LINEAR ALGEBRA MIDTERM

SOLUTION

1

a) FALSE. For exaple, $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$ is onto since $\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}$ has a pivot in every row (or just because $T\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$ covers all possible $\begin{pmatrix}x\\y\end{pmatrix}$ in $\mathbb{R}^2$).

b) TRUE. If $A = [\vec{a}_1, \dots, \vec{a}_n]$, and the $\vec{a}_1, \dots, \vec{a}_n$ are linearly independent, then

$$A\vec{x} = \vec{0} \iff x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0} \underset{\substack{\uparrow \\ \text{because} \\ \text{of liner independence}}}{\iff} x_1 = 0, \dots, x_n = 0$$

Therefore, the only solution of $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

c) TRUE: write $T(\vec{x}) = A\vec{x}$, and let B be an echelon form of A.

T onto $\iff$ B has pivots in every row $\iff$ B has 3 pivots $\iff$ B has pivots in every column $\iff$ T is one to one.

d) FALSE Take $A = I$, $B = I$. Both are invertible, but $A - B = 0$, which is not.

(1 cont)

e) FALSE. $T(0,0) = (0,1,0)$. Therefore it is not linear

---

2 a) $\{\vec{v}_1, \dots, \vec{v}_n\}$ linearly independent if the only solution of $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{0}$ is
$x_1 = 0,\ x_2 = 0, \dots, x_n = 0$.

b) $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

$\left\{ \underset{\vec{u}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}, \underset{\vec{v}}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\vec{w}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \right\}$ is linearly dependent (because $1\cdot\vec{u} + (-1)\cdot\vec{v} + (-1)\cdot\vec{w} = \vec{0}$).

---

3 a) Range(T) = Span {columns of A}.

Find a basis:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -4 \\ -3 & 3 & -6 & -9 \end{pmatrix} \xrightarrow{3^{rd} \to 3^{rd} + 3\cdot 1^{st}} \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↑ ↑ basis. ↑ ↑ pivot columns

$\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\} \text{ is a basis of Range}(T).}$

(b) Need to solve $A \cdot \vec{x} = \vec{0}$. Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ -3 & 3 & -6 & -9 & 0 \end{array}\right) \xrightarrow{\text{as before}} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{1^{st} = 1^{st} + 2^{nd}} \left(\begin{array}{cccc|c} 1 & 0 & 4 & -1 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{aligned} x_1 &= -4s_3 + s_4 \\ x_2 &= -2s_3 + 4s_4 \\ x_3 &= s_3 \\ x_4 &= s_4 \end{aligned} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s_3 \begin{pmatrix} -4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s_4 \begin{pmatrix} 1 \\ 4 \\ 0 \\ 1 \end{pmatrix}.$$

$$\boxed{\text{Basis of } \ker(T): \left\{ \begin{pmatrix} -4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

4

$$\begin{cases} x + 3y = 2 \\ 2x + hy = k \end{cases} \xrightarrow{2^{nd} = 2^{nd} - 2 \cdot 1^{st}} \begin{cases} x + 3y = 2 \\ 0 + (h-6)y = k - 4 \end{cases}$$

b) If $h \neq 6 \Rightarrow$ Unique solution ($y = \frac{k-4}{h-6}$, then find x).

a) If $h = 6$, $k \neq 4 \Rightarrow$ no solution ($0 \cdot y = k - 4 \neq 0$, impossible).

c) If $h = 6$, $k = 4 \Rightarrow \infty$ many solutions

(2^{nd} equation is $0 = 0$, get $\begin{cases} x = 2 - 3s_2 \\ y = s_2 \end{cases}$ $s_2 \in \mathbb{R}$)

5

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -1 & -3 & 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{2^{nd} - 2\cdot 1^{st},\; 3^{rd} + 1^{st}} \left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{1^{st} + 3\cdot 2^{nd}} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -5 & 3 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow[2^{nd}\cdot(-1)]{1^{st} + 2\cdot 3^{rd}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3 & 2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\boxed{\begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$$

6

a) $$A^T A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}}$$

b) $$A A^T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & -1 & 2 \\ -1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}}.$$

c) $$(A^T A)^{-1} = \frac{1}{5\cdot 2 - (-1)\cdot(-1)} \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

d) The three columns of (AA^T) are linear combinations of the two columns of A. Therefore, the columns of (AA^T) are <u>not</u> linearly independent, so (AA^T) is not invertible.

7 a) For example:

(i) A invertible

(ii) The rows of A are linearly independent

(iii) B has a pivot in every column.

(iv) T is one to one and onto

b) B has a pivot in every column $\Leftrightarrow$ B has n pivots

$\Leftrightarrow$ B has a pivot in every row

$\Leftrightarrow$ The equation $A\vec{x} = \vec{b}$ has a unique solution

$\Leftrightarrow$ $T(\vec{x}) = A\vec{x}$ is one to one and onto.

$\Leftrightarrow$ T invertible $\Leftrightarrow$ A invertible.

---

8) See pic

---

9) a) $B^2 = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)$

$$= A \underbrace{\boxed{(A^TA)^{-1}(A^TA)}}_{= I} (A^TA^{-1})A^T$$

$$= AI(A^TA^{-1})A^T = A(A^TA^{-1})A^T = \underline{\underline{B}}.$$

b) $BA = A\underbrace{(A^TA)^{-1}(A^TA)}_{I} = AI = \underline{\underline{A}}.$