Fall 2019 MTH 42 Midterm Exam Review. Luis Fernandez.

Topical Outline

Chapter 1

- Elementary row operations and row reduction.
- Echelon form and reduced row echelon form.
- Consistent and inconsistent systems of equations.
- Parametric descriptions of solutions.

Chapter 2

- Vectors. Algebraic properties of vectors.
- Linear combinations of vectors.
- Geometric representation of vectors.
- Span.
- Multiplication of vectors by matrices.
- Vector equations.
- Linear independence and linear dependence (definitions, examples, be able to connect this back to solutions to matrix equations, pivots, span etc.).
- Homogeneous systems, nontrivial solutions.
- Theorem 2.17: Given a particular solution of a linear system $A\vec{x} = \vec{b}$, all the other solutions can be written as the sum of the given solution and a solution of the associated homogeneous system $A\vec{x} = \vec{0}$.

Chapter 3

- Linear transformations.
  - Test if a transformation is linear.
  - Turn a transformation into a matrix.
  - One-to-one and onto transformations.
  - Theorem: T is one-to-one iff $T(x) = 0$ has only the trivial solution.
  - Theorem: T is onto iff the columns of associated matrix A span the codomain.
  - Theorem: T is one-to-one iff the columns of A are linearly independent.
- Matrix operations (add, multiply by scalar, multiply matrices, transpose).
- Properties of matrix multiplication.
- Transposes and their properties
- Invertible matrices.
- How to find the inverse of a matrix.

Chapter 4, Sec. 4.1

- Definition of subspace. How to determine if a subset of $\mathbb{R}^n$ is a subspace.
- Kernel and range of a linear transformation.

Review Problems

1. True or False?

   a. Any system of n linear equations in n variables has at most n solutions.

   b. If the augmented matrix $[A \ |\mathbf{b}]$ has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

   c. If the equation $A\mathbf{x} = \mathbf{0}$ only has the trivial solution, then the columns of the matrix A are linearly independent.

   d. The linear system with equation $A\mathbf{x} = \mathbf{0}$ is always consistent.

   e. Let A be an $m \times n$ matrix, and assume $m > n$ and the matrix has n pivots. Then for any vector $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ is consistent.

   f. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.

   g. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and if $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in Span$\{\mathbf{x}, \mathbf{y}\}$.

   h. Subsets of linearly independent sets are linearly independent.

   i. The transformation T defined by $T(x_1, x_2) = (2x_1 - 3x_2, x_1 + 4, 5x_2)$ is linear.

   j. A transformation T is linear if and only if $T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)$ for all $\mathbf{v}_1$ and $\mathbf{v}_2$ in the domain of T and for all scalars c_1 and c_2.

   k. If a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is onto, then it is one-to-one as well.

   l. If A is a 3×2 matrix, the the transformation $\mathbf{x} \to A\mathbf{x}$ cannot be one-to-one.

   m. If for some matrix A, and some vectors $\mathbf{x}$ and $\mathbf{b}$, we have $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the column vectors of A.

   n. If A and B are invertible $n \times n$ matrices, then so is $A + B$.

   o. If A and B are invertible $n \times n$ matrices, then so is AB.

   p. If A and B are $n \times n$ matrices and $AB = I_n$, then $BA = I_n$.

   q. If A and B are $n \times n$ matrices, then $A^2 = B^2$ implies $A = B$ or $A = -B$.

2. Define and present two examples and one non-example of the following:

   1. Span of a finite set of vectors
   2. Linear independence of a finite set of vectors
   3. Linear transformation

3. Know the big theorem, version 4, and be able to reproduce it in the exam:

   Theorem: Let $\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$. Let $A = [\vec{a}_1, \cdots, \vec{a}_n]$ be the matrix whose columns are the vectors $\vec{a_1}, \ldots, \vec{a}_n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent.

   1. $\mathcal{A}$ spans $\mathbb{R}^n$.
   2. $\mathcal{A}$ is linearly independent.
   3. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in $\mathbb{R}^n$.
   4. T is onto.

5. T is one-to-one.
6. A is invertible.
7. $\ker(T) = \{\vec{0}\}$.

4. Solve the following systems:

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 2 \\ x_1 + x_2 - 8x_3 = 4 \end{cases}$$

5. Find h and k such that the system

$$\begin{array}{ccccc} x & + & 2y & = & k \\ x & + & hy & = & 1 \end{array}$$

has
(a) no solution.
(b) a unique solution.
(c) infinitely many solutions.

6. Given three vectors $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Show that $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathrm{Span}(\mathbf{u}, \mathbf{v})$

7. Suppose

$$A = \begin{pmatrix} 1 & 5 & 0 & 5 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Give all solutions to the equation $A\mathbf{x} = \mathbf{0}$, in parametric vector form. Are the columns of A linearly independent?

8. Let $A = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}$, and $C = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -2 \end{pmatrix}$.

Determine whether or not the three vectors listed above are linearly independent or linearly dependent. If they are linearly dependent, determine a non-trivial linear relation.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + 3x_2 + 2x_3, 3x_1 + 8x_2 + 2x_3).$$

Find the standard matrix representation of T. Is T invertible? If so, find T^{-1}.

10. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 3x_3, 6x_1 + x_2 - x_3).$$

Find the standard matrix representation of T. Is T one-to-one? Is T onto?

11. List four equivalent conditions for an $n \times n$ matrix to be invertible.

12. Given

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Calculate: AB, BA, $(AB)^{-1}$ (if it exists),$(BA)^{-1}$ (if it exists).

13. Find the inverse of:

a) $\begin{pmatrix} 1 & 3 \\ 2 & -5 \end{pmatrix}$ **b)** $\begin{pmatrix} 4 & -3 \\ 7 & -9 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 3 & 0 \\ 2 & 7 & -1 \\ 6 & 0 & 4 \end{pmatrix}$ **d)** $\begin{pmatrix} 5 & 2 & 0 \\ -3 & -9 & -1 \\ 2 & 1 & 5 \end{pmatrix}$

14. Book exercises, Section 4.1: 1, 3, 5, 23, 25, 29, 37, 39.

15. Book exercises, Section 3.3: 35, 37, 39.

16. Book exercises, Section 2.3: 49–54.