SOLUTION. MATH 42 QUIZ 2

① Recall: the matrix of change of coordinates from $\mathcal{B}$ to $\mathcal{E}$ is

$$[I]_{\mathcal{E},\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ -2 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{2 \leftrightarrow 3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 1 \end{array}\right) \xrightarrow{R_3 \to -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & -1 \end{array}\right)$$

So, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix}$, and

$$[I]_{\mathcal{E},\mathcal{B}} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \\ -2 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} -2 & -6 & -4 \\ 1 & 0 & 1 \\ 2 & 9 & 6 \end{pmatrix}}$$

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② $$\vec{v} = 1 \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 3 \\ 3 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \\ 4 \\ -2 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 10 \\ 16 \\ -5 \end{pmatrix}}.$$

(3) Want to solve

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & -2 \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_2-R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right).$$ Therefore $$\boxed{\left[\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.}$$

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(4)

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 3 & -1 \\ -1 & 0 & -2 & 3 \\ 0 & 1 & 3 & 1 \end{vmatrix} \overset{C4-C1}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 3 & -3 \\ -1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -3 \\ 0 & -2 & 4 \\ 1 & 3 & 1 \end{vmatrix} \overset{R3-R1}{=} \begin{vmatrix} 1 & 3 & -3 \\ 0 & -2 & 4 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 4 \\ 0 & 4 \end{vmatrix} = \boxed{-8}$$

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(5) $\mathrm{Dim}(\mathrm{Ker}(T)) + \mathrm{Dim}(\mathrm{Range}(T)) = 7$, so

$$\boxed{\mathrm{Dim}(\mathrm{Ker}(T)) = 7-3 = 4}$$

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(6)

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \overset{\substack{R_3 = R_3 - x_1 R_2 \\ R_2 = R_2 - x_1 R_1}}{=} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2-x_1 & x_2^2-x_1x_2 \\ 1 & x_3-x_1 & x_3^2-x_1x_3 \end{vmatrix} = \begin{vmatrix} x_2-x_1 & x_2(x_2-x_1) \\ x_3-x_1 & x_3(x_3-x_1) \end{vmatrix}$$

Factor out from 1st & 2nd row

$$= (x_2-x_1)(x_3-x_1)\begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = \boxed{(x_2-x_1)(x_3-x_1)(x_3-x_2)}$$