MATH 42 - Linear Algebra

QUIZ 3. Time allowed: one hour. Professor Luis Fernández

NAME:____

INSTRUCTIONS: Solve the following exercises. **You must show work** in order to receive any credit.

[20] **1.** For the matrix $\begin{pmatrix} 3 & -2 & 2 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}$,

a) Find the characteristic polynomial and the eigenvalues.

b) Find ONE of the eigenvectors.

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[20] **2.** Diagonalize the matrix $\begin{pmatrix} 9 & -4 \\ 12 & -5 \end{pmatrix}$.

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[10] **3.** The eigenvalues of a 3×3 matrix A are $1, 0,$ and -1. The corresponding eigenvectors are:

For $\lambda = 1,$ eigenvector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. For $\lambda = 0,$ eigenvector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. For $\lambda = -1,$ eigenvector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Find the matrix A. You can leave it expressed as a product of matrices (you do not need to evaluate products of matrices or inverses).

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[30] **4.** Let V be the vector space of 2×2 matrices. Recall that the trace of a matrix, written $\mathrm{Tr}(A)$, is the sum of its diagonal entries:

$$\mathrm{Tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \quad \text{(for a } 2 \times 2 \text{ matrix).}$$

a) Prove that the subset S of V consisting of matrices of trace equal to 0 is a subspace of V.

b) Find a basis for S.

c) What is the dimension of S?

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[20] **5.** Remember that $\mathbf{P}^2$ is the vector space of polynomials of degree less than or equal to 2.

a) Write down a basis of $\mathbf{P}^2$.

b) What is the dimension of $\mathbf{P}^2$?

c) Is the set of elements of $\mathbf{P}^2$ given by $\mathcal{V} = \{x^2 + 1,\ 2x + 5,\ x^2 + x - 2,\ 7x - 2\}$ linearly independent?

NOTE: you do not need to do any calculation in this exercise, but you need to explain why.

(Turn over for bonus exercise.)

[15] **6.** BONUS: Let A be a 3×3 matrix with only one eigenvalue λ of multiplicity 3, and only one eigenvector. Suppose that $\vec{w}$ is a vector that satisfies

$$(A - \lambda I)^3 \vec{w} = \vec{0}, \quad \text{but} \quad (A - \lambda I)^2 \vec{w} \neq \vec{0}.$$

Let $\vec{v}$ and $\vec{u}$ be defined by

$$\vec{v} = (A - \lambda I)\vec{w} \quad \text{and} \quad \vec{u} = (A - \lambda I)\vec{v}.$$

a) Show that $\vec{u}$ is an eigenvector of A with eigenvalue λ.

b) Show that $A\vec{v} = \vec{u} + \lambda\vec{v}$.

c) Show that $A\vec{w} = \vec{v} + \lambda\vec{w}$.

d) Show that A can be written as

$$A = [\,\vec{u} \quad \vec{v} \quad \vec{w}\,] \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} [\,\vec{u} \quad \vec{v} \quad \vec{w}\,]^{-1}.$$