MTH 42. Linear Algebra. Midterm exam. Fall 2025.

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Name: SOLUTION

INSTRUCTIONS:

- This exam contains 8 questions, 6 pages for a total of 100 points plus a 10-point bonus question.
- You have 110 minutes to complete the exam.
- **You must show all your work** in order to get credit.
- You can use a non-graphing scientific calculator. No other electronic devices, notes or books are permitted.

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1. (18 points) Determine if the following statements are true or false (circle **T** or **F**). You do not need to justify your answer.

   (a) **T** **F** If A, B are $n \times n$ matrices, $\det(A+B) = \det(A) + \det(B)$.

   (b) **T** **F** If A, B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

   (c) **T** **F** $(AB)^T = B^T A^T$.

   (d) **T** **F** If A is a square matrix, and the only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, then $\det(A) \neq 0$.

   (e) **T** **F** If A, B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

   (f) **T** **F** If A is an $n \times n$ matrix, and $k \in \mathbb{R}$, then $\det(kA) = k^n \det(A)$.

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2. (10 points) Multiply $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 & 1\cdot 1 + 2\cdot 2 + 3\cdot 1 \\ 3\cdot 1 + 0\cdot 2 + 1\cdot 3 & 3\cdot 1 + 0\cdot 2 + 1\cdot 1 \\ 2\cdot 1 + 2\cdot 2 - 2\cdot 3 & 2\cdot 1 + 2\cdot 2 - 2\cdot 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 14 & 8 \\ 6 & 4 \\ 0 & 4 \end{bmatrix}}$$

3. (16 points) Find the values of h and k so that the linear system $\begin{cases} x - 3hy = 2 \\ 2x + hy = 3k \end{cases}$ has

(a) No solution.

(b) A unique solution.

(c) Infinitely many solutions.

$$\left[\begin{array}{cc|c} 1 & -3h & 2 \\ 2 & h & 3k \end{array}\right] \xrightarrow{2^{nd} - 2\cdot 1^{st}} \left[\begin{array}{cc|c} 1 & -3h & 2 \\ 0 & 7h & 3k-4 \end{array}\right]$$

If $\boxed{h \neq 0,}$ $\left[\begin{array}{cc|c} 1 & -3h & 2 \\ 0 & 7h & 3k-4 \end{array}\right]$ Unique solution

If $h = 0$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 3k-4 \end{array}\right]$.

If $h=0$, $k \neq \frac{4}{3}$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 3k-4 \end{array}\right]$ No solution

If $h=0$, $k = \frac{4}{3}$, $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$ ∞ many solutions.

So:

(a) If $h=0$, $k \neq \frac{4}{3}$.

(b) If $h=0$, $k = \frac{4}{3}$.

(c) If $h \neq 0$.

4. (16 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -1 & -3 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2\cdot R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2\cdot(-1)]{R_1+3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -5 & 3 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 3 & 2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} -3 & 3 & 2 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}.$$

5. (16 points) Use Cramer's rule to find the solution of the system $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} \overset{C_3+C_2}{=} \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = -(-1)\cdot\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1\cdot(1-2) = \boxed{-1}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}}{-1} \overset{R_3+R_2}{=} \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}}{-1} = \frac{1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{-1} = \frac{(1-1)}{1} = \boxed{0}$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}{-1} = \frac{1\cdot\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{-1} = \frac{1\cdot(1-2)}{-1} = \boxed{1}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix}}{-1} = \frac{-(-1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{-1} = \frac{1\cdot(1-2)}{-1} = \boxed{1}$$

Solution is $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$

6. (14 points) When x and y are real numbers, we know that $(x+y)(x-y) = x^2 - y^2$. However, this is not true for matrices:

(a) Give an example of two 2×2 matrices A and B (not necessarily different) such that $(A+B)(A-B) \neq A^2 - B^2$

(b) Find the correct expansion of $(A+B)(A-B)$.

(a) For exaple, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$(A+B)(A-B) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

whereas

$$A^2 - B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

not equal

(b) $(A+B)(A-B) = A^2 \underbrace{- AB + BA}_{\text{this is not necessarily } 0} - B^2$.

AB

A^2

7. (10 points) Find the determinant of the matrix $\begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$

$$= 2 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{vmatrix} = 2\cdot(-2) \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 2\cdot(-2)\cdot 2 \begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= -8\cdot(8-2) = \boxed{-48}$$

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8. (10 points (bonus)) Let A be an $n \times m$ matrix with $n > m$, and suppose that the $m \times m$ matrix $A^T A$ is invertible. Consider the matrix $B = A(A^T A)^{-1} A^T$.

   (a) Show that $B^2 = B$.

   (b) Show that $BA = A$.

$(A^T A)^{-1}(A^T A) = I$, and $A \cdot I = A$

(a) $B^2 = A\boxed{(A^T A)^{-1}(A^T \cdot A)}(A^T A)^{-1} A^T$

$= \underbrace{A(A^T A)^{-1} A^T}_{=B} = \boxed{B}$

(b) $BA = A\underbrace{(A^T A)^{-1}(A^T \cdot A)}_{= I} = AI = \underline{\underline{A}}$