MTH 42. Linear Algebra. Test 2. Take-home exam.

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Print Name: ______________________________

INSTRUCTIONS:

- This exam contains 10 questions, 3 pages, 107 points (points over 100 count as extra credit).
- You have until Monday 12/08 to complete the exam.
- **You must show all your work** in order to get credit.
- You can use notes and/or books, but **you cannot** copy or look for answers online.
- The exam is due on Monday 12/08 at 10am in class. Make sure to write the solutions carefully and staple all pages together, including this page.

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1. (8 points) Determine if the following statements are true or false (circle **T** or **F**). Justify your answer.
   (a) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be onto.
   (b) If the columns of a matrix A are linearly independent, then the nullspace of A is $\{\vec{0}\}$.
   (c) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is onto, then it must also be one-to-one.
   (d) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 - 2x_2, x_2 + 1, 5x_2)$ is linear.

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2. (15 points) Vector spaces:
   (a) Write down carefully the definition of vector space.
   (b) Prove carefully that the space $\mathbb{R}^\infty$ of sequences of real numbers is a vector space.
   (Recall that a sequence of real numbers is an infinite list of real numbers $\boldsymbol{a} = (a_1, a_2, a_3, \dots)$, with addition and multiplication by scalars defined component by component. That is
   - $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots)$ and
   - $k\boldsymbol{a} = (ka_1, ka_2, ka_3, \dots)$ if $k \in \mathbb{R}$).

   (c) Prove that the set $S \subset \mathbb{R}^\infty$ of all sequences of the form $\boldsymbol{a} = (a, 2a, 4a, 8a, 16a, \dots)$ is a subspace of $\mathbb{R}^\infty$.

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3. (9 points) Linear independence:
   (a) Show that the three vectors $\vec{u}_1 = (1, 1, -1)$, $\vec{u}_2 = (0, 2, 1)$, $\vec{u}_3 = (4, 1, 3)$ in $\mathbb{R}^3$ are linearly independent.
   (b) Show that the three vectors $\vec{v}_1 = (0, 3, 1, -1)$, $\vec{v}_2 = (6, 0, 5, 1)$, $\vec{v}_3 = (4, -7, 1, 3)$ in $\mathbb{R}^4$ are linearly dependent.
   (c) Express each one of the vectors as a linear combination of the other two.

4. (10 points) Find a basis for the space $\text{Sym}(5,\mathbb{R})$ of 5×5 symmetric matrices, and then find its dimension. What would be the dimension of $\text{Sym}(n,\mathbb{R})$ for a general n?

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5. (15 points) In the space P_3 of polynomials of degree 3, consider the following bases:

$$\begin{aligned} B &= \{-t^3+9t^2-18t+6, t^2-4t+2, -t+1, 1\} \\ C &= \{8t^3-12t, 4t^2-2, 2t-2, 1\} \end{aligned}$$

(a) Find the transition matrix $P_{C\to B}$ and $P_{B\to C}$.

(b) Let $p = t^3 - 5t + 3t - 1$. Find $[p]_B$.

(c) Use (a) and (b) to find $[p]_C$.

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6. (10 points) Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -3 \end{bmatrix}.$$

(a) Construct a matrix whose nullspace is the span of $\{\vec{v}_1, \vec{v}_2\}$.

(b) Construct a matrix whose column space is the span of $\{\vec{v}_1, \vec{v}_2\}$.

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7. (10 points) Find bases for the row space, the column space, and the null space, of the matrix

$$\begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 1 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 2 & -9 \end{bmatrix}$$

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8. (10 points) Find the values of r and s for which the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r+1 & -2 \\ 0 & s-1 & r+2 \\ 0 & -1 & 2 \end{bmatrix}$$

(a) Has rank 1 (if any).

(b) Has rank 2 (if any).

(c) Has rank 3 (if any).

(d) Has rank 4 (if any).

9. (10 points) Let $T_1(x_1, x_2) = (x_1 + x_2,\, x_1 - x_2)$ and $T_2(x_1, x_2) = (3x_1,\, 2x_1 + 4x_2)$ be linear operators in $\mathbb{R}^2$.

   (a) Find the standard matrices $[T_1]$ and $[T_2]$.

   (b) Find the standard matrices $[T_2 \circ T_1]$ and $[T_1 \circ T_2]$.

   (c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2))$.

   (d) Find a basis for the kernel of T_1

   (e) Find a basis for the range of T_1.

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10. (10 points) Let W be the space spanned by $\mathbf{f} = \sin x$ and $\mathbf{g} = \cos x$.

   (a) Show that for any value of θ, $\mathbf{f}_1 = \sin(x + \theta)$ and $\mathbf{g}_1 = \cos(x + \theta)$ are vectors in W.

   (b) Show that $\mathbf{f}_1$ and $\mathbf{g}_1$ form a basis for W.
