Review Sheet

CSI 30: Discrete Mathematics I

DEPARTMENT OF MATHEMATICS and COMPUTER SCIENCE

Logic and Proofs

1. Write each of these statements as an English sentence of the form "if p, then q"

   (a) I will remember our appointment only if you remind me.

   (b) The team will win the game if their regular pitcher recovers.

   (c) It is necessary to pay for a ticket to get into the movies.

   (d) You can take out more library books only if you pay your fines.

   (e) To pass this course it is sufficient that you study hard.

2. Determine if the biconditional is true or false.

   $2^4 > 10$ if and only if the earth has shape of a plate.

3. Let p and q be the propositions

   p:"You drive over 65 miles per hour"

   q: "You get speeding ticket"

   Write these propositions using p and q and logical connectives.

   a) Driving over 65 miles is sufficient for getting a speeding ticket.

   b) You get a speeding ticket, but you didn't drive over 65 miles per hour

   c) You drive over 65 miles per hour, but you don't get a speeding ticket.

4. Let p and q be the propositions "The election is decided" and "The votes have been counted", respectively. Express each of these compound propositions as an English sentence.

   a) $\neg p \wedge q$ **b)** $p \leftrightarrow q$ **c)** $\neg p \rightarrow \neg q$

5. Use De Morgan's Laws to find the negation of each of the following statements.

   (a) Yoshiko knows Java and calculus

   (b) Rita will move to Oregon or Washington

6. Construct a truth table for the compound proposition

   $(\neg p \rightarrow \neg q) \wedge (q \leftrightarrow p)$

7. Show that the compound proposition

   $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

   is a tautology using a truth table.

8. Show that the compound proposition

   $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

   is a tautology without using truth tables, but using laws instead.

9. Show that

   $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$

   are logically equivalent.

10. Determine the truth value of each of these statements if the domain of each variable consists of all integers. Explain your answer.

    a) $\exists n\ (n^2 = 81)$

    b) $\forall n\ (n^2 \neq n)$

11. Let $T(x)$ be the statement 'x has a Cable TV' and $C(x, y)$ be the statement 'x and y watch the same TV show'. The domain for the variables x and y consists of all students in your class. Use quantifiers and logical connectives to express each statement:
    a. Flora doesn't have a Cable TV

    b. Exactly one student in your class has a Cable TV.

    c. No one in the class has watched the same TV show with Joe.

12. Express the given statement using predicates, quantifiers, logical connectives, and mathematical operators. Domain: all real numbers.

    'There exists a real number such that if any real number is multiplied by it, we get 0 '

13. Let I(x) be the statement "x has an Internet connection" and C(x,y) be the statement "x and y have chatted over the Internet", where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.

    a) There are two students in your class who have not chatted with each other over the Internet.

    b) There is a student in your class who has chatted with everyone in your class over the Internet.

14. Rewrite the statement

    $\neg\exists y(\forall x\ \exists z\ T(x, y, z)\ \vee\ \exists x \forall z\ U(x, y, z))$

    so that negations appear only within predicates.

15. Write the converse, inverse and the contrapositive of the statement "If you are a Computer Science major, then you know Discrete Mathematics."

16. Here are three premises:

    1. Every bird has two feet.
    2. Every insect has six feet.
    3. Polly has two feet.

    If we conclude "Polly is a bird," have we made a valid argument? If not, why not?

17. Which of the following statements is NOT logically equivalent to the biconditional $p \leftrightarrow q$?

    (a) $(p \rightarrow q) \wedge (q \rightarrow p)$

    (b) $(\neg p \vee q) \wedge (\neg q \vee p)$

    (c) $\neg q \rightarrow \neg p$

Sets

18. List the members of the following set.

    $\{x|x$ `is an integer such that` $x^2 < 25\}$

19. For sets $A = \{1, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 5\}$, and $C = \{1, 3, 5, 7, 9, 11\}$, determine which of the following statements are true or false.

    a) $A \subseteq B$

    b) $A \subseteq C$

    c) $B \subseteq C$

20. Determine whether the following statements are true or false.

    a) $3 \in \{\emptyset, 3, \{3\}\}$

    b) $\{3, 1\} \subseteq \{1, \{2\}, \{3\}, \{1, 3\}\}$

21. Let the universal set be $U = \{1, 2, 3, 4, 5, 6, 7\}$, with subsets $A = \{1, 2\}$, $B = \{2, 3, 4\}$, $C = \{4, 5, 6, 7\}$. Find

    (a) $A \times C$

    (b) $A \bigcap B \bigcap C$

    (c) $|C|$

    (d) $\overline{B}$

    (e) $A - B$

    (f) $(A \bigcup C) \bigcap B$

22. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 5, 6, 7, 8\}$, $B = \{2, 3, 5, 7\}$. Give bitstring representations of the following sets:

(a)	A	(b)	$A \cap B$
(c)	B	(d)	$A \cup B$
(e)	$B - A$	(f)	$\emptyset$
(g)	$A - B$	(h)	$\mathcal{U}$

23. Use set identities to prove the following identity:

    $\overline{A \cup \overline{B}} = \overline{A} \cap B$

Functions

24. Determine whether the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^5 + 1$, is

    a) one-to-one

    b) onto

    c) a bijection

25. Determine the range of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 3$

26. Let $\mathcal{B}$ be the set of all bit strings. Prove or show false by giving a counterexample: The function $f : \mathcal{B} \to \mathbb{Z}$ defined by $f(s) =$ the sum of the bits in the string s, or 0, if s is the empty string, is

    (a) one-to-one

    (b) onto

27. Let $\mathcal{B}$ be the set of all bitstrings. Which of the following defines a function from $\mathcal{B}$ to $\mathbb{Z}$? If a function is not defined, what part of the definition of 'function' fails?

    (a) f(S) = the position of the first 1 bit in the string S.

    (b) g(S) = the sum of all the bits in the string S.

    (c) h(S) = the position of the last 0 bit in the string S.

28. Let $f(x) = 2x$ and $g(x) = 3x + 5$, be functions from **R** to **R**. Find formulas for the composite functions $f \circ g$ and $g \circ f$.

29. Is function $g(x) = x^3 - 4$, with domain and codomain $\mathbb{R}$, invertible? If yes, find its inverse function. If not, explain why. How do your answers change If the domain and codomain is reduced to $\mathbb{Z}$?

Algorithms

30. Answer the following questions about the algorithm:

```
procedure it(n:positive integer)
sum := 0
For i:= 1 to n
    sum := sum + i*i
End-for
Return(sum)
```

    (a) How many multiplication operations will be done? (The answer is an expression in terms of `n`.)

    (b) If $n = 3$, what value will be returned?

31. Given the algorithm:

```
procedure thing(a_1,a_2,a_3,...a_n:integers)
sum1 := 0
sum2 := 0
For i := 1 to n
   If (a_i > 0), sum1 := sum1 + a_i
   If (a_i < 0), sum2 := sum2 + a_i
End-for
Return(sum1,sum2)
```

For the set of values {1,5,-2,-9,2,5,-7} as input for the above algorithm, what are values of `sum1` and `sum2` that will be returned?

32. Consider the algorithm:

```
procedure foo(n:integer)
If n > 10, print('A')
If (n <= 10) and (n > -10), print('B')
Else print('C')
```

(a) What will be printed if the procedure `foo` is run on `n=4`?

(b) What will be printed if the procedure `foo` is run on `n=-4`?

(c) What will be printed if the procedure `foo` is run on `n=24`?

33. How many comparisons does the linear search algorithm make in finding 13 in the following list? $1, 7, 2, 3, 6, 8, 13, 4, 89$

34. Use binary search to find the location of 14 in the following list: $1, 6, 8, 9, 13, 14, 16, 22, 36, 38$

35. Use binary search to find the location of 10 in the following list: $9, 10, 14, 16, 22, 36, 56, 59, 61$

36. Use **bubble sort** to put the list $5, 2, 4, 1, 3$ into increasing order. How many steps are performed?

37. Find the greatest common divisor $GCD(231, 42)$ using the Euclidean Algorithm:

`procedure GCD (a, b:  positive integers)`

$x := a$

$y := b$

While $y \neq 0$

$r := x \textbf{ mod } y$

$x := y$

$y := r$

Return$(x = \mathtt{GCD}(a, b))$

Integers

38. Find

    a) 243 **div** 13 **b)** 243 **mod** 13 **c)** (-100) **div** 23 **d)** (-100) **mod** 23

39. State True or False, explain why.

    $12 \equiv 54(\mathbf{mod}\ 7)$

40. Find the base 5 expansion of 187.

41. Find the base 16 expansion of 187. (Use the symbols $A = 10$, $B = 11$, $\ldots$, $F = 15$.)

42. Find the binary expansion of 49.

43. Compute the decimal representation of $(12112)_3$

44. Compute the decimal representation of $(77501)_8$

45. Compute the decimal representation of $(A501)_{16}$. ($A = 10$, $B = 11$, $\ldots$, $F = 15$.)

46. Compute 5^6 **mod** 7 without use of calculator.

47. What is the value of $26 + 28$ **mod** 30?

48. What is the value of 16×20 **mod** 3?

49. Check which integers are multiplicative inverses of 5 `mod` 59.

    a) 12 **b)** 46 **c)** 71

50. Consider the following generator of pseudo-random numbers:

    $x_n = (5x_{n-1} + 7) \ \mathtt{mod}\ 12$, with seed $x_0 = 4$.

    What sequence of pseudo-random numbers does it generate?

Counting

51. At a bakery, there are four shapes of bread that can be made: roll, bowl, loaf, and extra large loaf. Each bread can be of type: Ciabatta, Country, Whole Grain, Classic White, and Focaccia. Each bread can be sliced or not. How many different choices of bread are available?

52. How many bit-strings of length 16 are there, that start with 0 or end with 11?

53. List all 3-permutations and 3-combinations (3-subsets) of the set $S = \{1, 2, 3, 4, 5\}$.

54. Find the values of $P(16, 5)$ and $C(26, 4) = \binom{26}{4}$.

55. How many bit-strings of length 14 contain

   (a) five 0's in the beginning?
   (b) at least four 1's?
   (c) number of 1's and the number of 0's are the same?

56. In how many ways can the letters of the word 'COLUMN' be arranged?

57. In how many ways can the letters of the word 'ABRAKADABRA' be arranged?

58. There are 10 men and 8 women in a room. In how many ways we can choose 4 men and 4 women from the room?

59. How many integers from 1 to 50 have no prime divisors other than 2 or 3?

60. In a group of 50 students, 24 like tea and 36 like coffee. Each student likes at least one of the two drinks. How many like both coffee and tea?

61. How many passwords of length 5, that contain only lower case letters and four special symbols $`, -, +, \_$ can be made?

62. What is the eighth row of Pascal's Triangle?

63. What is the coefficient of x^2y^5 in the expansion of the binomial $(3x - 2y)^7$?

64. What is the probability that a 5-card poker hand contains (a) exactly three hearts? (b) at least three hearts?

Answers

Logic and Proofs. 1.(a) If I remember our appointment, then you remind(ed) me. (b) If their regular pitcher recovers, the team will win the game. (c) If you get into the movies, then you paid for a ticket. (d) If you take out more library books, then you paid your fines. (e) If you study hard, then you (will) pass this course. **2**. True: the biconditional $p \leftrightarrow q$ is true in all cases except when p is true and q is false. In this example both p and q are false. **3**. (a) $p \to q$. (b) $q \wedge \neg p$. (c) $p \wedge \neg q$. **4.** (a) "The votes have been counted, but the election is not decided." (Note: 'but' is a form of 'and.') (b) "The election is decided if and only if the votes have been counted." (c) "If the election is not decided, then the votes have not been counted." **5**. (a) "Yoshiko doesn't know Java, or doesn't know calculus." (b) "Rita will not move to Oregon, and will not move to Washington."

6.

p	q	$\neg p$	$\neg q$	$\neg p \to \neg q$	$q \leftrightarrow p$	$(\neg p \to \neg q) \wedge (q \leftrightarrow p)$
t	t	f	f	t	t	t
t	f	f	t	t	f	f
f	t	t	f	f	f	f
f	f	t	t	t	t	t

7.

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$\neg q \wedge (p \to q) \to \neg p$
t	t	f	f	t	f	t
t	f	f	t	f	f	t
f	t	t	f	t	f	t
f	f	t	t	t	t	t

8. Modus Tollens. Alternatively, using $p \to q \equiv \neg p \vee q$,

$$\begin{aligned}
(\neg q \wedge (p \to q)) \to \neg p &\equiv \neg q \wedge (\neg p \vee q) \to \neg p \\
&\equiv (\neg q \wedge \neg p) \vee (\neg q \vee q) \to \neg p && \text{Distributive Law} \\
&\equiv (\neg q \wedge \neg p) \vee \mathbb{T} \to \neg p && \text{Negation Law} \\
&\equiv (\neg q \wedge \neg p) \to \neg p \\
&\equiv \neg p \to \neg p \\
&\equiv \neg p \vee p \\
&\equiv \mathbb{T} && \text{Idempotent Law}
\end{aligned}$$

9.

p	q	r	$p \to q$	$p \to r$	$p \to q \vee p \to r$	$q \vee r$	$p \to (q \vee r)$
t	t	t	t	t	t	t	t
t	t	f	t	f	t	t	t
t	f	t	f	t	t	t	t
t	f	f	f	f	f	f	f
f	t	t	t	t	t	t	t
f	t	f	t	t	t	t	t
f	f	t	t	t	t	t	t
f	f	f	t	t	t	f	t

The sixth and eighth columns have identical truth values, so the corresponding statements are logically equivalent. **10**. (a) True: $n = 9$ or $n = -9$ are instantiations. (c) False: $n = 1$ is a counterexample. **11**. (a) $\neg T(\text{Flora})$. (b) $\exists! x\ T(x)$. Alternative: $\exists x\ T(x) \wedge [(T(y) \to (y = x)]$. (c) $\forall x\ \neg T(x, \text{Joe})$. **12**. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R},\ xy = 0$. **13.** (a) $\exists x \exists y[(x \neq y) \wedge \neg C(x, y)]$. (b) $\exists x\ \forall y\ C(x, y)$. **14**. $\forall y(\exists x \forall z \neg T(x, y, z) \wedge \forall x \exists z \neg U(x, y, z))$. **15**. Converse: "If you know Discrete Mathematics, then you are a Computer Science Major." Inverse: "If you are not a Computer Science Major, then you don't know Discrete Mathematics." Contrapositive: "If you don't know Discrete Mathematics, then you are not a Computer Science major." **16**. Not valid: fallacy of affirming the conclusion of premise 1. **17.** (c) is equivalent to $p \to q$, but not equivalent to $q \to p$.

Sets. 18. $-4, -3, -2, -1, 0, 1, 2, 3, 4$. **19**. (a) False. (b) True. (c) False. **20** (a) True. (b)

False. **21**. (a) $\{(1,4),(1,5),(1,6),(1,7),(2,4),(2,5),(2,6),(2,7)\}$. (b) $\emptyset$. (c) 4. (d) $\{1,5,6,7\}$. (e) $\{1\}$. (f) $\{2,4\}$. **22.** (a) 10001111. (b) 00001010. (c) 01101010. (d) 11101111. (e) 01100000. (f) 00000000. (g) 10000101. (h) 11111111. **23.** $\overline{A \cup \overline{B}} = \overline{A} \cap \overline{\overline{B}}$, by DeMorgan's Law. Since $\overline{\overline{B}} = B$, we obtain $\overline{A} \cap B$.

Functions. 24. (a) By definition, f is one-to-one if $[f(x) = f(y)] \to [x = y]$. Suppose $x^5 + 1 = y^5 + 1$. Then $x^5 = y^5$, which implies $\sqrt[5]{x} = \sqrt[5]{y}$, which implies $x = y$, since every real number has a unique 5th root. Hence f is one-to-one. (b) By definition, f is onto if $\forall(y \in \mathbb{R})\ \exists(x \in \mathbb{R})$ such that $x^5 + 1 = y$. Given y, the appropriate choice of x is $x = \sqrt[5]{y-1}$, which exists since every real number has a unique 5th root. Hence f is onto. (c) By definition, f is a bijection if it is one-to-one and onto. By (a) and (b), f is a bijection. **25**. Since $\forall x$, $x^2 \geq 0$, it follows that $\forall x, f(x) \geq 3$. Hence the range of f is a subset of the interval $[3,\infty]$. Given any $y \in [3,\infty]$, $x = \sqrt{y-3}$ exists, and $f(x) = y$. Hence the range of f is all of the interval $[3,\infty)$. (Alternative: the graph of f is an upward-opening parabola with vertex at $(0,3)$; the range is evident from the graph.) **26.** f is not one-to-one since, e.g., $01 \neq 010$ but $f(01) = f(010) = 1$. f is not onto since, e.g., $\forall s \in \mathcal{B}$, $f(s) \neq -1$. **27.** A function must assign a unique element of the codomain to each element of the domain. (a) does not define a function since it does not assign any integer to a string of all 0's or the to the empty string. (b) does not define a function since it does not assign any integer to the empty string. (c) does not define a function since it does not assign any integer to a string of all 1's or to the empty string. **28**. $(f \circ g)(x) = f(g(x)) = 2(3x+5) = 6x+10$. $(g \circ f)(x) = g(f(x)) = 3(2x)+5 = 6x+5$. **29**. Since $g : \mathbb{R} \to \mathbb{R}$ is one-to-one and onto, it has an inverse function. To find a formula for g^{-1}, solve $y = x^3 - 4$ for x. We obtain $x = \sqrt[3]{y+4}$. Interchanging the roles of x and y, we obtain the formula $g^{-1}(x) = \sqrt[3]{x+4}$. If the domain and codomain are restricted to $\mathbb{Z}$, then g is not onto, since not every integer can be expressed in the form $x^3 - 4$, $x \in \mathbb{Z}$. For example, 1 cannot be expressed in this form. Hence, in this case, g is not onto, therefore, not invertible.

Algorithms. 30. (a) At the ith step, the multiplication i^2 is performed, so there are n multiplications in total. (b) 14. **31**. `sum1` $= 14$, `sum2` $= -18$. **32.** (a) B, (b) B, (c) A.

33. The algorithm makes 7 comparisons within the `While` loop, and 1 in the `If ... Then` statement, for a total of 8 comparisons.

34.

i	j	m
1	10	5
6	10	8
6	8	7
6	7	6
6	6	(Break)

location = 6.

35.

i	j	m
1	9	5
1	5	3
1	3	2
1	2	1
2	2	(Break)

location = 2.

36.

i	j	a_1	a_2	a_3	a_4	a_5
1	1	5	2	4	1	3
1	2	2	5	4	1	3
1	3	2	4	5	1	3
1	4	2	4	1	5	3
2	1	2	4	1	3	5
2	2	2	4	1	3	5
2	3	2	1	4	3	5
3	1	2	1	3	4	5
3	2	1	2	3	4	5
4	1	1	2	3	4	5

steps = 4+3+2+1=10.

37.

x	y	r
231	42	21
42	21	0
21	0	(Break)

GCD = 21.

38. (a) 18 (b) 9 (c) -5 (d) 15. **39**. True: 54 and 12 differ by a multiple of 7. **40**. Successively dividing by 5 and recording the remainders, we get $2, 2, 2, 1$. Hence $(187)_{10} = (1222)_5$. **41**. Successively dividing by 16 and recording the remainders, we get $7, 11$ or $7, B$. Hence $(187)_{10} = (B7)_{16}$. **42**. Successively dividing by 2 and recording the remainders we get $1, 0, 0, 0, 1, 1$. Hence $(49)_{10} = (110001)_2$. **43**. $(12112)_3 = 2\times3^0+1\times3^1+1\times3^2+2\times3^3+1\times3^4 = 149$. **44.** $(77501)_8 = 1 \times 8^0 + 5 \times 8^2 + 7 \times 8^3 + 7 \times 8^4 = 1 + 320 + 3584 + 28672 = 32577$. **45**. $(A501)_{16} = 1 \times 16^0 + 5 \times 16^2 + 10 \times 16^3 = 42241$. **46.** The steps are: $5^2 = 4 \bmod 7$; $5^3 \equiv 5 \times 4 \equiv 6 \bmod 7$; $5^4 \equiv 5 \times 6 \equiv 2 \bmod 7$; $5^5 \equiv 5 \times 2 \equiv 3 \bmod 7$; $5^6 \equiv 5 \times 3 \equiv 1 \bmod 7$. **47.** 24. **48**. 2. **49**. (a) Yes: $5 \times 12 = 60 \equiv 1 \bmod 59$. (b) No: $5 \times 46 = 230 \not\equiv 1 \bmod 59$. (c) Yes: $5 \times 71 = 355 \equiv 1 \bmod 59$. **50**. $4, 3, 10, 9, 4, 3, 10, 9, \ldots$. **51**. $4 \times 5 \times 2 = 40$ choices. **52**. By the sum rule, $2^{15} + 2^{14} - 2^{13} = 16384$. **53**. 3-combinations: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$, $\{1, 4, 5\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$. 3-permutations: each of the previous can be reordered in $3! = 6$ ways, yielding a total of 60 ordered triples. **54**. $P(16, 5) = \dfrac{16!}{11!} = 16 \times 15 \times 14 \times 13 \times 12 = 524160$. $C(26, 4) = \dfrac{26!}{22!4!} = \dfrac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2 \times 1} = 14950$. **55**. (a) $2^9 = 512$. (b) $2^{14} - (C(14, 0) + C(14, 1) + C(14, 2) + C(14, 3)) = 15914$. Explanation: From the total number of strings, take away the number containing exactly 0, 1, 2 and 3 ones. (c) $2 \times C(14, 7) = 6864$. Explanation: Choose 7 positions for the ones; the other positions must contain zeroes. This can be done in $C(14, 7)$ ways. Switch the ones with the zeroes in each case; this doubles the number of strings. **56**. Since the letters are all different, the letters can be rearranged in $6! = 720$ ways. **57.** There are 11 letters, but not all are different: there are 5 A's, 2 B's, 1 D, 1 K and 2 R's. Hence the letters can be rearranged in $\dfrac{11!}{5!2!1!1!2!} = 83160$ distinguishable ways. **58**. $C(10, 4) \times C(8, 4) = 14700$. **59**. 14. **60**. Let T, C be the sets of students who like Tea, Coffee (respectively). Each student likes at least one of the two, so $\mid T \cup C \mid = 50$. With the given information $|T| = 24$, $|C| = 36$, we solve the sum rule $\mid T \cup C \mid = |T| + |C| - \mid T \cap C \mid$ for $\mid T \cap C \mid$, obtaining $\mid T \cap C \mid = 10$. **61**. $30^5 = 24300000$. **62**. 1 8 28 56 70 56 28 8 1. **63**. -6048. **64**. (a) $\dfrac{C(13, 3) \times C(39, 2)}{C(52, 5)} \approx .0815$.
(b) $\dfrac{C(13, 3) \times C(39, 2)}{C(52, 5)} + \dfrac{C(13, 4) \times C(39, 1)}{C(52, 5)} + \dfrac{C(13, 5) \times C(39, 0)}{C(52, 5)} \approx .0928$.

AW/NN June 2017

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