frac{59400}{11} = 5400.\end{aligned}$$

The total enrollment is 5400 students. To check, verify that 55% of 5400 is 2970. □

Example 141. Full-time tuition at a university increased from \$2,850 to \$3,000. What was the percent increase in the tuition? (Round to the nearest tenth of a percent.)

Solution. The tuition increased by $\$3000 - 2850 = \$150$. The percent increase is the ratio

$$\frac{\text{amount of increase}}{\text{original tuition}} = \frac{150}{2850},$$

expressed as a percent. Equivalently, we want to answer the question "\$150 is what percent of \$2850?" We solve the proportion

$$\begin{aligned}\frac{150}{2850} &= \frac{P}{100} \qquad (4.1)\\ \frac{3}{57} &= \frac{P}{100}\\ 57P &= 300\\ P &\approx 5.26\end{aligned}$$

Rounded to the nearest tenth of a percent, the tuition increase was approximately 5.3%. □

4.3.1 Exercises

1. 12 is 20% of what number?
2. 12 is 30% of what number?
3. 12 is 40% of what number?
4. 90 is what percent of 225?
5. 90 is what percent of 300?
6. 90 is what percent of 375?
7. What is 125% of 600?
8. What is 175% of 600?

9. 250 is what percent of 325? (Round to the nearest tenth of a percent.)
10. 108 is 80% of what number?
11. A baseball team won 93 games, or 62% of the games it played. How many games did the team play?
12. New York State sales tax is 8.25%. If the sales tax on a DVD player is \$16.50, what is the (before-tax) price of the player?
13. Marina's annual salary last year was \$56,000. This year she received a raise of \$4,480. By what percent did her salary increase?
14. A town's population decreased from 13,000 to 12,220. By what percent did the population decrease?

4.4 Rates

There are lots of real-world quantities which compare in a fixed ratio. For example, for any given car, the ratio of miles driven to gallons of gas used,

$$\frac{\text{miles}}{\text{gallon}} \qquad \text{or} \qquad \text{“miles per gallon”}$$

is essentially unchanging, or fixed. If we know, say, that 5 gallons of gas was needed to drive 150 miles, we can predict the amount that will be needed to drive any other distance, by solving a simple proportion.

Example 142. Maya used 5 gallons of gas to drive 150 miles. How many gallons will she need to drive 225 miles?

Solution. We solve the proportion

$$\frac{150 \text{ miles}}{5 \text{ gallon}} = \frac{225 \text{ miles}}{x \text{ gallons}},$$

where x represents the number of gallons she will need. Before setting the cross-products equal, reduce the fraction on the left side to $\frac{30}{1}$. Then

$$\begin{aligned} 30x &= 225 \\ x &= \frac{225}{30} = 7\frac{1}{2}. \end{aligned} \tag{4.2}$$

She will need $7\frac{1}{2}$ gallons of gas to drive 225 miles. □

Miles per gallon is an example of a **rate**, or comparison of unlike quantities by means of a ratio. In the example above, the miles per gallon rate for Maya's car was

$$\frac{150 \text{ miles}}{5 \text{ gallons}} = \frac{150}{5} = \frac{30}{1} \qquad \text{or 30 miles per gallon.}$$

Other examples of rates are: dollars per hour (pay rate for an hourly worker), dollars per item (price of an item for sale), calories per minute (energy use by an athlete). You can undoubtedly think of many others.

Example 143. A runner burns 375 calories running 3.5 miles. (a) How many calories will she burn in running a marathon (approximately 26 miles)? (b) What is her rate of energy use (calories per mile)? Round off to the nearest whole unit.

Solution. (a) The ratio

$$\frac{\text{calories}}{\text{miles}}$$

is assumed fixed and, from the given information, is equal to

$$\frac{375}{3.5}.$$

Let x denote the number of calories the runner burns in running a marathon. Then

$$\begin{aligned} \frac{375}{3.5} &= \frac{x \text{ calories}}{26 \text{ miles}} \qquad \text{(setting cross-products equal)} \\ (375)(26) &= 3.5x \\ x &= \frac{(375)(26)}{3.5} \approx 2786. \end{aligned}$$

She will burn approximately 2786 calories running the marathon. (b) Her rate of energy use is

$$\frac{375 \text{ calories}}{3.5 \text{ miles}} \approx 107 \quad \text{calories per mile.}$$

□

Example 144. A painting crew can paint three apartments in a week. If a building contains 40 equal size apartments, how long will it take the crew to paint all the apartments?

Solution. The ratio

$$\frac{\text{apartments painted}}{\text{time in weeks}}$$

is assumed fixed at $\frac{3}{1}$. If y is the number of weeks needed to paint 40 apartments, then

$$\frac{3}{1} = \frac{40}{y},$$

and $y = \frac{40}{3} = 13\frac{1}{3}$ weeks. □

4.4.1 Exercises

Use proportions to solve the following problems.

1. On a map, $\frac{3}{4}$ inch represents 14 miles. If two cities are 42 miles apart, how far apart are they on the map?

2. A truck burns $2\frac{1}{2}$ quarts of oil on an 1800 mile trip. How many quarts will be burned on a cross-country trip of 3240 miles?

3. An investment of $\$2,000$ earns $\$48$ in interest over a year. How much would need to be invested to earn $\$200$ in interest? (Round to the nearest dollar.)

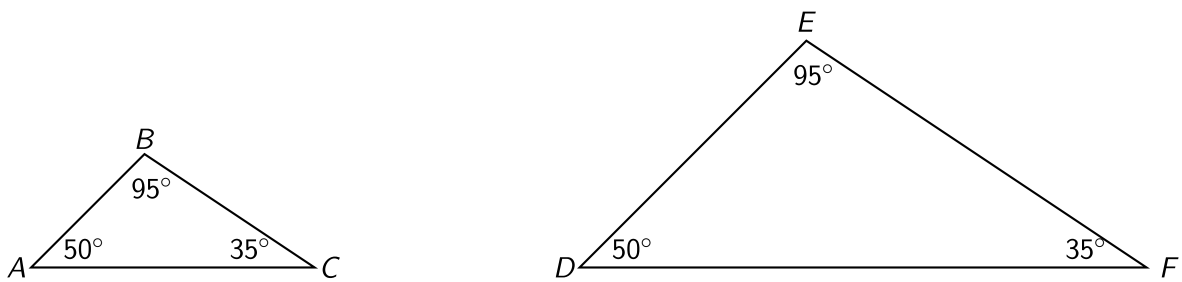
4. In a sample of 600 bottles, 11 were found to be leaking. Approximately how many bottles would you expect to be leaking in a sample of $20,000$ bottles?

5. The ratio of the weight of lead to the weight of an equal volume of aluminum is 21 : 5. If a bar of aluminum weighs 15 pounds, how much would a bar of lead of the same size weigh?

4.5 Similar triangles

Two triangles which have the same shape but possibly different sizes are called **similar**. Having the same shape means that the three angles of one triangle are equal to the three corresponding angles in the other.

The triangles below are similar, because the angle at A in the small triangle is equal to the angle at D in the big triangle ($50°$); the angle at B in the small triangle is equal to the angle at E in the big triangle ($95°$); and similarly the angle at C in the small triangle is equal to the angle at F in the large triangle ($35°$).



Given any triangle, we can obtain a similar one by enlarging (or reducing) all the side lengths in a fixed ratio. In the example above, the bigger triangle was obtained from the smaller using the enlargement ratio 2 : 1. If $\overline{AB}$ stands for the length of the side AB, etc., then

$$\overline{DE} = 2 \times \overline{AB}, \qquad \overline{EF} = 2 \times \overline{BC}, \qquad \text{and} \quad \overline{DF} = 2 \times \overline{AC}.$$

Equivalently,

$$\frac{\overline{DE}}{\overline{AB}} = \frac{\overline{EF}}{\overline{BC}} = \frac{\overline{DF}}{\overline{AC}} = \frac{2}{1}.$$

In words: sides which are opposite equal angles (called **corresponding sides**) have length ratio equal to the ratio 2 : 1. We must be careful to compare side lengths in the proper order. In this case, we used the order larger : smaller, since that is the order in the ratio 2 : 1. We could have used the reverse order, smaller : larger, but only if we also used the reversed ratio 1 : 2.

If we know, say, that $\overline{BC}$ is 12 feet, we can solve the proportion

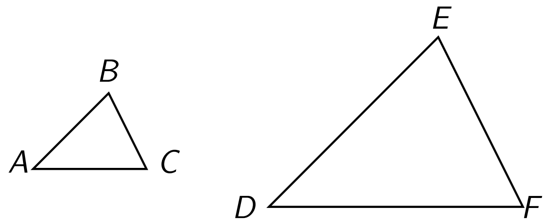
$$\frac{2}{1} = \frac{\overline{EF}}{12}$$

and determine that $\overline{EF}$ is 24 feet.

This is the key fact about similar triangles.

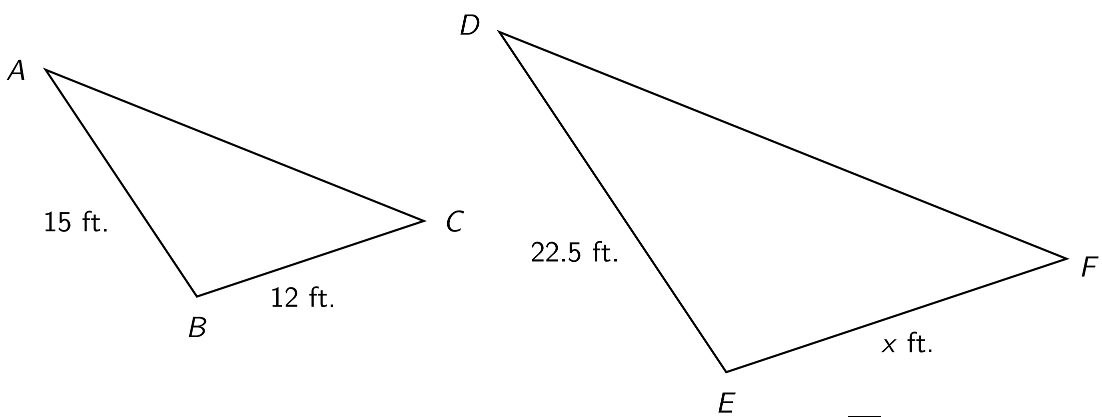
Corresponding sides of similar triangles are proportional:

If: $\angle A = \angle D, \quad \angle B = \angle E, \quad \text{and} \quad \angle C = \angle F,$



$$\text{Then: } \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}} = \frac{\overline{AB}}{\overline{DE}}.$$

Example 145. The triangles below are similar, with $\angle D = \angle A$, $\angle E = \angle B$ and $\angle F = \angle C$. $\overline{AB} =$ 15 feet, $\overline{BC} = 12$ feet, and $\overline{DE} = 22.5$ feet. Find $\overline{EF}$.



Solution. The ratios of the corresponding sides opposite $\angle F = \angle C$ ($\frac{\overline{DE}}{\overline{AB}}$) and the corresponding sides opposite $\angle D = \angle A$ ($\frac{\overline{EF}}{\overline{BC}}$) are equal:

$$\frac{\overline{DE}}{\overline{AB}} = \frac{\overline{EF}}{\overline{BC}}.$$

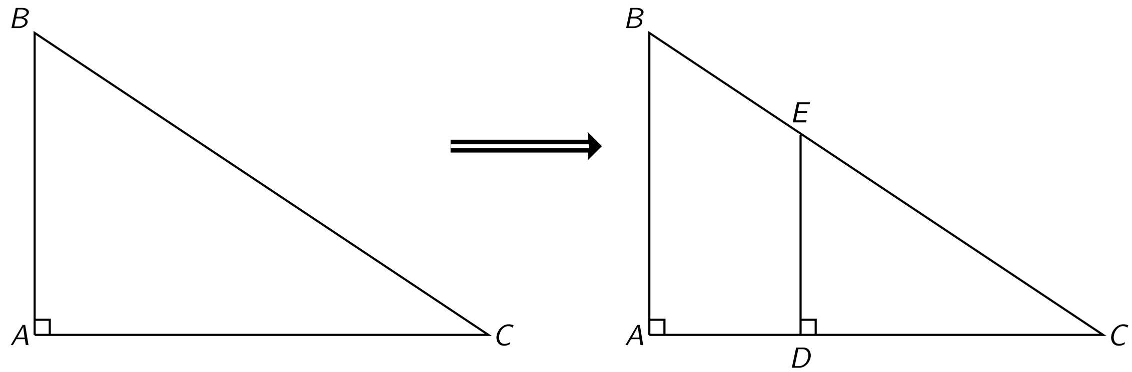
Filling in the given information, and using x to represent the unknown side length $\overline{EF}$, we have

$$\frac{22.5}{15} = \frac{x}{12}$$
$$(22.5)(12) = 15x$$
$$270 = 15x$$
$$x = \frac{270}{15} = 18.$$

$\overline{EF} = 18$ feet. □

We now describe two ways of obtaining a pair of similar triangles. Both involve the notion of **parallel line segments**. Two line segments are parallel if they do not cross, even if extended infinitely in either direction (think of straight train tracks).

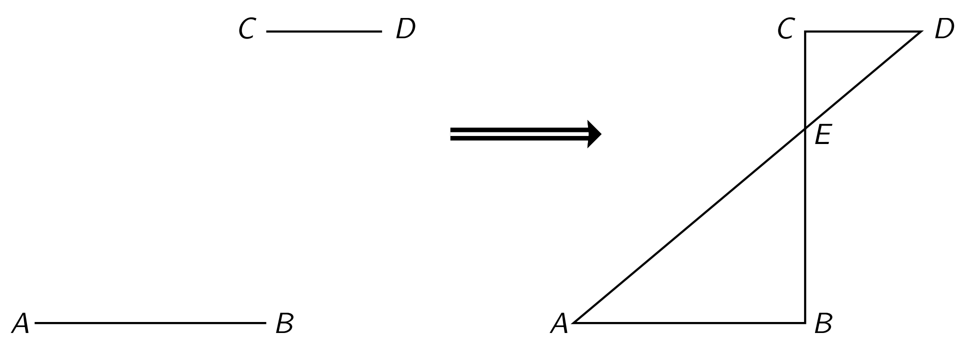
1. If two sides of a triangle are joined by a line segment *parallel to the third side*, the resulting (smaller) triangle is similar to the original triangle.



Starting with triangle ABC on the left, we drew the segment $\overline{DE}$, parallel to $\overline{AB}$, creating (smaller) triangle DEC, similar to ABC. Both triangles are visible on the right (the smaller overlapping the larger), so there was really no need to draw the first picture. By similarity, the corresponding sides are proportional:

$$\frac{\text{smaller triangle}}{\text{larger triangle}} \qquad \frac{\overline{DE}}{\overline{AB}} = \frac{\overline{CD}}{\overline{CA}} = \frac{\overline{CE}}{\overline{CB}}.$$

2. If a pair of *parallel* line segments is "cross-connected" as shown, the two triangles formed are similar.



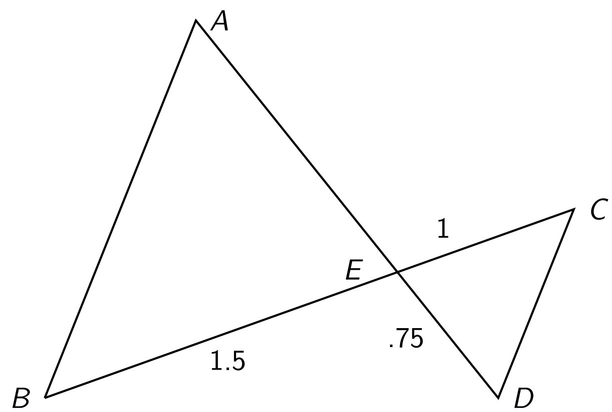
Starting with the parallel segments $\overline{AB}$ and $\overline{CD}$, we connected D to A and C to B by means of straight lines (called **transversals**) which cross at point E. The upper and lower angles at E are equal. In addition, $\angle D = \angle A$ and $\angle C = \angle B$. (These facts are visually plausible; they would be proved in a geometry class.) It follows that triangles ABE and CDE are similar and hence

$$\frac{\text{upper triangle}}{\text{lower triangle}} \qquad \frac{\overline{CD}}{\overline{AB}} = \frac{\overline{EC}}{\overline{EB}} = \frac{\overline{ED}}{\overline{EA}}.$$

In the examples and exercises, it will be convenient to use the following symbolism:

$$\begin{array}{ccc} \| & \longleftrightarrow & \text{“is parallel to”} \\ \sim & \longleftrightarrow & \text{“is similar to”} \\ \triangle & \longleftrightarrow & \text{“triangle”} \end{array}$$

Example 146. In the picture below, $\overline{AB} \parallel \overline{CD}$. Some of the side lengths are given in inches. Find the length of the side $\overline{AE}$.



Solution. Because $\overline{AB} \parallel \overline{CD}$, $\triangle ABE \sim \triangle CDE$. Corresponding sides are proportional, hence

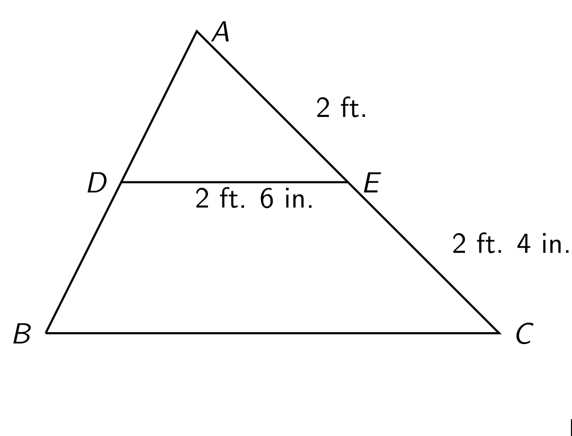
$$\frac{\text{smaller triangle}}{\text{bigger triangle}} \qquad \frac{\overline{CD}}{\overline{AB}} = \frac{\overline{EC}}{\overline{EB}} = \frac{\overline{ED}}{\overline{EA}}.$$

In particular, using $\overline{EC} = 1$, $\overline{EB} = 1.5$, and $\overline{ED} = .75$, and x for the length of $\overline{AE}$,

$$\frac{1}{1.5} = \frac{.75}{x}$$

or $x = (1.5)(.75) = 1.125$. The length of $\overline{AE}$ is 1.125 inches. ☐

Example 147. In the following figure, $\overline{DE} \parallel \overline{BC}$. Find the length of $\overline{BC}$.



Solution. $\triangle ABC \sim \triangle ADE$ because $\overline{DE} \parallel \overline{BC}$. Corresponding sides are proportional, hence

$$\frac{\overline{AC}}{\overline{AE}} = \frac{2 \text{ ft.} + 2 \text{ ft. } 4 \text{ in.}}{2 \text{ ft.}} = \frac{\overline{BC}}{\overline{DE}} = \frac{\overline{BC}}{2 \text{ ft. } 6 \text{ in.}}.$$

Converting all measurements to inches, and letting x represent the length of $\overline{BC}$ in inches,

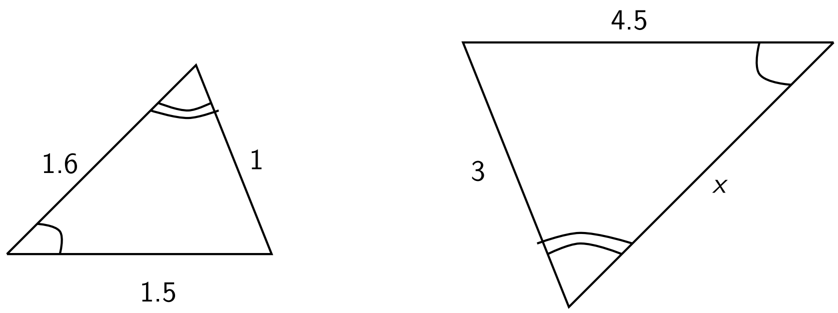
$$\frac{52}{24} = \frac{x}{30}$$

which gives $x = 65$ inches, or 5 ft. 5 in. □

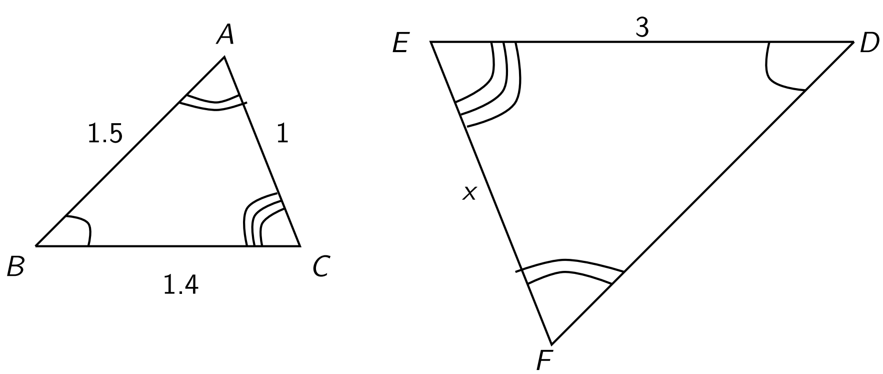
4.5.1 Exercises

In the first two exercises, the triangles are similar, and similarly marked angles are equal. Find the length x in each case.

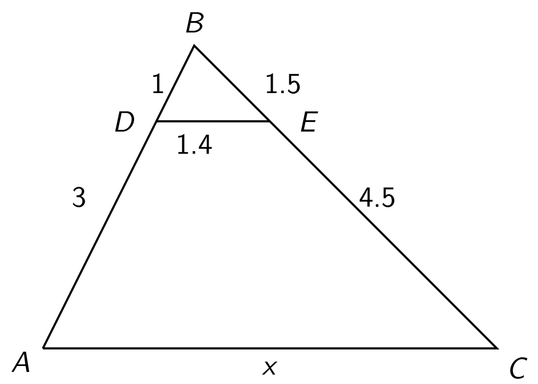
1.



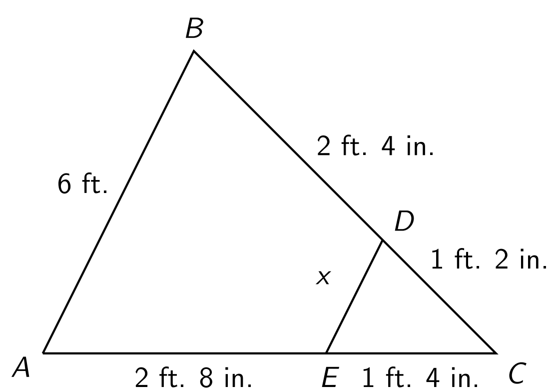
2.



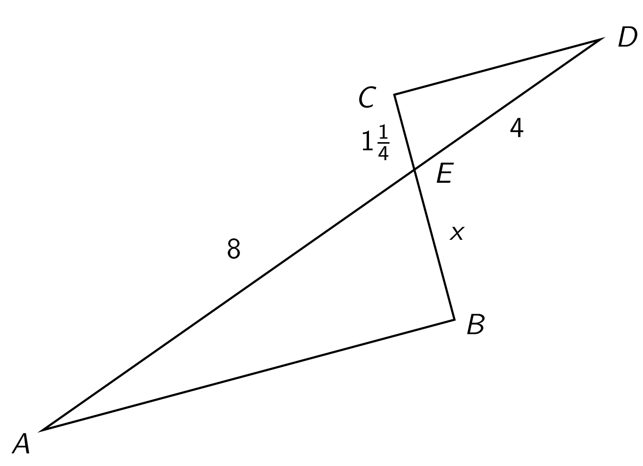
3. In the figure below, $\overline{AC} \parallel \overline{DE}$. Find the side length x.



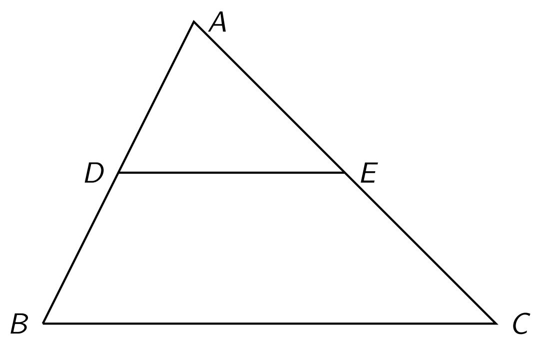
4. In the figure below, $\overline{AB} \parallel \overline{DE}$. Find the side length x.



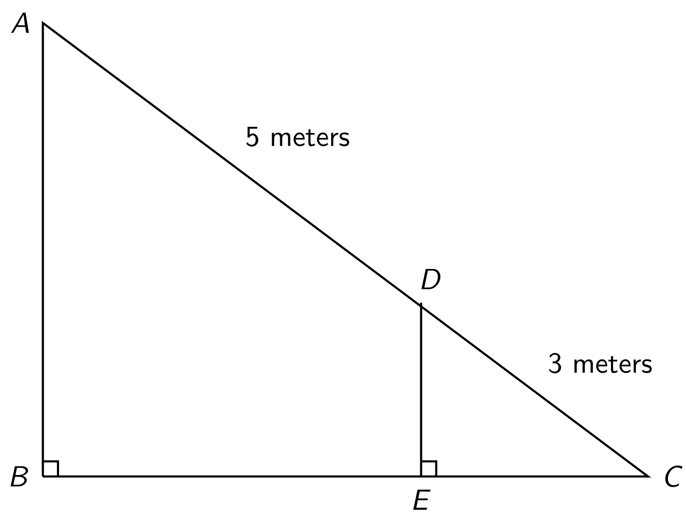
5. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find the side length x.



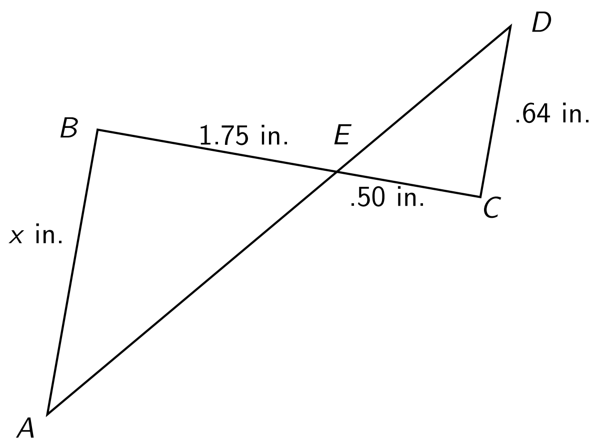
6. In the figure, $\overline{DE} \parallel \overline{BC}$. D cuts $\overline{AB}$ exactly in half. If $\overline{BC}$ is 12 centimeters long, how long is $\overline{DE}$?



7. In the figure below, $\overline{AB} \parallel \overline{DE}$, and length measures are given in meters. How long is $\overline{AB}$ if $\overline{DE}$ has a length of $2\frac{2}{5}$ meters?



8. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find the side length x.

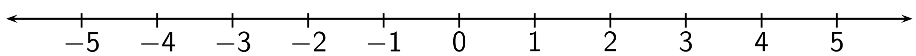


9. A 6 foot man casts an 8 foot shadow on the ground. How long is the shadow of a nearby 32 foot tree? Draw a figure involving similar triangles which illustrates the situation.

Chapter 5

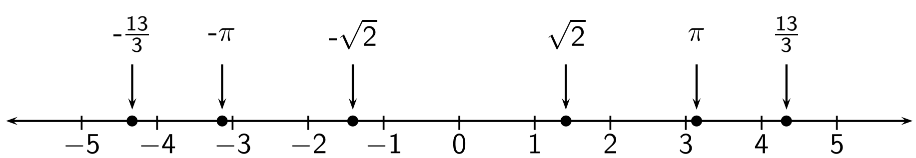
Signed Numbers

In this chapter we expand the set of numbers available for doing arithmetic. We now include **negative** numbers, which lie to the left of 0 on the number line, forming a kind of "mirror image" of the positive numbers. On the expanded number line, each number (except 0) has a **sign**. The negative numbers have sign $-$, and the positive numbers have sign $+$. The $+$ sign for positive numbers is understood and usually omitted. 0 has no sign.



The rule of thumb "left is less" continues to hold on the expanded number line. Thus, for example, $-3 < -1$, since -3 lies to the left of -1 on the number line.

The set of negative numbers (like the set of positive numbers), includes not just negative *integers*, but also negative fractions such as $-\frac{13}{3} = -4\frac{1}{3}$, and negative *irrational* numbers, such as $-\sqrt{2} \approx -1.41$ and $-\pi \approx -3.14$.



The **absolute value** of a number is its distance from 0 on the number line. The symbol for absolute value is a pair of vertical lines, $|\ |$. Thus $|\,0\,| = 0$ and

$$|-1| = |1| = 1, \qquad |-\sqrt{2}| = |\sqrt{2}| = \sqrt{2}, \qquad |-\pi| = |\pi| = \pi, \quad \text{etc.}$$

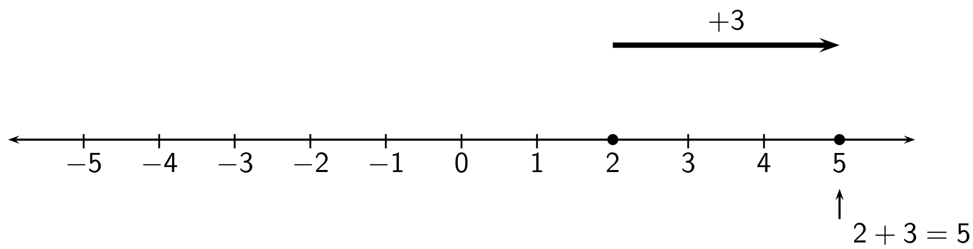
The absolute value of a number cannot be negative, even if the number itself is negative. This is because, intuitively, distance is a nonnegative quantity.

5.1 Adding signed numbers

To do arithmetic with signed numbers, we extend the ordinary operations of addition, subtraction, multiplication and division so that they remain consistent with the familiar operations with nonnegative numbers. We define the extended addition operation in terms of motion along the number line, as follows:

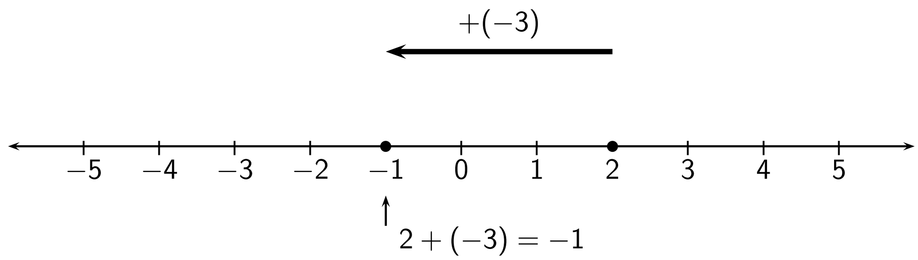
To add a *positive* number,	move to the *right*;
To add a *negative* number,	move to the *left*.

Thus, to add 3 to 2, we imagine starting at 2 on the number line and moving 3 "steps" to the right, arriving at 5.



Example 148. Perform the signed number addition $2 + (-3)$.

Solution. We start at 2 as before, but now we move 3 steps *left*, taking the sign of -3 into account.

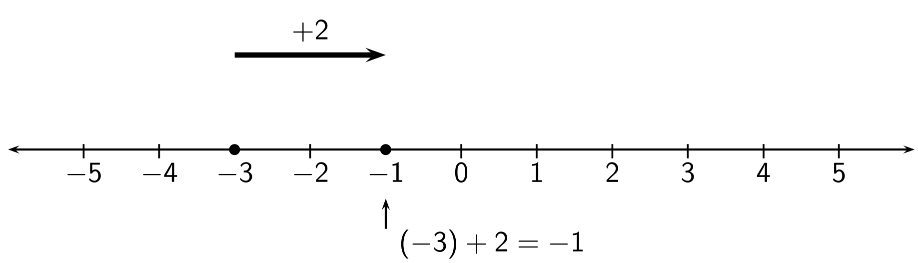


As the picture shows, we end up at -1. We conclude that $2 + (-3) = -1$. □

When two signed numbers are added, the order of addition does not affect the sum.

Example 149. Perform the signed number addition $(-3) + 2$.

Solution. This is just the previous example, with the addition in the reverse order. Now we *start* at -3, and move 2 steps *right*, taking into account the (positive) sign of 2. We arrive at the same result, -1:



□

More generally, for *any* two signed numbers,

$$a + b = b + a.$$

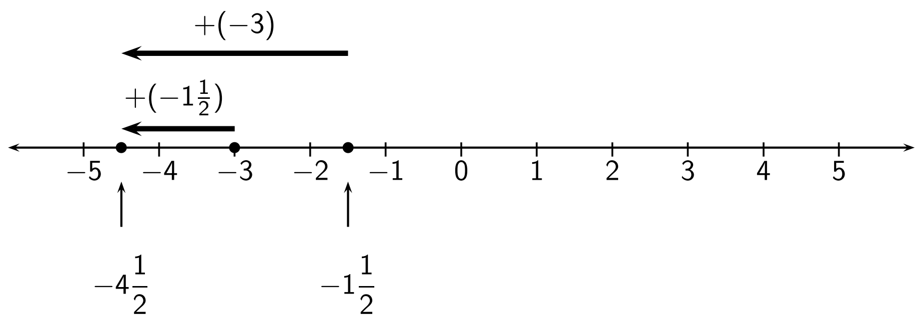
Addition, extended to signed numbers, remains **commutative**.

Example 150. Perform the signed number addition $-1\frac{1}{2} + (-3)$ in both possible orders.

Solution. Both numbers are negative, so we move consistently left on the number line. We either start at $-1\dfrac{1}{2}$ and move 3 steps left, arriving at

$$\left(-1\frac{1}{2}\right)+(-3)=-\left(1\frac{1}{2}+3\right)=-\left(\frac{3}{2}+\frac{6}{2}\right)=-\frac{9}{2}=-4\frac{1}{2}$$

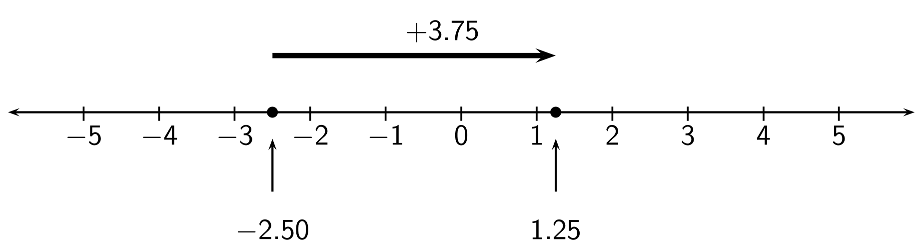
(upper arrow in the picture below); or we start at -3 and move $1\dfrac{1}{2}$ steps left (lower arrow). Either way, we arrive at $-4\dfrac{1}{2}$.



□

Example 151. Perform the signed number addition $-2.50+3.75$.

Solution. Starting at -2.50, we move a distance of 3.75 to the right, ending at 1.25, as shown.



□

In the last example, the *absolute value* of the sum was the *difference* obtained by subtracting the smaller absolute value from the larger ($3.75-2.50=1.25$), while the *sign* of the sum $(+)$ was the same as the sign of the number with the larger absolute value. This is no accident, but illustrates a general rule about adding numbers with opposite signs.

When two signed numbers are added

- if the numbers have **opposite** signs,
  1. the *sign* of the sum is the sign of the number with the larger absolute value;
  2. the *absolute value* of the sum is the difference bewteen the two individual absolute values (larger − smaller).
- if the two numbers have the **same** sign,
  1. the *sign* of the sum is the common sign of the summands;
  2. the *absolute value* of the sum is the sum of the individual absolute values.

Example 152. Add $15 + (-18)$.

Solution. The numbers have opposite signs, so the sign of the sum is the same as the sign of the number with the larger absolute value (-18), i.e., $-$. The absolute value of the sum is the difference $|-18| - |15| = 18 - 15 = 3$. Thus the sum is -3. This reasoning is summarized as

$$15 + (-18) = -(18 - 15) = -3.$$

□

Example 153. Add $-7 + (-9)$.

Solution. The numbers have the same sign, $-$, so that is also the sign of the sum. The absolute value of the sum is the sum of the individual absolute values, $|-7| + |-9| = 7 + 9 = 16$. Thus the sum is -16. The reasoning is summarized as

$$-7 + (-9) = -(7 + 9) = -16.$$

□

Example 154. Add $-3.6 + 4.9$.

Solution. The numbers have opposite signs, and the number with the larger absolute value (4.9) determines the sign of the sum $(+)$. The absolute value of the sum is the difference $|4.9| - |-3.6| = 4.9 - 3.6$. Thus

$$-3.6 + 4.9 = +(4.9 - 3.6) = +1.3 = 1.3.$$

□

Example 155. Add $4\frac{3}{5} + \left(-7\frac{1}{3}\right)$.

Solution. The numbers have opposite signs, and the number with the larger absolute value is negative, so the sum will be negative. The absolute value of the sum is the difference in the individual absolute values,

$$\begin{aligned}
&= 7\frac{1}{3} - 4\frac{3}{5} \\
&= 7\frac{5}{15} - 4\frac{9}{15} \qquad \text{(using the LCD = 15)} \\
&= 6\frac{20}{15} - 4\frac{9}{15} \qquad \left(\text{borrowing } 1 = \frac{15}{15} \text{ from } 7\right) \\
&= 2\frac{11}{15}.
\end{aligned}$$

Finally, recalling that the sign of the sum is negative, we have

$$4\frac{3}{5} + \left(-7\frac{1}{3}\right) = -2\frac{11}{15}.$$

□

5.1.1 Exercises

Add.

1. $-6 + 19$
2. $12 + (-4)$
3. $-34 + (-28)$
4. $266 + (-265)$
5. $5\frac{3}{5} + \left(-4\frac{1}{2}\right)$
6. $-13.38 + (-9.03)$
7. $-1001.36 + 909$
8. $\left(-\frac{3}{4}\right) + 2$
9. $\left(-\frac{5}{6}\right) + (-5)$
10. $\left(3\frac{1}{5}\right) + \left(2\frac{5}{8}\right)$

Use an appropriate signed number addition for the following.

11. Find the temperature at noon in Anchorage if the temperature at dawn was -11° F and the temperature subsequently rose by 36° F.
12. Find the height (in feet above ground level) of an elevator which started 30 feet below ground level and subsequently rose 70 feet.

5.1.2 Opposites, Identity

There are exactly two numbers at any given non-zero distance from 0, one negative and the other positive. Pairs of numbers such as $\{-\frac{13}{3}, \frac{13}{3}\}$, $\{-\sqrt{2}, \sqrt{2}\}$, $\{-\pi, \pi\}$, which are unequal but *equidistant* from 0, and hence have the same absolute value, are called **opposites**. (0 is the only number which is its own opposite, having absolute value 0.) To find the opposite of a nonzero number, we simply change its sign.

Example 156. (a) The opposite of 11 is -11. (b) The opposite of -1.7 is 1.7.

Since there are only two possible signs, the *opposite of the opposite* of a number is the number we started with.

> The opposite of the opposite of N is N:
>
> $$-(-N) = N.$$

Example 157. (a) The opposite of the opposite of $\frac{5}{8}$ is $-\left(-\frac{5}{8}\right) = \frac{5}{8}$. (b) the opposite of the opposite of -5 is $-(-(-5)) = -5$.

Suppose a nonzero number is added to its opposite, for example, $3 + (-3)$. The rule for adding signed numbers with opposite signs doesn't seem to work here: it is not clear what the sign of the sum should be. But in fact it doesn't matter since the *absolute value* of the sum is 0, which has no sign. Accordingly, $-3 + 3 = 0$. (Intuitively: if we start at 3 and move 3 steps left, we arrive at 0.) In general,

> The sum of a number and its opposite is 0:
>
> $$N + (-N) = 0.$$

Example 158. (a) $\frac{1}{7} + \left(-\frac{1}{7}\right) = 0$. (b) $-23.04 + 23.04 = 0$.

Recall that 0 is the **additive identity** for ordinary addition because, when 0 is added to a number, the result is the identical number, i.e, the number does not change. This remains true for *signed* numbers.

Example 159. $-4 + 0 = -4$.

5.1.3 Exercises

Find the following:

1. The opposite of 0.062
2. The opposite of $-2\frac{5}{7}$

3. $-\left(-\left(-\frac{2}{7}\right)\right)$

4. The sum of 99 and its opposite.

5. The sum of $-\pi$ and its opposite.

6. The opposite of the opposite of -0.25.

7. The sum of -5 and the opposite of 5.

8. $-258 + (-(-258))$

9. $-9.1 + (-9.1)$

10. $-\dfrac{3}{8} + 0$

11. $0 + 5.5$

12. $4 + \left(2\dfrac{2}{3} + \left(-2\dfrac{2}{3}\right)\right)$

5.1.4 Associativity

Another important property of addition, **associativity**, extends to signed number addition. Associativity of addition means that when three or more numbers are added, it doesn't matter how you associate them into groups for addition: $x + y + z = (x + y) + z = x + (y + z)$. Recall that this property allowed us to add long columns of nonnegative numbers.

We can make use of column addition with signed numbers, too, by associating and adding all the positive numbers, and, separately, associating and adding the *absolute values* of all the negative numbers. Then we add the two subtotals, treating the subtotal associated with the negative numbers as negative. This way, we apply the signed number rule just once, at the end.

Example 160. Add: $43.6 + (-5.8) + (-135) + 69.5 + (-134) + 158.7 + (-162.3)$

Solution. We add all the positive numbers,

$$\begin{array}{r} 43.6 \\ 69.5 \\ +\ 158.7 \\ \hline 271.8 \end{array}$$

and the absolute values of all the negative numbers,

$$\begin{array}{r} 5.8 \\ 135. \\ 134. \\ +\ 162.3 \\ \hline 437.1 \end{array}$$

Treating the subtotal associated with the negative numbers as negative, we add $271.8 + (-437.1)$. By the rule for adding signed numbers with opposite signs,

$$271.8 + (-437.1) = -(437.1 - 271.8) = -165.3.$$

□

If we think of positive numbers as "profits," and negative numbers as "losses," then signed number addition is like "balancing the books." There is a net gain if the total profits are larger than the total losses (in absolute value); otherwise there is a net loss.

Example 161. Last week, a business received checks from clients in the amounts of \$350.65, \$461.00 and \$900.78, and paid bills in the amounts of \$261.50, \$551.00 and \$78.70. What was the net profit or loss for the week?

Solution. Bills paid are losses, so we treat them as negative numbers; checks received are profits, counted as positive numbers. The total profits are

$$\begin{array}{rrrrrr} & 3 & 5 & 0 & . & 6\ 5 \\ & 4 & 6 & 1 & . & 0\ 0 \\ + & 9 & 0 & 0 & . & 7\ 8 \\ \hline 1, & 7 & 1 & 2 & . & 4\ 3 \end{array}$$

and the total losses are

$$\begin{array}{rrrrrr} & 2 & 6 & 1 & . & 5\ 0 \\ & 5 & 5 & 1 & . & 0\ 0 \\ + & & 7 & 8 & . & 7\ 0 \\ \hline & 8 & 9 & 1 & . & 2\ 0 \end{array}$$

Affixing a negative sign to the total losses, we calculate the net profit (or loss) as the sum

$$1712.43 + (-891.20) = +(1712.43 - 891.20) = 821.23.$$

There was a net profit of \$821.23 for the week. □

5.1.5 Exercises

Perform the additions.

1. $1 + (-1) + 1 + (-1) + 1 + (-1)$
2. $-8.2 + (-198.5) + 8.2$
3. $44 + (-5.5) + 28.8 + 36 + (-19.1) + (-8)$
4. $\frac{1}{2} + \left(-\frac{1}{3}\right) + \frac{6}{7}$
5. $\left(-6\frac{3}{8}\right) + 4\frac{3}{4} + \left(-2\frac{1}{2}\right) + \frac{7}{8}$

6. $\left(-1\frac{2}{3}\right) + \left(-2\frac{5}{6}\right) + \left(-8\frac{1}{3}\right)$
7. $18.50 + (-21.25) + (-69.95) + 13.50 + 79.99 + (-86.50)$
8. $5 + \left(-3\frac{1}{2}\right) + 4\frac{1}{2} + \left(-2\frac{1}{2}\right) + (-4) + \left(-\frac{1}{2}\right) + 7$
9. Find the average noon temperature for the first week of January in Barrow, Alaska if the noon temperatures were: Monday: $-8°$ F, Tuesday: $2°$ F, Wednesday: $-3°$ F, Thursday: $-5°$ F, Friday: $1°$ F, Saturday: $4°$ F, Sunday: $9°$ F.
10. Find the net profit (or loss) of a business that received checks in the amounts of \$453.05, \$865.50 and \$300.25, and paid bills in the amounts of \$561.50, \$449.25, \$798.75 and \$75.25.

5.2 Subtracting signed numbers

The phrase "subtract A from B" indicates the operation $B - A$. We extend subtraction to signed numbers by defining it in terms of addition.

> To subtract a signed number A from a signed number B, add B to the *opposite* of A:
>
> $$B - A = B + (-A).$$

With this definition, there is no longer any real need for a separate operation called subtraction. For example, instead of subtracting $17 - 9$, we could simply add $17 + (-9)$ (verify that the results are the same). Despite this, it is convenient to retain the operation of subtraction.

Example 162. (a) Subtract 15 from 9. (b) Subtract 8 from -2.

Solution. (a) $9 - 15 = 9 + (-15)$. Applying the rule for adding signed numbers with opposite signs,

$$9 + (-15) = -(15 - 9) = -6.$$

(b) $-2 - 8 = -2 + (-8)$. Applying the rule for adding signed numbers with the same sign,

$$-2 + (-8) = -(2 + 8) = -10.$$

□

Example 163. Subtract -17 from 12.

Solution. $12 - (-17) = 12 + (-(-17))$. Remembering that $-(-17) = 17$ (opposite of the opposite), we have

$$12 - (-17) = 12 + 17 = 29.$$

□

Example 164. Subtract (-15) from -32.

Solution. $-32-(-15) = -32+(-(-15)) = -32+15$. Applying the rule for adding signed numbers with opposite signs,

$$-32-(-15) = -32+15 = -(32-15) = -17.$$

□

Example 165. Subtract $2\frac{7}{8}$ from $1\frac{5}{6}$.

Solution. Adding the opposite of $2\frac{7}{8}$ to $1\frac{5}{6}$, we get

$$\begin{aligned} 1\frac{5}{6} + \left(-2\frac{7}{8}\right) &= -\left(2\frac{7}{8} - 1\frac{5}{6}\right) \\ &= -\left(2\frac{21}{24} - 1\frac{20}{24}\right) \qquad (\text{LCD} = 24) \\ &= -1\frac{1}{24}. \end{aligned}$$

□

Example 166. Perform the subtraction $3.359 - 10.08$.

Solution. $3.359-10.08 = 3.359+(-10.08) = -(10.08-3.359)$. We perform the subtraction of absolute values vertically.

$$\begin{array}{rrrrrrr} & 1 & 0 & . & 0 & 8 & 0 \\ - & & 3 & . & 3 & 5 & 9 \\ \hline & & 6 & . & 7 & 2 & 1 \end{array}$$

Remembering that the sign was negative, $3.359 - 10.08 = -6.721$. □

Formerly "impossible" subtractions, such as

$$7-12$$

can now be easily performed.

$$7-12 = 7+(-12) = -(12-7) = -5.$$

Examples like this show that subtraction is **not commutative**: *changing the order of subtraction changes the result to its opposite.* In general, for any two signed numbers,

$$B - A = -(A - B).$$

Example 167. $22 - 100 = -78 = -(100 - 22)$.

We often need to find the *difference* of quantities such as temperature or altitude which can take both negative and positive values.

Example 168. The summit of Mount Whitney in California is 14505 feet above sea level. Not far away, in Death Valley, the lowest point is 282 feet below sea level. What is the altitude difference between Mount Whitney's summit and the lowest point in Death Valley?

Solution. Assigning a negative altitude to a point below sea level, the difference is

$$14505 - (-282) = 14505 + 282 = 14787 \text{ feet.}$$

□

5.2.1 Exercises

Perform the subtractions.

1. Subtract 31 from 7
2. $98 - 100$
3. $.65 - (-6.4)$
4. Subtract 5.5 from 2.4
5. Subtract -53 from 68.6
6. $-8.88 - (-1.11)$
7. Subtract 2.2 from $\frac{1}{5}$
8. $-87 - 23$
9. $-87 - (-23)$
10. $5\frac{3}{8} - 11$
11. $5\frac{3}{8} - (-11)$
12. Subtract $\frac{3}{5}$ from 0.1 and express the difference as a fraction.
13. $-2.5 - 1\frac{4}{5}$
14. $\left(-\frac{1}{2}\right) - \left(-\frac{2}{3}\right)$
15. A snowball at -5° C is heated until it melts and then boils. If water boils at $100^\circ C$, by how much did the temperature of the snowball rise?
16. A sunken car is salvaged from the bottom of a lake. The elevation of the lake bottom is -66.2 feet. The car is lifted by a crane to a height 79.5 feet above lake level. Through what vertical distance was the car lifted?

5.3 Multiplying Signed Numbers

Multiplication of positive numbers was defined in terms of repeated addition. For example,

$$3 \times 4 = 4 + 4 + 4 = 12.$$

There is no problem extending this definition to the product of a negative number by a positive number.

$$3 \times (-4) = (-4) + (-4) + (-4) = -12.$$

Products such as

$$(-2) \times 5$$

pose no problem if, for consistency, we define signed number multiplication to be *commutative* (we do):

$$(-2) \times 5 = 5 \times (-2) = (-2) + (-2) + (-2) + (-2) + (-2) = -10.$$

The general rule is:

> *The product of two numbers with opposite signs is the negative of the product of their absolute values.*

Example 169. Find the products (a) $7 \times (-11)$ and (b) $(-12) \times 5$.

Solution. We take the negative of the product of the absolute values in each case. (a) $7 \times (-11) = -(7 \times 11) = -77$. (b) $(-12) \times 5 = -(12 \times 5) = -60$. □

When it comes to the product of two negative numbers, our intuition fails. It makes no sense to "repeatedly" add a number to itself when the number of repeats is *negative*! How should we define $(-2) \times (-3)$?

It is best to give up on intuition and let consistency rule. Look at the pattern below:

$$\begin{aligned}
4 \times (-3) &= -12 && = \text{the opposite of } 4 \times 3 \\
3 \times (-3) &= -9 && = \text{the opposite of } 3 \times 3 \\
2 \times (-3) &= -6 && = \text{the opposite of } 2 \times 3 \\
1 \times (-3) &= -3 && = \text{the opposite of } 1 \times 3 \\
0 \times (-3) &= 0 && = \text{the opposite of } 0 \times 3 \\
(-1) \times (-3) &= && = ? \\
(-2) \times (-3) &= && = ??
\end{aligned}$$

It looks like the pattern "ought" to continue as follows:

$$\begin{aligned}
(-1) \times (-3) &= 3 && \text{the opposite of } (-1) \times 3 \\
(-2) \times (-3) &= 6 && \text{the opposite of } (-2) \times 3
\end{aligned}$$

We make the following general definition:

> *The product of two negative numbers is the (positive!) product of their absolute values.*

Example 170. Find the product $(-8) \times (-12)$.

Solution. $(-8) \times (-12) = 8 \times 12 = 96$. □

Note that the product of two *positive* numbers could also be defined as the (positive) product of their absolute values. So our definition is consistent with what we already know about positive numbers, and applies whenever numbers *with the same sign* are multiplied.

[OPTIONAL: If you find it hard to accept that the product of two negative numbers is positive, the following discussion might help. An important axiom (fact) of arithmetic states that multiplication "distributes" over addition. That is, for any three numbers, A, B and C,

$$A(B + C) = AB + AC.$$

If signed number arithmetic is to be consistent with the arithmetic of nonnegative numbers, this axiom must continue to hold. In particular, if A and B are positive,

$$(-A)(B+(-B)) = (-A)(B) + (-A)(-B).$$

Since the left hand side equals 0 (why?) so does the right hand side. It follows that $(-A)(-B)$ must be the opposite of $(-A)(B) = -(-AB) = AB$. In other words, it must be true that $(-A)(-B) = AB$.]

Here is a summary of the rules for multiplying signed numbers.

When two signed numbers are **multiplied**

- if the numbers have the **same** sign, the product is positive.
- if the numbers have **opposite** signs, the product is negative.

In both cases, the absolute value of the product is the product of the individual absolute values.

In the examples and exercises below, we freely use all three ways of symbolizing multiplication: $\cdot$, $\times$, and juxtaposition. It is easiest to determine the sign of the product first, and then compute with the absolute values.

Example 171. Multiply $(-16) \times 5$.

Solution. $(-16) \times 5 = -80$ (negative since the numbers have opposite signs). $\square$

Example 172. Find the product $(-11) \cdot (-12)$.

Solution. $(-11) \cdot (-12) = 132$ (positive since the numbers have the same signs). $\square$

Example 173. Find the product $(-6.30)(-2.05)$.

Solution. The product is positive since the numbers have the same sign. Temporarily ignoring the decimal points,

$$\begin{array}{rr} & 630 \\ \times & 205 \\ \hline & 3150 \\ & 1260 \\ \hline & 129150 \end{array}$$

Inserting four decimal places counting from the right, and remembering that the product is positive, we conclude that

$$(-6.30)(-2.05) = 12.915.$$

$\square$

Example 174. Find the product $\frac{3}{4} \cdot \left(-\frac{6}{5}\right)$.

Solution. The numbers have opposite signs, so the product is negative.

$$-\left(\frac{3}{\cancel{4}_{2}} \cdot \frac{\cancel{6}^{3}}{5}\right) = -\frac{9}{10} \qquad \text{(pre-cancelling 2).}$$

□

Example 175. Find the product $\left(-2\frac{1}{2}\right)\left(-3\frac{1}{4}\right)$.

Solution. The numbers have the same sign so the product is positive, and we compute with the absolute values. Converting the mixed numbers to improper fractions for multiplication,

$$\frac{5}{2} \cdot \frac{13}{4} = \frac{65}{8} = 8\frac{1}{8}.$$

□

Recall the **zero property**, which states that

$$0 \cdot x = 0 \quad \text{and} \quad x \cdot 0 = 0, \qquad \text{for any number } x.$$

This remains true for signed numbers.

Example 176. $(-17.1) \cdot 0 = 0$.

Also, 1 continues to be the **multiplicative identity** , that is,

$$1 \cdot x = x \quad \text{and} \quad x \cdot 1 = x, \qquad \text{for any number } x.$$

Example 177. $\left(-\frac{15}{31}\right) \cdot 1 = -\frac{15}{31}$.

Multiplication, like addition, continues to be **associative** when extended to signed numbers. Thus we can multiply several signed numbers without worrying how we group them for multiplication. One consequence is that the sign of a product of several signed numbers can be quickly determined in advance by simply determining whether the number of negative factors is *even* or *odd*.

> The sign of a product of several factors is
>
> - positive if the number of negative factors is *even*,
> - negative if the number of negative factors is *odd*.

This is true because every *pair* of negative factors has a positive product. If there is an even number of negative factors, they can be "paired off" (in any convenient order), producing a product of positive numbers which is, of course, positive. But if the number of negative factors is odd, there is always one "unpaired" negative factor, which makes the total product negative.

Example 178.

$(-1)(2)(-3)(-4) = -24$ (an odd number (three) of negative factors makes a negative product),

$(-1)(-2)(-3)(-4) = 24$ (an even number (four) of negative factors makes a positive product).

5.3.1 Exercises

Find the products.

1. $8 \times (-6)$
2. -9×7
3. $(-6) \times (-9)$
4. $(-1) \times 5$
5. $(91.4)(-1)$
6. $(0)(-88.8)$
7. $(-1) \times (-1)$
8. $(-1) \times (-1) \times (-1)$
9. $6.5 \times (-31)$
10. $(-5.03) \times (-0.6)$
11. $\left(\frac{7}{9}\right) \times 0$
12. $\left(\frac{6}{7}\right)\left(-\frac{7}{6}\right)$
13. $\left(-1\frac{6}{7}\right)\left(-1\frac{1}{2}\right)$
14. $(-1.62)(1000)$
15. $1(-1)$
16. $(-3)(-50)(-2)$
17. $(-3)(0.5)(-0.7)(1)$
18. $(3)(-10)(2)(-5)$
19. $(-6)(-5)(-4)(-3)(0)$
20. $\left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{3}{8}\right)\left(\frac{4}{5}\right)$

5.4 Dividing Signed Numbers

Recall that two numbers are *reciprocal* if their product is 1. We extend this definition unchanged to signed numbers. For example, since

$$(-2) \times \left(-\frac{1}{2}\right) = 1,$$

-2 and $-\frac{1}{2}$ are reciprocal. Similarly, $-\frac{3}{8}$ and $-\frac{8}{3}$ are reciprocal, since

$$\left(-\frac{3}{8}\right)\left(-\frac{8}{3}\right) = 1.$$

Division by a signed number $N \neq 0$ can then be defined (as with positive numbers) as *multiplication by the reciprocal* of N, i.e.,

$$M \div N = M \cdot \frac{1}{N}$$

for any number M. Since a number and its reciprocal must have the same sign (why?), it follows that the rules for dividing signed numbers are exactly analogous to the rules for multiplying them.

> When two signed numbers are **divided** (and the divisor is nonzero)
>
> - if the dividend and divisor have the **same** sign, the quotient is positive
> - if the dividend and the divisor have **opposite** signs, the quotient is negative.
>
> In both cases, the absolute value of the quotient is the quotient of the individual absolute values.

Recall that a fraction bar indicates division, that is, $\frac{x}{y} = x \div y$. By the rule for division of signed numbers, if either x or y (but not both) is negative, the fraction represents a negative number. For example, suppose a, b and c are positive. Then

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

All three fractions are equivalent, but it is customary to avoid the form in the middle (with a negative number in the denominator). In the examples and exercises, we indicate division by the $\div$ symbol or the fraction bar interchangeably. It is easiest to determine the sign first, and then compute with the absolute values. For divisions of whole numbers, explicit conversion to reciprocal multiplication is not necessary.

Example 179. Divide 24 by -6.

Solution. The quotient $24 \div (-6)$ is negative since the numbers have opposite signs. The absolute value is $24 \div 6 = 4$. Thus

$$24 \div (-6) = -4.$$

□

Example 180. Express the fraction $\frac{-33}{5}$ as a mixed number.

Solution. The fraction represents division of numbers with opposite signs, so the result is negative. For the division $33 \div 5$, the quotient is 6 and the remainder is 3, so

$$\frac{-33}{5} = -6\frac{3}{5}.$$

$\square$

Example 181. Find $\left(-2\frac{5}{9}\right) \div \left(-\frac{1}{6}\right)$.

Solution. Because the numbers have the same sign, the quotient is positive. We convert the mixed number to an improper fraction and multiply by the reciprocal of the divisor.

$$\begin{aligned}\left(-2\frac{5}{9}\right) \div \left(-\frac{1}{6}\right) &= \left(\frac{-23}{9}\right) \cdot \left(\frac{-6}{1}\right) \\ &= \left(\frac{23}{\cancelto{3}{9}}\right) \cdot \left(\frac{\cancelto{2}{6}}{1}\right) \\ &= \frac{46}{3} \\ &= 15\frac{1}{3}.\end{aligned}$$

$\square$

Example 182. Divide $3.738 \div (-100)$.

Solution. The quotient is negative because the numbers have opposite signs. Thus

$$3.738 \div (-100) = -0.03738.$$

(Recall that dividing a decimal by a power of 10 (10^2 in this case) simply results in a leftward shift of the decimal point by a number of places equal to the exponent on 10.) $\square$

Example 183. Divide $16 \div (-0.25)$.

Solution. The quotient is negative because the signs are opposite. We can either perform the long division

$$0.25\overline{)16} \qquad \text{equivalent to} \quad 25\overline{)1600},$$

or convert 0.25 to the fraction $\frac{1}{4}$ and multiply by its reciprocal. The latter seems easier:

$$16 \div (-0.25) = -\left(16 \div \frac{1}{4}\right) = -(16 \times 4) = -64.$$

$\square$

5.4.1 Division and 0

Recall that 0 can be the numerator of a fraction, but it cannot be the denominator. This remains true for signed fractions. Fractions are divisions, and conversely, every division can be represented as a fraction. A fraction like $\frac{0}{-3}$ makes sense and is equal to 0, so

$$0 \div (-3) = \frac{0}{-3} = -\frac{0}{3} = -0 = 0.$$

But the fraction $\frac{-3}{0}$, and therefore the division $(-3) \div 0$, is undefined. (You may wish to review the discussion in Section 2.2.1, where all attempts to make sense of a fraction with denominator 0 led to failure or contradiction.) We summarize the properties of 0 with respect to division.

For a nonzero signed number N

- $0 \div N = 0$
- $N \div 0$ is undefined.

$0 \div 0$ is also undefined.

Example 184. $0 \div (-23)$ and $\frac{0}{-71}$ are both equal to 0.

$1.62 \div 0$ and $\frac{-16}{0}$ are undefined.

5.4.2 Exercises

Perform the divisions, or state that they are undefined.

1. $(-24) \div (-8)$
2. $(-24) \div 8$
3. $66 \div 0$
4. $30 \div (-6)$
5. $(-30) \div 6$
6. $\frac{-26}{13}$
7. $\frac{-19}{0}$
8. $\frac{-72}{-18}$
9. $\frac{9.5}{-1.9}$

10. $601.03 \div (-1000)$

11. $0 \div (-1000)$

12. $\left(\frac{4}{5}\right) \div \left(-\frac{25}{32}\right)$

13. $\left(-4\frac{1}{2}\right) \div \left(-1\frac{7}{8}\right)$

14. $-10\frac{3}{4} \div 5$

15. $100 \div \frac{1}{4}$

16. $-110.98 \div 4$

17. $.5 \div .4$

18. $50 \div (-.4)$

5.5 Powers of Signed Numbers

Since exponents indicate repeated multiplication, there is no problem applying exponents to signed numbers.

$$(-4)^3 = (-4)(-4)(-4) = -64 \qquad \text{and} \quad (-2)^4 = (-2)(-2)(-2)(-2) = 16.$$

Example 185. Evaluate $\left(-\frac{3}{2}\right)^3$.

Solution.

$$\begin{aligned}\left(-\frac{3}{2}\right)^3 &= \left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right)\\ &= -\frac{3^3}{2^3} \qquad \text{(the product of three negative factors is negative)}\\ &= -\frac{27}{8} = -3\frac{3}{8}.\end{aligned}$$

□

Example 186. Evaluate $(-0.2)^4$.

Solution. $(-0.2)^4 = (-0.2)(-0.2)(-0.2)(-0.2) = 0.0016$. (The product of four negative factors is positive.) □

The previous examples show that

A power of a negative number is

- negative if the exponent is *odd*,
- positive if the exponent is *even*.

When applying exponents to signed numbers, it is essential to put parentheses around the number, including its sign. For example, the square of -3 is written

$$(-3)^2 = (-3)(-3) = 9.$$

Without parentheses, as in

$$-3^2,$$

the exponent applies only to 3, not to -3, and this is *not* a power of -3. Rather, -3^2 represents the *opposite* of 3^2, so

$$-3^2 = -9.$$

Example 187. Evaluate $\left(-\frac{3}{4}\right)^2$ and $-\left(\frac{3}{4}\right)^2$

Solution. The first expression is the square of a negative number, which is positive,

$$\left(-\frac{3}{4}\right)^2 = \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) = \frac{9}{16},$$

while the second is the opposite of the square of a positive number, which is negative:

$$-\left(\frac{3}{4}\right)^2 = -\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = -\frac{9}{16}.$$

□

The behavior of 0 in an exponential expression, in particular, the interpretation of 0 as an exponent, extends unchanged to signed numbers. (You may wish to review Section 1.4 on powers of whole numbers in Chapter 1.)

For a nonzero signed number N,

- $N^0 = 1$
- $0^N = 0$

(0^0 is undefined.)

Example 188. $(-3.5)^0 = 1$. $\left(-\frac{1}{2}\right)^0 = 1$. $0^4 = 0$. 0^0 is undefined.

5.5.1 Exercises

Find the value of each expression.

1. 8^2
2. $(-8)^2$
3. -8^2
4. -6^3
5. $(-6)^3$
6. 0^5
7. $\left(-\frac{2}{5}\right)^3$
8. $-(-3^3)$
9. $-(-2)^4$
10. $(.01)^3$
11. $(10)^0$
12. $(-9.08)^1$
13. $(-23)^0$
14. $-(-1.4)^3$
15. $(-1)^{59}$
16. 0^0
17. $\left(2\frac{1}{2}\right)^2$
18. $\left(-\frac{1}{10}\right)^4$

5.6 Square Roots of Signed Numbers

If there is a number whose square is the number n, it is called a **square root** of n. For example, since $4^2 = 16$, 4 is a square root of 16. It is also true that $(-4)^2 = 16$, so -4 is *another* square root of 16.

Every positive number N has two square roots, which are opposites:

- The positive square root is denoted $\sqrt{N}$.
- The negative square root is denoted $-\sqrt{N}$.

As usual, 0 is a special case. It has only one square root, itself:

$$\sqrt{0} = 0.$$

Whole numbers whose square roots are also whole numbers are called **perfect squares**.

Example 189. The first few perfect squares are 0, 1, 4, 9, 16 and 25, because

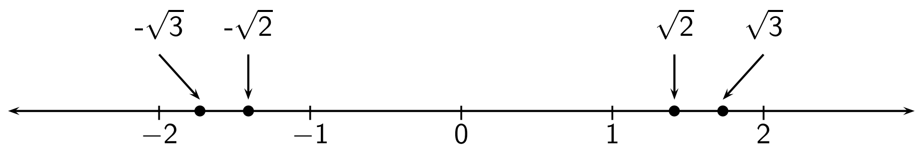
$$\sqrt{0} = 0, \quad \sqrt{1} = 1, \quad \sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{16} = 4, \quad \text{and } \sqrt{25} = 5.$$

The square root of a whole number which is not a perfect square has a decimal expansion which neither terminates nor repeats. (Such numbers are called *irrational*.) A calculator will inform you that

$$\sqrt{2} \approx 1.41 \qquad \text{and} \qquad \sqrt{3} \approx 1.73 \qquad \text{(rounded to the nearest hundredth).}$$

Example 190. Between what two whole numbers do $-\sqrt{2}$ and $-\sqrt{3}$ lie? Which number is larger?

Solution. On the number line, $-\sqrt{2} \approx -1.41$ and $-\sqrt{3} \approx -1.73$ both lie between -1 and -2.



Recalling that "left is less" on the number line,

$$-\sqrt{3} < -\sqrt{2},$$

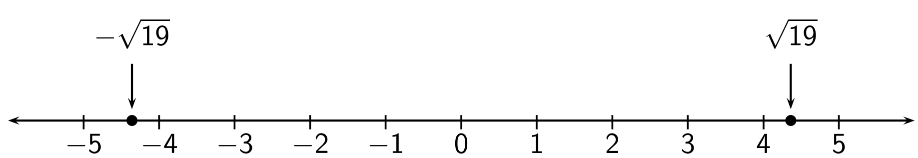
in other words, $-\sqrt{2}$ is larger. □

Example 191. Between what two integers does $-\sqrt{19}$ lie?

Solution. We first locate the largest perfect square less than 19, and the smallest perfect square greater than 19. We find that 19 is between 4^2 and 5^2. That is, $16 < 19 < 25$. It follows (by a geometric argument we gave in Section 1.4.3) that

$$4 < \sqrt{19} < 5.$$

This relationship has a "mirror image" on the negative half of the number line, as indicated below.



We see that $-\sqrt{19}$ lies between -5 and -4. More precisely, $-\sqrt{19}$ is greater than -5 and less than -4,

$$-5 < -\sqrt{19} < -4.$$

□

The square root of a positive fraction is not usually a quotient of whole numbers. The exceptions occur when the numerator and denominator of the fraction are both perfect squares. For example, $\sqrt{\dfrac{4}{9}} = \dfrac{2}{3}$ because $\left(\dfrac{2}{3}\right)^2 = \dfrac{4}{9}$. But $\sqrt{\dfrac{2}{5}}$ cannot be evaluated in this way. We can, however, write

$$\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}} \qquad \left(\text{since } \left(\frac{\sqrt{2}}{\sqrt{5}}\right)^2 = \frac{2}{5}\right).$$

This expresses $\sqrt{\dfrac{2}{5}}$ as a quotient, but not a quotient of whole numbers.

We come now to a troubling fact.

> Square roots of negative numbers *cannot be defined* within the system of signed numbers.

Suppose, for example, there were a number which, when squared, yields -4. That number must have absolute value 2, so the only possibilities are 2 and -2. But

$$2^2 = (-2)^2 = 4 \qquad (\text{not } -4).$$

A similar argument applies to any negative number. It is possible to expand the set of signed numbers so as to remedy this defect. This involves adjoining some "imaginary" (as opposed to "real") numbers. We leave that for a more advanced course. For now, when we encounter the square root of a negative number, we simply say that it is not a real number.

Example 192. $\sqrt{-5}$ is not a real number.

5.6.1 Exercises

Find the square roots, or state that they are not real numbers.

1. $\sqrt{9}$
2. $-\sqrt{25}$
3. $\sqrt{-25}$
4. $-\sqrt{49}$
5. $\sqrt{\dfrac{81}{4}}$
6. $-\sqrt{\dfrac{121}{144}}$
7. $\sqrt{100}$
8. $\sqrt{0}$
9. $-\sqrt{\dfrac{25}{36}}$

10. $\sqrt{\dfrac{-64}{-49}}$

11. $-\sqrt{\dfrac{1}{16}}$

Between what two integers do the following square roots lie?

11. $\sqrt{7}$

12. $-\sqrt{7}$

13. $-\sqrt{30}$

Insert the appropriate inequality ($<$ or $>$) between the following pairs of numbers.

14. $-\sqrt{10} \qquad -\sqrt{8}$

15. $\sqrt{12} \qquad \sqrt{15}$

5.7 Evaluating Expressions

A mathematical **expression** is any meaningful combination of numbers, letters, operation symbols (such as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$), and grouping symbols (such as parentheses, brackets, fraction bars and the extended square root symbol like $\sqrt{}$). For example, the expression

$$\frac{x+y}{x-y}$$

indicates a fraction whose numerator is the sum of two unspecified numbers, x and y, and whose denominator is their difference. Letters in a mathematical expression are called **variables** because they represent unspecified numbers that can take various different values. Expressions can be **evaluated** (assigned a numerical value) if numerical values are assigned to all the variables appearing in the expression.

Example 193. Evaluate the expression $\dfrac{x+y}{x-y}$ if $x = 2$ and $y = -6$.

Solution. We replace each letter by its assigned numerical value, enclosed in parentheses,

$$\frac{x+y}{x-y} = \frac{(2)+(-6)}{(2)-(-6)},$$

and simplify the resulting expression

$$\frac{(2)+(-6)}{(2)-(-6)} = \frac{-4}{8} = -\frac{1}{2}.$$

□

Assigning different values to the letters in an expression usually changes the value of the expression.

Example 194. Evaluate the expression $\dfrac{x+y}{x-y}$ if $x = -3$ and $y = 3$.

Solution. Replacing x and y by their assigned values,

$$\frac{x+y}{x-y} = \frac{(-3)+(3)}{(-3)-(3)} = \frac{0}{-6} = 0.$$

$\square$

The reason for replacing letters with their assigned values *in parentheses* is to avoid pitfalls such as the ones highlighted in the next two examples.

Example 195. Evaluate x^2 if $x = -4$.

Solution. Forgetting parentheses, we would write $x^2 = -4^2 = -16$, which is wrong, since the square of any nonzero number is positive. The correct evaluation is

$$x^2 = (-4)^2 = 16.$$

$\square$

Example 196. Evaluate ab if $a = 3$ and $b = -4$.

Solution. Without parentheses, we might think $ab = 3-4 = -1$, mistakenly turning multiplication into subtraction. The correct evaluation is

$$ab = (3)(-4) = -12.$$

$\square$

To evaluate complicated expressions consistently, we must follow the **order of operations**, which is restated below for convenience.

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

Recall that "*in order of appearance*" means in order *from left to right*, and that grouping symbols include parentheses, brackets, braces (curly brackets), the square root symbol, and the fraction bar.

Example 197. Evaluate the expression $x - 3y$ if $x = 4$ and $y = -\frac{1}{2}$.

Solution. Substituting the assigned values, and multiplying first according to the order of operations,

$$\begin{aligned} x - 3y &= (4) - 3\left(-\frac{1}{2}\right) \\ &= 4 + \frac{3}{2} \\ &= \frac{8}{2} + \frac{3}{2} \quad = \quad \frac{11}{2} \quad = \quad 5\frac{1}{2}. \end{aligned}$$

$\square$

Example 198. Evaluate $a - b - c$ and $a - (b - c)$ if $a = 2$, $b = -11$ and $c = 10$.

Solution. The first expression is evaluated as follows:

$$\begin{aligned} a - b - c &= (2) - (-11) - (10) \\ &= 13 - 10 \qquad \text{(left subtraction before right subtraction)} \\ &= 3. \end{aligned}$$

For the second expression,

$$\begin{aligned} a - (b - c) &= (2) - \Big((-11) - (10)\Big) \\ &= 2 - (-21) \qquad \text{(subtraction within grouping symbols first)} \\ &= 23. \end{aligned}$$

$\square$

Example 199. Evaluate $3 - (p - r \div t)$ if $p = \frac{3}{4}$, $r = -\frac{1}{8}$ and $t = \frac{5}{12}$.

Solution. Within grouping symbols, division comes first.

$$\begin{aligned} 3 - (p - r \div t) &= 3 - \left[\frac{3}{4} - \left(-\frac{1}{8}\right) \div \left(\frac{5}{12}\right)\right] \quad \text{(changing parentheses to brackets for clarity)} \\ &= 3 - \left[\frac{3}{4} - \left(-\frac{1}{\cancelto{2}{8}}\right) \cdot \left(\frac{\cancelto{3}{12}}{5}\right)\right] \quad \text{(division as multiplication by the reciprocal)} \\ &= 3 - \left[\frac{3}{4} - \left(-\frac{3}{10}\right)\right] \\ &= 3 - \left[\frac{3}{4} + \frac{3}{10}\right] \quad \text{(subtraction within grouping symbols first)} \\ &= 3 - \left[\frac{15}{20} + \frac{6}{20}\right] \quad \text{(using the LCD = 20)} \\ &= 3 - \left[\frac{21}{20}\right] \quad = \quad \frac{60}{20} - \frac{21}{20} \quad = \quad \frac{39}{20} \quad = \quad 1\frac{19}{20}. \end{aligned}$$

$\square$

Example 200. Evaluate $\dfrac{3x^3 - 5x^2 + 13x - 4}{x^2 + 2}$ if $x = 0$.

Solution. $\dfrac{3(0)^3 - 5(0)^2 + 13(0) - 4}{0^2 + 2} = \dfrac{-4}{2} = -2.$ $\square$

Example 201. Evaluate the expressions $\sqrt{a^2 + b^2}$ and $a + b$ if $a = -0.6$, and $b = 0.8$.

Solution. For the first expression,

$$\begin{aligned} \sqrt{a^2 + b^2} &= \sqrt{(-0.6)^2 + (0.8)^2} \\ &= \sqrt{.36 + .64} = \sqrt{1} = 1. \end{aligned}$$

For the second expression,

$$\begin{aligned} a + b &= -0.6 + 0.8 \\ &= 0.2. \end{aligned}$$

This example shows that, in general,

$$\sqrt{a^2 + b^2} \neq a + b.$$

□

5.7.1 Exercises

Evaluate the expressions for the given values of the variables. Reduce fractions to lowest terms and express improper fractions as mixed numbers.

1. $2a - b$ if $a = 11$ and $b = -5$
2. $2rt$ if $r = 5$ and $t = -6$
3. $-6x^2$ if $x = -\frac{2}{3}$
4. $p^2 - q^2$ if $p = -2$ and $q = 3$
5. $a - ab - b$ if $a = -0.6$ and $b = 0.8$
6. $-x^2 + 3y^2 - 8y + 5$ if $x = -2$ and $y = 0$
7. $(d - e)(d^2 + ed + e^2)$ if $d = -1$ and $e = -4$
8. $\dfrac{2x + y}{x - 2y}$ if $x = 4$ and $y = -6$
9. $2pq - q^2$ if $p = 1.2$ and $q = 2.3$
10. $\dfrac{3a - 2b}{5a + b}$ if $a = 1\frac{1}{2}$ and $b = \frac{1}{4}$
11. $\sqrt{x^2 + y^2}$ if $x = \frac{2}{3}$ and $y = -\frac{1}{2}$
12. $-b + \sqrt{b^2 - 4ac}$ if $a = -2$, $b = 7$ and $c = -3$

5.8 Using Formulae

A **formula** expresses one quantity in terms of others. For example, the formula for the area A of a rectangle of length l and width w is

$$A = lw.$$

If we know the length and width of a rectangle, we can calculate its area, using the formula. For example, the area of a rectangle of width 17 feet and length 50 feet is

$$A = lw = (17)(50) = 850 \text{ square feet.}$$

Example 202. The formula for the distance s (in feet) that an object falls in a time t (in seconds) is

$$s = 16t^2.$$

Use the formula to determine the distance an object falls in (a) 2 seconds; (b) 4 seconds.

Solution. We substitute $t = 2$ and $t = 4$ into the formula. In 2 seconds the object falls

$$s = 16(2)^2 = 64 \text{ feet},$$

while, in 4 seconds it falls

$$s = 16(4)^2 = 256 \text{ feet}.$$

$\square$

Example 203. The formula which converts temperature in degrees Celsius ($^\circ C$) to temperature in degrees Fahrenheit ($^\circ F$) is

$$F = \frac{9}{5}C + 32.$$

Find the temperature in $^\circ F$ if a thermometer reads $-10^\circ C$.

Solution. Substituting (-10) for C in the formula, we find

$$\begin{aligned} F &= \frac{9}{5}(-10) + 32 \\ &= -\frac{9}{\cancel{5}^{1}} \cdot \frac{\cancel{10}^{2}}{1} + 32 \\ &= -18 + 32 = 14. \end{aligned}$$

The temperature is 14° F. $\square$

Example 204. The *Pythagorean theorem* gives a formula for the length c of the hypotenuse of a right triangle in terms of the lengths a, b of its legs.

$$c = \sqrt{a^2 + b^2}.$$

Find the length of the hypotenuse of a right triangle whose legs have lengths 5 and 12 feet.

Solution. We put $a = 5$ and $b = 12$ in the Pythagorean formula.

$$c = \sqrt{a^2 + b^2} = \sqrt{(5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \text{ feet}.$$

$\square$

Example 205. The *semi-perimeter* of a triangle with side lengths a, b and c is given by the formula

$$s = \frac{1}{2}(a + b + c).$$

Find the semi-perimeter of a triangle with side lengths $a = 6$ ft 8 in, $b = 9$ ft 4 in, and $c = 11$ ft 10 in.

Solution. We convert all measurements to inches, using the fact that 1 ft = 12 in. Thus

$$a = 80 \text{ in}, \qquad b = 112 \text{ in}, \qquad \text{and} \qquad c = 142 \text{ in}.$$

The semi-perimeter is

$$\begin{aligned} s = \frac{1}{2}(a+b+c) &= \frac{1}{2}(80+112+142) \\ &= 167 \text{ in} \\ &= 13 \text{ ft} \quad 11 \text{ in}. \end{aligned}$$

□

Example 206. The child's dosage for a medicine is given by the formula

$$C = \frac{t}{t+12} \cdot A,$$

where C is the child's dosage, A is the adult dosage, and t is the child's age in years. Find the dosage for a four-year old child if the adult dosage is 48 mg (milligrams).

Solution. Substituting $t = 4$ and $A = 48$,

$$C = \frac{4}{4+12} \cdot 48 = \frac{1}{4} \cdot \frac{48}{1} = 12.$$

The child's dosage is 12 mg. □

5.8.1 Exercises

1. Find the area of a rectangle whose length is 4.8 meters and whose width is 3.6 meters. Use the formula $A = lw$, where A is the area, l is the length, and w is the width.
2. Find the perimeter of the rectangle in the preceding exercise, using the formula $P = 2l + 2w$, where P is the perimeter and l, w are the length and width, respectively.
3. Find the length of the hypotenuse of a right triangle whose legs are 0.3 yards and 0.4 yards. Use the Pythagorean theorem.
4. How far does an object fall in 3 seconds? Use the formula $s = 16t^2$, where s is the distance fallen (in ft), and t is the time (in sec).
5. If a thermometer reads 22°C, find the temperature in °F. Use the formula $F = \frac{9}{5}C + 32$.
6. Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, gives the area ($A$) of a triangle (not necessarily a right triangle) with side lengths a, b and c, and semi-perimeter s. Use Heron's formula and the semi-perimeter formula $s = \frac{1}{2}(a+b+c)$ to find the area of a triangle with $a = b = c = 2$ ft.
7. The formula $A = P(1+r)^t$ gives the amount A of money in a bank account t years after an initial amount P is deposited, when the annual interest rate is r. Find A after 2 years if the interest rate is 5% ($r = .05$), and the initial deposit was $P = \$500$.
8. Find the child's dosage for a 10-year old, if the adult dosage is 2.2 grams. Use the formula $C = \frac{t}{t+12} \cdot A$, where C is the child's dosage, t is the child's age, and A is the adult dosage.

5.9 Linear Equations in One Variable

An equation is a statement that two mathematical expressions are equal. If the statement has just one variable, and if

- the variable does not appear in the denominator of a fraction,
- the variable does not appear under a $\sqrt{\ }$ symbol,
- the variable is not raised to a power other than 1,

then we have a **linear equation in one variable**. Here are some examples of linear equations in one variable:

$$2x + 4 = 8 \qquad -3y = 12 \qquad z - 9 = -1 \qquad 2 - \frac{2}{3}t = 0.$$

A **solution** to an equation in one variable is a number which, when substituted for the variable, makes a true statement.

Example 207. Show that -4 is a solution to the equation $-3y = 12$

Solution. When we substitute -4 for y in the equation, we get

$$\begin{aligned} -3(-4) &= 12 \\ 12 &= 12, \qquad \text{a } \textit{true} \text{ statement.} \end{aligned}$$

□

Example 208. Show that 10 is *not* a solution to the equation $z - 9 = -1$.

Solution. When we substitute 10 for z in the equation, we get

$$\begin{aligned} 10 - 9 &= -1 \\ -1 &= 1, \qquad \text{a } \textit{false} \text{ statement.} \end{aligned}$$

□

Example 209. Show that 3 is a solution to the equation $2 - \frac{2}{3}t = 0$, but 4 is not.

Solution. When we substitute 3 for t, we obtain

$$\begin{aligned} 2 - \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{1} &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0, \qquad \text{a } \textit{true} \text{ statement.} \end{aligned}$$

When we substitute 4 for t, we obtain

$$\begin{aligned} 2 - \frac{2}{3} \cdot \frac{4}{1} &= 0 \\ 2 - \frac{8}{3} &= 0, \\ -\frac{2}{3} &= 0, \qquad \text{a } \textit{false} \text{ statement.} \end{aligned}$$

□

Here is the formal definition of the subject of this section.

> A linear equation in one variable x is an equation that can be written in the form
>
> $$ax + b = c$$
>
> for some numbers a, b and c, with $a \neq 0$.

Such an equation always has a **unique** (i.e., exactly one) solution.

5.9.1 Finding solutions

We find solutions to equations using two common-sense principles:

- Adding equals to equals produces equals,
- Multiplying equals by equals produces equals.

For example, 2 and 2 are equals, and so are 3 and 3. Adding equals to equals,

$$\begin{aligned} 2 &= 2 \\ 3 &= 3 \\ 2+3 &= 2+3 \end{aligned}$$

produces equals: $5 = 5$.

Consider the linear equation

$$x - 3 = 6.$$

Although we cannot say that $x - 3$ and 6 are "equals" (without knowing the value of x) we can say that *if* they are equal *for some* x, then adding equals to both sides, or multiplying both sides by equals, will produce a new pair of equals *for the same* x. In particular, if $x - 3 = 6$ is true for some x, so is the equation obtained by adding 3 to both sides:

$$\begin{array}{rcl} x - 3 &=& 6 \\ \underline{\quad +3} && \underline{+3 \quad} \\ x &=& 9. \end{array}$$

The solution to the last equation is obvious (and obviously unique): x must be 9. It is slightly less obvious that 9 is a solution of the original equation. Could there be some *other* solution to the original equation, say, $x = p$? If so, then $p - 3 = 6$, and adding 3 to both sides yields $p = 9$. So p is no different from the solution we already found. We conclude that 9 is the **unique** solution of $x - 3 = 6$.

Example 210. Find the solution of the equation $9 = -\frac{1}{5}z$ by multiplying both sides by equals. Check that you have indeed obtained the unique solution.

Solution. If we multiply both sides of the equation by -5, we get

$$\begin{aligned} -5(9) &= -5\left(-\frac{1}{5}\right)z \\ -45 &= z. \end{aligned}$$

To check that -45 is the solution, substitute -45 for z in the original equation, and verify that a true statement results.

$$\begin{aligned} 9 &= -\frac{1}{5}z \\ 9 &\stackrel{?}{=} -\frac{1}{5}(-45) \\ 9 &\stackrel{?}{=} \frac{\cancelto{9}{45}}{\cancelto{1}{5}} \\ 9 &= 9 \qquad\qquad \text{a true statement.} \end{aligned}$$

$\square$

Equations which have the same solution are called **equivalent**. As the previous examples show, a strategy for finding the solution of an equation is to systematically transform it into simpler equivalent equations, until we arrive at an equation like $x = 8$, or $-45 = z$, whose solution is obvious.

> To find the solution (solve) a linear equation in one variable, we can do one or both of the following:
>
> - add the same number to both sides of the equation;
> - multiply both sides of the equation by the same nonzero number.

We can replace "add" by "subtract," and "multiply" by "divide," if convenient. (This is because subtracting a number is the same as adding the opposite number, and dividing by a nonzero number is the same as multiplying by the reciprocal number.)

Example 211. Solve the equation $2x + 4 = -10$ and check that the solution is correct.

Solution. First subtract 4 from both sides:

$$\begin{aligned} 2x + 4 &= -10 \\ -4 \quad &\quad -4 \\ \hline 2x &= -14. \end{aligned}$$

Then divide both sides by 2:

$$\begin{aligned} 2x &= -14 \\ \frac{\cancel{2} \cdot x}{\cancel{2}} &= \frac{\cancelto{-7}{-14}}{\cancelto{1}{2}} \\ x &= -7. \end{aligned}$$

The solution is -7. To check, we substitute -7 for x in the original equation, and see if a true statement results.

$$\begin{aligned} 2x + 4 &= -10 \\ 2(-7) + 4 &\stackrel{?}{=} -10 \\ -14 + 4 &\stackrel{?}{=} -10 \\ -10 &= -10 \qquad \text{a true statement} \end{aligned}$$

□

Example 212. Solve the equation $26 = 5 - \frac{7}{8}x$ and check the solution.

Solution. At some point it will be necessary to multiply both sides by the reciprocal of $-\frac{7}{8}$. But it is better not to do that immediately. If we first subtract 5 from both sides, the computations are shorter.

$$\begin{aligned} 26 &= 5 - \frac{7}{8}x \\ \underline{-\quad 5} &\ \ \underline{-\quad 5\qquad} \\ 21 &= -\frac{7}{8}x \end{aligned}$$

Now we multiply both sides by $\left(-\frac{8}{7}\right)$

$$\begin{aligned} -\frac{8}{7} \cdot 21 &= \left(-\frac{8}{7}\right)\left(-\frac{7}{8}x\right) \\ -\frac{8}{\cancel{7}} \cdot \cancel{21}^{3} &= \left(-\frac{\cancel{8}}{\cancel{7}}\right)\left(-\frac{\cancel{7}}{\cancel{8}}x\right) \\ -24 &= x \end{aligned}$$

□

A linear equation given in the standard form

$$ax + b = c \qquad (a \neq 0)$$

is solved in two steps: *first,* subtract b from both sides; *second,* divide both sides by a.

Example 213. Solve the equation $-x + 5 = 12$.

Solution. Note that $-x$ means $-1x$. Following the two step procedure, we first subtract 5 from both sides, and then divide both sides by -1.

$$\begin{aligned} -x + 5 &= 12 \\ -x &= 7 \qquad \text{(subtracting 5 from both sides)} \\ x &= -7 \qquad \text{(dividing or multiplying both sides by } -1) \end{aligned}$$

Note that dividing by -1 is the same as multiplying by -1, since -1 is its own reciprocal! □

Example 214. Solve $3x + \frac{1}{2} = \frac{1}{2}$.

Solution. We first subtract $\frac{1}{2}$ from both sides.

$$\begin{aligned} 3x + \frac{1}{2} &= \frac{1}{2} \\ 3x &= \frac{1}{2} - \frac{1}{2} \\ 3x &= 0 \\ x &= 0. \end{aligned}$$

At the last step, we divided both sides by 3. Recall that $\frac{0}{3} = 0$. □

Example 215. Four times a number is -5.6. What is the number?

Solution. Let x represent the unknown number. The sentence "Four times a number is -5.6" is a verbal form of the equation

$$4x = -5.6.$$

Dividing both sides by 4 yields

$$x = \frac{-5.6}{4} = -1.4.$$

The number is -1.4. It is straightforward to check that four times -1.4 is indeed -5.6. □

5.9.2 Exercises

Determine if the given value of the variable is a solution of the equation.

1. $y + 1.6 = 14.4, \quad y = 2$
2. $2x + 3 = 19, \quad x = 8$

Solve the equations and check the solutions.

3. $x - 5 = 19$
4. $6 + y = -2$
5. $9x = 45$
6. $\frac{2}{5}t = 24$
7. $-y - 14 = 5$
8. $\frac{x}{8} = -4$
9. $19t - 5 = -5$
10. $4x - 5 = 7$
11. $2y - 1.6 = 13.2$

12. $\frac{2}{3} - \frac{x}{5} = \frac{-14}{15}$

13. Twelve times a number is 108. What is the number?

14. Five more than a number is $-9\frac{1}{2}$. What is the number?