MATH 42 - Linear Algebra

Test 1. Time allowed: one hour. Professor Luis Fernández

NAME: SOLUTION

INSTRUCTIONS: Solve the following exercises. **You must show all your work** in order to receive credit.

[12] **1.** Determine if the following statements are true or false (circle **T** or **F**). You do not need to justify your answer.

a) **T** **F** If A is an invertible $n \times n$ matrix, then the system $A\vec{x} = \vec{0}$ has infinitely many solutions.

b) **T** **F** If the reduced row echelon form of an $n \times n$ matrix A is I_n, then A is invertible.

c) **T** **F** If A is invertible, then the only solution of the system $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

d) **T** **F** If A and B are invertible $n \times n$ matrices, then AB is also invertible.

---

[22] **2.** Solve the system of equations $\begin{cases} x - y + 4z = 1 \\ 3x - 2y - 5z = 3 \end{cases}$ Express your answer in parametric form.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 3 & -2 & -5 & 3 \end{array}\right] \xrightarrow{2^{nd} - 3 \cdot 1^{st}} \left[\begin{array}{ccc|c} 1 & -1 & 4 & 1 \\ 0 & 1 & -17 & 0 \end{array}\right] \xrightarrow{1^{st} + 2^{nd}} \left[\begin{array}{ccc|c} 1 & 0 & -13 & 1 \\ 0 & 1 & -17 & 0 \end{array}\right]$$

$\Rightarrow$ get $\begin{array}{l} x - 13z = 1 \\ y - 17z = 0 \end{array}$, or $\boxed{\begin{array}{l} x = 13t + 1 \\ y = 17t \\ z = t \end{array}}$

or $\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 13 \\ 17 \\ 1 \end{bmatrix}}$

[22] **3.** Find the value(s) of k so that the following linear system is consistent (that is, has at least one solution):

$$\begin{cases} 3x_1 - 5x_2 = 4 \\ 9x_1 + kx_2 = -1 \end{cases}$$

Find the solution when $k = -2$.

$$\left[\begin{array}{cc|c} 3 & -5 & 4 \\ 9 & k & -1 \end{array}\right] \xrightarrow{2^{nd} - 3 \cdot 1^{st}} \left[\begin{array}{cc|c} 3 & -5 & 4 \\ 0 & k+15 & -13 \end{array}\right]$$

The last equation is $(k+15)x_2 = -13$. If $k = -15$, this would be $0 = -13$, which is absurd.

Therefore, if $k = -15 \Rightarrow$ no solution

otherwise, get $x_2 = \frac{-13}{k+15}$, $x_1 = \frac{1}{3}\left(4 - \frac{5 \cdot 13}{k+15}\right)$.

If $k = -2$, then $(-2+15)x_2 = -13 \Rightarrow \boxed{x_2 = -1}$

$3x_1 = 5x_2 + 4 = -1 \Rightarrow \boxed{x_1 = -\frac{1}{3}}$

[22] **4.** Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -1 & -3 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[3^{rd}+1^{st}]{2^{nd}-2\cdot 1^{st}} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2^{nd}\cdot(-1)]{1^{st}+3\cdot 2^{nd}} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -5 & 3 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{1^{st}+2\cdot 3^{rd}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3 & 2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

Therefore, $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$

Check

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} -3 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3+6-2 & +3-3+0 & 2+0-2 \\ -6+10-4 & 6-5+0 & 4+0-4 \\ 3-6+3 & -3+3+0 & -2+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

[22] **5.** For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$, find

a) $A^T A$.

b) AA^T.

c) $(A^T A)^{-1}$.

d) [BONUS, +5pt] Show that AA^T is not invertible.

a) $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1+0+4 & -1+0+0 \\ -1+0+0 & 1+1+0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}}$

b) $AA^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+1 & 0-1 & 2+0 \\ 0-1 & 0+1 & 0+0 \\ 2+0 & 0+0 & 4+0 \end{bmatrix}$

$= \boxed{\begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 4 \end{bmatrix}}$

c) $(A^T A)^{-1} = \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} = \frac{1}{5 \cdot 2 - (-1)(-1)} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$

$= \boxed{\frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}}$

d) Try to find inverse:

$\left[\begin{array}{ccc|ccc} 2 & -1 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{1 \leftrightarrow 2} \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 1 & 0 \\ +2 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 & 2 & 1 \end{array}\right]$

$\xrightarrow[3^{rd} - 2 \cdot 2^{nd}]{1^{st} \cdot (-1)} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & -2 & -4 & 1 \end{array}\right]$

impossible to get 0 0 1 here.
So not invertible.