Harmonic maps from the 2-sphere to the m-sphere

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(an algebraic recipe)

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HARMONIC MAPS

A map $F : M \to N$ is *harmonic* if it is a critical point of the *energy* functional

$$F \longrightarrow \frac{1}{2} \int_D \|dF\|^2 dvol_M,$$

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For maps $F : S^2 \to S^m \subset \mathbb{R}^{m+1}$, the Euler-Lagrange equation is

$$\boxed{f : S^2 \to S^m \subset \mathbb{R}^{m+1} \text{ harmonic } \iff \frac{\partial^2 f}{\partial z \partial \bar{z}} = \lambda f}$$

where

- $(z, \bar{z})$ are holomorphic coordinates in S^2 (say stereographic projection).
- λ is a scalar function.

INITIAL GOAL

Find (all) maps

$$f : S^2 \to S^m$$

satisfying

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \lambda f.$$

STEP 1

If the image of f in S^m lies in a totally geodesic subsphere $S^k \subsetneq S^m$, then $f = i \circ \varphi$, where

- $\varphi : S^2 \to S^k$ is *linearly full*
- $i : S^k \to S^m$ is an isometric immersion.

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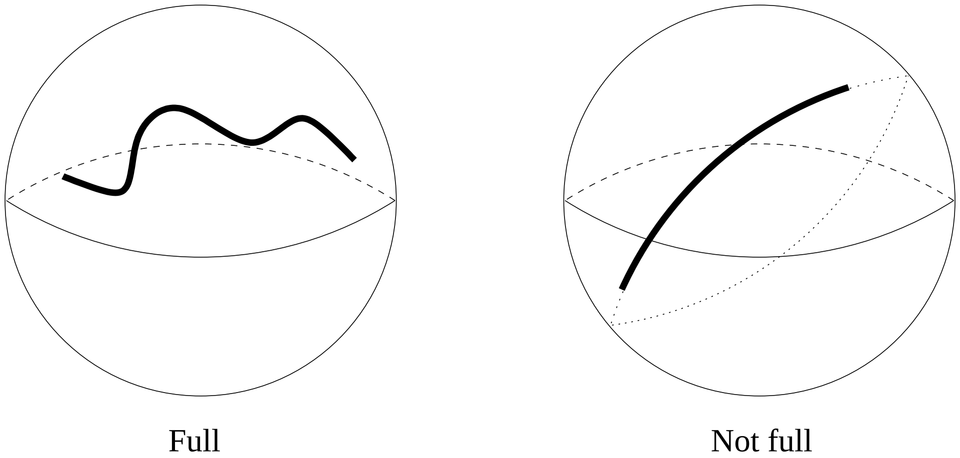
- $\varphi : S^2 \to S^k$ is *linearly full*
- $i : S^k \to S^m$ is an isometric immersion.

A map $\varphi : S^2 \to S^k$ is *linearly full* (or *full* for short) if its image does not lie in a proper geoedesic subsphere of S^k. Pictorially:



Hence, we only need to study linearly full maps $\varphi : S^2 \to S^k$.

MODIFIED GOAL 1

Find (all) linearly full maps

$$\varphi : S^2 \to S^k$$

satisfying

$$\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = \lambda \varphi.$$

FACTS (Calabi, 1967)

If $\varphi : S^2 \to S^k \subset \mathbb{R}^{k+1} \subset \mathbb{C}^{k+1}$ is harmonic and linearly full,

---

1. $\left(\dfrac{\partial^i \varphi}{\partial z^i}, \dfrac{\partial^j \varphi}{\partial z^j}\right) = \left(\dfrac{\partial^i \varphi}{\partial \bar{z}^i}, \dfrac{\partial^j \varphi}{\partial \bar{z}^j}\right) = 0,$ for $i + j \geq 1$.

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NOTATION:

For $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \in \mathbb{C}^N,$

$$(\vec{u}, \vec{v}) = \sum_{k=1}^{N} u_k v_k \quad \text{and} \quad \langle \vec{u}, \vec{v} \rangle = \sum_{k=1}^{N} u_k \bar{v}_k.$$

A subspace $V \subset \mathbb{C}^N$ is called *isotropic* if $(\vec{u}, \vec{v}) = 0 \;\; \forall \, \vec{u}, \vec{v} \in V$.

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2. k is even, say $k = 2n$. So we assume $\varphi : S^2 \to S^{2n} \subset \mathbb{R}^{2n+1} \subset \mathbb{C}^{2n+1}$.

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3. The *osculating plane*
$$\psi = \text{Span}\left\{\frac{\partial \varphi}{\partial \bar{z}}, \frac{\partial^2 \varphi}{\partial \bar{z}^2}, \ldots, \frac{\partial^n \varphi}{\partial \bar{z}^n}\right\}$$
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4. This induces a map $\psi : S^2 \to \mathcal{Z}_n \subset \mathrm{Gr}(n, \mathbb{C}^{2n+1})$, where
   - $\mathcal{Z}_n = \{P \in \mathrm{Gr}(n, \mathbb{C}^{2n+1}) : (\vec{u}, \vec{v}) = 0 \ \ \forall \vec{u}, \vec{v} \in P\}$ is a regular, compact, projective variety.
   - There is a surjective Riemannian submersion $\pi : \mathcal{Z}_n \to S^{2n}$ such that $\varphi = \pi \circ \psi$.

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5. Then: $$\left\{ \begin{matrix} \varphi : S^2 \to S^{2n} \\ \text{harmonic, full} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \psi : S^2 \to \mathcal{Z}_n \\ \text{holomorphic, horizontal, full} \end{matrix} \right\}$$

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6. Conversely:

$$\left\{\begin{array}{c}\psi : S^2 \to \mathcal{Z}_n \\ \text{holomorphic, horizontal, full}\end{array}\right\} \Rightarrow \left\{\begin{array}{c}\varphi = \pm\pi \circ \psi : S^2 \to S^{2n} \\ \text{harmonic, full}\end{array}\right\}$$

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5. Then: $$\left\{ \begin{matrix} \varphi : S^2 \to S^{2n} \\ \text{harmonic, full} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \psi : S^2 \to \mathcal{Z}_n \\ \text{holomorphic, horizontal, full} \end{matrix} \right\}$$
6. Conversely: $$\left\{ \begin{matrix} \psi : S^2 \to \mathcal{Z}_n \\ \text{holomorphic, horizontal, full} \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} \varphi = \pm\pi \circ \psi : S^2 \to S^{2n} \\ \text{harmonic, full} \end{matrix} \right\}$$
7. Furthermore (Barbosa, 1974) $$\text{Area}(\varphi(S^2)) = 4\pi d,$$ where d is the *algebraic degree* of ψ in $\mathcal{Z}_n$.

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NOTE: there exist such ψ if and only if $d \geq \dfrac{n(n+1)}{2}$ (Barbosa, 1976).

STEP 3

Use the following parametrization of $\mathcal{Z}_n$: given an orthonormal basis of $\mathbb{C}^{2n+1}$,

$$\beta = \{E_0, E_1, \ldots, E_n, \bar{E}_1, \ldots, \bar{E}_n\}$$

define the bi-rational map

$$\begin{aligned} b_\beta : \mathbb{CP}^{\frac{n(n+1)}{2}} &\quad \rightarrow \quad \mathcal{Z}_n \\ [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}] &\quad \rightarrow \quad \text{span}\left\{ \frac{\alpha_i}{s} E_0 + E_i - \sum_{j=1}^{n} \frac{\alpha_i \alpha_j + s\tau_{ij}}{2s^2} \right\}_{i=1}^{n} \end{aligned}$$

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Then a computation shows:

$$\left\{ \begin{array}{c} \psi : S^2 \to \mathcal{Z}_n \\ \text{holomorphic} \\ \\ \\ \end{array} \right\} \iff \left\{ \begin{array}{c} \widetilde{\psi} := b_\beta^{-1} \circ \psi : S^2 \to \mathbb{CP}^{\frac{n(n+1)}{2}} \\ \text{holomorphic} \\ \\ \\ \end{array} \right\}$$

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Find (all) holomorphic maps $\widetilde{\psi} : S^2 \to \mathbb{CP}^{\frac{n(n+1)}{2}}$ given by

$$z \to \widetilde{\psi}(z) = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$$

of degree d, satisfying, for $1 \leq i, j \leq n$,

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Then $\psi = b_\beta \circ \widetilde{\psi} : S^2 \to \mathcal{Z}_n$ will be a holomorphic, horizontal, linearly full map of degree d. (And all such ψ will have this form for some basis β.)

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We can assume:

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FURTHER, it suffices to consider the case where

- s has d simple complex zeros $z_1, z_2, \ldots, z_d$.
- $\alpha_1(z_\ell) \neq 0$ for all $\ell = 1, 2, \ldots, d$.

PARTICULAR CASE $n = 2$ and $n = 3$

For $n = 2$, the equations above read

$$\alpha_1'\alpha_2 - \alpha_1\alpha_2' = s\tau_{12}' - s'\tau_{12}$$

For $n = 3$,

$$\begin{cases} \alpha_1'\alpha_2 - \alpha_1\alpha_2' & = & s\tau_{12}' - s'\tau_{12} \\ \alpha_2'\alpha_3 - \alpha_2\alpha_3' & = & s\tau_{23}' - s'\tau_{23} \\ \alpha_3'\alpha_1 - \alpha_3\alpha_1' & = & s\tau_{31}' - s'\tau_{31} \end{cases}$$

Remember that We can assume

- s, α_i, τ_{jk} are polynomials of maximum degree d in a complex variable z.
- s has d simple complex zeros $z_1, z_2, \ldots, z_d$.
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ANALYSIS OF THE SYSTEM

$$\alpha_i'\alpha_j - \alpha_j'\alpha_i = s\tau_{ij}' - s'\tau_{ij} \quad \Longleftrightarrow \quad \frac{\alpha_i'\alpha_j - \alpha_j'\alpha_i}{s^2} = \left(\frac{\tau_{ij}}{s}\right)'$$

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Since s has simple zeros at $z_1, \ldots, z_d$,

- The second condition is a consequence of the first.
- The residue condition can be written

$$\left((z - z_\ell)^2\, \frac{\alpha_i'\alpha_j - \alpha_j'\alpha_i}{s^2}\right)'_{|z=z_\ell} = 0,\ 1 \leq \ell \leq d.$$

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A simple computation gives that this is equivalent to the condition

$$s \mid \mathrm{Wr}(s, \alpha_i, \alpha_j).$$

ANALYSIS OF THE SYSTEM

Write

$$s = \prod_{\ell=1}^{d} (z - z_\ell), \qquad \alpha_i = a_{i0}\, s + \sum_{\ell=1}^{d} a_{i\ell}\, \frac{s}{z - z_\ell}, \qquad \tau_{ij} = t_{ij0}\, s + s \int \frac{\alpha_i' \alpha_j - \alpha_i \alpha_j'}{s^2}\, dz$$

(Note that t_{ij0} are arbitrary complex numbers.)

ANALYSIS OF THE SYSTEM

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$$s = \prod_{\ell=1}^{d} (z - z_\ell), \qquad \alpha_i = a_{i0}\, s + \sum_{\ell=1}^{d} a_{i\ell}\, \frac{s}{z - z_\ell}, \qquad \tau_{ij} = t_{ij0}\, s + s \int \frac{\alpha_i' \alpha_j - \alpha_i \alpha_j'}{s^2}\, dz$$

(Note that t_{ij0} are arbitrary complex numbers.)

The condition $s \mid \mathrm{Wr}(s, \alpha_i, \alpha_j)$ becomes

$$a_{i\ell} \sum_{k \neq \ell} \frac{a_{jk}}{(z_\ell - z_k)^2} - a_{j\ell} \sum_{k \neq \ell} \frac{a_{ik}}{(z_\ell - z_k)^2} = 0, \ 1 \leq \ell \leq d$$

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Another view: set of integral elements of the exterior differential system generated by the forms

$$\omega_\ell = \sum_{k \neq \ell} \frac{dz_k \wedge dz_\ell}{(z_\ell - z_k)^2},\ 1 \leq \ell \leq d.$$

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Since $0 \neq \alpha_1(z_\ell) = a_{1\ell} s'(z_\ell)$, we must have $a_{1\ell} \neq 0$, and then

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where $\lambda_\ell = \dfrac{1}{a_{1\ell}} \sum_{k \neq \ell} \dfrac{a_{1k}}{(z_\ell - z_k)^2}$.

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In matrix notation this can be written with the single equation

$$\begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{nd} \end{pmatrix} = 0,$$

where λ_m, $1 \leq m \leq d$, are implicitly defined.

MODIFIED GOAL 4 - FINAL!

Find (all) solutions of the algebraic equations

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where the last matrix has rank n (to guarantee linear fullness).

Then, taking your favorite complex numbers a_{i0} and t_{ij0} and defining

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gives a solution of the equation

$$\alpha_i'\alpha_j - \alpha_j'\alpha_i = s\tau_{ij}' - s'\tau_{ij}.$$

(And all other solutions arise the same way.)

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5. Write $\widetilde{\psi} = [s : \alpha_1 : \cdots : \alpha_n : \tau_{12} : \cdots : \tau_{n-1,n}]$. Then
$$\varphi = \pi \circ b_\beta \circ \widetilde{\psi}$$
is a linearly full harmonic map. (And all such arise this way!)

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What is the dimension of the set of complex numbers z_ℓ, λ_ℓ, $1 \leq \ell \leq d$, s.t.

$$\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}} := \begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_1)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix}$$ has nullity at least n?

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This is *at least* $2d-1-\frac{n(n+1)}{2}$. Thus, the dimension of the set of complex numbers z_ℓ, λ_ℓ, $1 \le \ell \le d$, such that nul$(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}) \ge n$ is *at least*

$$2d - \frac{n(n+1)}{2}$$

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This is *at least* $2d - 1 - \frac{n(n+1)}{2}$. Thus, the dimension of the set of complex numbers z_ℓ, λ_ℓ, $1 \leq \ell \leq d$, such that nul$(\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}) \geq n$ is *at least*

$$2d - \frac{n(n+1)}{2}$$

Adding the degrees of freedom of the previous page we obtain that our recipe constructs a set of dimension *at least*

$$2d - \frac{n(n+1)}{2} + n^2 + \frac{n(n+1)}{2} = 2d + n^2.$$

MORE NUMEROLOGY

Consider the projections (well defined since $\alpha(z_\ell) \neq 0$, $1 \leq \ell \leq d$)

$$[s : \alpha_1 : \cdots : \alpha_{n-1} : \alpha_n : \tau_{12} : \cdots : \tau_{n-2,n-1} : \tau_{1n} : \cdots : \tau_{n-1,n}] \quad \in \quad \mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n+1)/2})$$

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\vdots & & \\
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[s : \alpha_1 : \alpha_2 : \tau_{12}] & \in & \mathrm{PD}_d^f(S^2, \mathbb{CP}^3) \\
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[s : \alpha_1] & \in & \mathrm{PD}_d^f(S^2, \mathbb{CP}^1)
\end{array}$$

MORE NUMEROLOGY

Consider the projections (well defined since $\alpha(z_\ell) \neq 0$, $1 \leq \ell \leq d$)

$$\begin{array}{ccc}
[s : \alpha_1 : \cdots : \alpha_{n-1} : \alpha_n : \tau_{12} : \cdots : \tau_{n-2,n-1} : \tau_{1n} : \cdots : \tau_{n-1,n}] & \in & \mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n+1)/2}) \\
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- The image of each p_n has codimension at least 1.
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By induction, assume dim$(\mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n-1)/2})) \leq 2d + (n-1)^2$. Then

$$\dim(\mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n+1)/2})) \quad \leq \quad \dim(\mathrm{Im}(p_{n-1})) + \dim(fiber)$$

MORE NUMEROLOGY

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$$\begin{aligned}
\text{dim}(\mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n+1)/2})) &\leq \text{dim}(\text{Im}(p_{n-1})) + \text{dim}(fiber) \\
&\leq 2d + (n-1)^2 - 1 + 2n = 2d + n^2.
\end{aligned}$$

SOME OPEN PROBLEMS

- These sets of harmonic maps are algebraic varieties. Are they manifolds?
- Adapt this recipe to maps from Riemann surfaces other than S^2.