What we have studied so far, in two pages

Suppose that A is a matrix, B is an echelon form of A, and $\vec{b}$ is any vector. Let us define T to be the transformation $T(\vec{x}) = A\vec{x}$. Then

- The columns of A are linearly independent $\iff$ B has a pivot in **every** column.
- B has a pivot in **every** column $\iff$ The equation $A\vec{x} = \vec{b}$ has **at most** one solution.
- By definition, T is one to one $\iff$ **for any** $\vec{b}$, the equation $T(\vec{x}) = \vec{b}$ has **at most** one solution.
  $\iff$ **for any** $\vec{b}$, the equation $A\vec{x} = \vec{b}$ has **at most** one solution.

Summarizing, the following statements are equivalent (which means that if one of them is true, the others are also true):

- T is **one to one**.
- The columns of A are **linearly independent**.
- The equation $A\vec{x} = \vec{b}$ has **at most** one solution, for all $\vec{b}$.
- The equation $A\vec{x} = \vec{0}$ has **only** the trivial solution ($\vec{x} = \vec{0}$).
- B has a pivot in **every** column.

NOTE ALSO: if A is an $n \times m$ matrix (which means that $T : \mathbb{R}^m \to \mathbb{R}^n$) then, if $n < m$ (that is, A has more columns than rows), the echelon form of A cannot have a pivot in every column, so:

T cannot be injective if it goes from a "bigger space" to a "smaller space".

At the other end, recall:

- The rows of A are linearly independent $\iff$ B has a pivot in **every** row.
- B has a pivot in **every** row $\iff$ The equation $A\vec{x} = \vec{b}$ has **at least** one solution.
- By definition, T is onto $\iff$ **for any** $\vec{b}$, the equation $T(\vec{x}) = \vec{b}$ has **at least** one solution.
  $\iff$ **for any** $\vec{b}$, the equation $A\vec{x} = \vec{b}$ has **at least** one solution.

Summarizing, the following statements are equivalent (which means that if one of them is true, the others are also true):

- T is **onto**.
- The rows of A are **linearly independent**.
- The equation $A\vec{x} = \vec{b}$ has **at least** one solution, for all $\vec{b}$.
- B has a pivot in **every** row.

NOTE ALSO: if A is an $n \times m$ matrix (which means that $T : \mathbb{R}^m \to \mathbb{R}^n$) then, if $n > m$ (that is, A has more rows than columns), the echelon form of A cannot have a pivot in every column, so:

T cannot be onto if it goes from a "smaller space" to a "bigger space".

Now suppose that A is an $n \times n$ **square** matrix, that is, n rows and n columns. As before, let B be an echelon form of A, and $\vec{b}$ any vector. Then Suppose that A is a matrix, and B is an echelon form of A. Let T be the transformation $T(\vec{x}) = A\vec{x}$. Then note that because A (and therefore B) has n rows and n columns,

$$\boxed{B \text{ has a pivot in every } \textbf{column} \iff B \text{ has } n \text{ pivots} \iff B \text{ has a pivot in every } \textbf{row}}$$

Note also that if B has a pivot in every column and in every row, then the equation $A\vec{x} = \vec{b}$ has, on the one hand, **at most** one solution, and on the other, **at least** one solution, from which we conclude that it has **exactly one** solution:

$$\boxed{B \text{ has a pivot in every row and column} \iff A\vec{x} = \vec{b} \text{ has exactly one solution}}$$

Using this and the results in the previous page we get the Big Theorem:

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T(\vec{x}) = A\vec{x}$, and if B is an echelon form of A, then

1. T is one to one.
2. The columns of A are **linearly independent**.
3. B has a pivot in **every** column.
4. B has a pivot in **every** row.
5. The equation $A\vec{x} = \vec{b}$ has **exactly** one solution, for all $\vec{b}$.
6. The columns of A span $\mathbb{R}^n$.
7. T is onto.