Linear Algebra
QUIZ 3 Solution.

① a) $\begin{vmatrix} 3-\lambda & -2 & 2 \\ 2 & -\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda)(-\lambda) + (-2)\cdot 2\cdot(-1) + 2\cdot 2\cdot 2$

$- (3-\lambda)\,2\cdot 2 - (-\lambda)(-2)\cdot 2 - 2(-\lambda)(-1)$

$= -\lambda^3 + 3\lambda^2 \cancel{+4} \cancel{+8} \cancel{-12} \cancel{+4\lambda} \cancel{-4\lambda} - 2\lambda$

$= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 - 3\lambda + 2)$

$= -\lambda(\lambda-1)(\lambda-2).$

<u>Eigenvalues</u>: $-\lambda(\lambda-1)(\lambda-2) = 0$

$\Rightarrow \boxed{\lambda = 0, 1, 2}.$

b) $\lambda = 0$: $\begin{pmatrix} 3 & -2 & 2 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 2 \\ -1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\to \begin{aligned} x_1 &= -s_3 \\ 2x_2 &= -s_3 \\ x_3 &= s_3 \end{aligned} \to \boxed{\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}}$

$\lambda = 1$: $\begin{pmatrix} 2 & -2 & 2 \\ 2 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{aligned} x_1 &= -s_3 \\ x_2 &= 0 \\ x_3 &= s_3 \end{aligned}$

$\boxed{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\lambda = 2$: $\begin{pmatrix} 1 & -2 & 2 \\ 2 & -2 & 2 \\ -1 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{aligned} x_1 &= 0 \\ x_2 &= s_3 \\ x_3 &= s_3 \end{aligned}$

$\boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$

(2) Diagonalize $\begin{pmatrix} 9 & -4 \\ 12 & -5 \end{pmatrix}$.

$$\begin{vmatrix} 9-\lambda & -4 \\ 12 & -5-\lambda \end{vmatrix} = \lambda^2+5\lambda-9\lambda-45+48 = \lambda^2-4\lambda+3$$
$$= (\lambda-3)(\lambda-1).$$

Eigenvalues : $\boxed{\lambda=1,\ \lambda=3}$.

For $\lambda=1$,
$$A-I = \begin{pmatrix} 8 & -4 \\ 12 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \to \vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda=3$,
$$A-3I = \begin{pmatrix} 6 & -4 \\ 12 & -8 \end{pmatrix} \to \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & -2 \\ 0 & 0 \end{pmatrix} \to \vec{u}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Therefore
$$\begin{pmatrix} 9 & -4 \\ 12 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \boxed{\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}}$$

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(3)
$$A = \boxed{\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}_{C=:}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}^{-1}}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} =$$

$$\left[\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)\right]$$

$$\boxed{A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 0 & -1 \\ -3 & 1 & -2 \end{pmatrix}}$$

④ a) Suppose that $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ are in S.

This means that $a_1+d_1=0$, $a_2+d_2=0$.

Take any linear combination

$$x_1 \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + x_2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} x_1a_1+x_2a_2 & x_1b_1+x_2b_2 \\ x_1c_1+x_2c_2 & x_1d_1+x_2d_2 \end{pmatrix}.$$

The trace of this matrix is

$$x_1a_1+x_2a_2+x_1d_1+x_2d_2 = x_1\underbrace{(a_1+d_1)}_{=0} + x_2\underbrace{(a_2+d_2)}_{=0}$$
$$=0.$$

Therefore, since its trace $=0$, any linear combination of elements of S is in S, and S is a subspace.

b) Any element of S can be written as $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

But $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Therefore, the set $\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$ spans S.

They are clearly linearly independent: if

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ then}$$

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow c_1=0,\ c_2=0,\ c_3=0.$$

Therefore it is a basis

c) Its dimension is 3.

5 a) A basis for P^2 is $\{1, x, x^2\}$:

b) Dimension is 3.

c) NO, because it is a set with 4 vectors in a space of dimension 3, and $4 > 3$.