MATH 42 - Linear Algebra

Midterm Exam. Time allowed: two hours. Professor Luis Fernández

NAME:________________________________

INSTRUCTIONS: Solve the following exercises. **You must show all your work** in order to receive any credit.

[10] **1.** Determine if the following statements are true or false, carefully **justifying your answer**: If the statement is true, give a short explanation of why. If it is false, give a counterexample.

a) A transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be onto.

b) If the columns of a matrix A are linearly independent, then the nullspace of A is $\{\vec{0}\}$.

c) If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is onto, then it must also be one-to-one.

d) If A and B are invertible matrices, then so is $A - B$.

e) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 - 2x_2, x_2 + 1, 5x_2)$ is linear.

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[10] **2. a)** Write the definition of linear independence of a set of vectors. Start with "A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent if ..."

b) Give an example of a linearly independent set of vectors in $\mathbb{R}^2$ and a linearly dependent set of vectors in $\mathbb{R}^3$.

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[15] **3.** Consider the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & -4 \\ -3 & 3 & -6 & -9 \end{pmatrix} \vec{x}.$$

a) Find a basis for the range of T.

b) Find a basis for the kernel of T.

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[15] **4.** Find the values of h and k so that the linear system $\begin{cases} x + 3y = 2 \\ 2x + hy = k \end{cases}$ has

a) No solution.

b) A unique solution.

c) Infinitely many solutions.

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[15] **5.** Find the inverse of the matrix $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{pmatrix}$

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[15] **6.** For the matrix $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$, find

a) $A^T A$.

b) AA^T.

c) $(A^T A)^{-1}$.

d) [BONUS, +5pt] Show that AA^T is not invertible.

[15] **7.** Below is part of the statement of the Big Theorem.

a) State 3 equivalent conditions to the given one.

b) For two of the conditions you write, explain why they are equivalent to the given one.

[NOTE: there are many options: you can give conditions on the columns of A, on the rows of A, on the pivots of B, on the transformation T... You only need to write three conditions, and for two of them, explain why they are equivalent to the given one.

(The Big Theorem) Let $\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$. Let B be an echelon form of A.

(i) (Given) The matrix A is invertible.

(ii)

(iii)

(iv)

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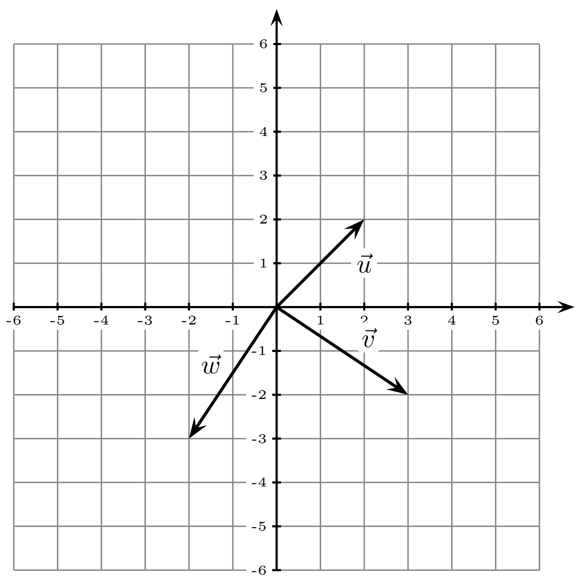
[10] **8.** Given the vectors $\vec{u}, \vec{v}, \vec{w}$ in the coordinate system below, draw the following vectors:

a) $\vec{v} + \vec{w}$

b) $\vec{u} - 2\vec{v}$

c) $\vec{u} + \vec{v} + \vec{w}$



[+5] **9.** [BONUS] Let A be an $n \times m$ matrix with $n > m$. Consider the matrix $B = A(A^TA)^{-1}A^T$.

a) Show that $B^2 = B$.

b) Show that $BA = A$.