

MTH 42. Linear Algebra. Midterm exam. Fall 2025.

Professor Luis Fernández

Name: _____

INSTRUCTIONS:

- This exam contains 8 questions, 5 pages for a total of 100 points plus a 10-point bonus question.
- You have 110 minutes to complete the exam.
- **You must show all your work** in order to get credit.
- You can use a non-graphing scientific calculator. No other electronic devices, notes or books are permitted.

1. (18 points) Determine if the following statements are true or false (circle **T** or **F**). You do not need to justify your answer.

- (a) **T** **F** If A, B are $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$.
- (b) **T** **F** If A, B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.
- (c) **T** **F** $(AB)^T = B^T A^T$.
- (d) **T** **F** If A is a square matrix, and the only solution of the equation $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, then $\det(A) \neq 0$.
- (e) **T** **F** If A, B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.
- (f) **T** **F** If A is an $n \times n$ matrix, and $k \in \mathbb{R}$, then $\det(kA) = k^n \det(A)$.

2. (10 points) Multiply $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$

3. (16 points) Find the values of h and k so that the linear system $\begin{cases} x - 3hy &= 2 \\ 2x + hy &= 3k \end{cases}$ has
- (a) No solution.
 - (b) A unique solution.
 - (c) Infinitely many solutions.

4. (16 points) Find the inverse of the matrix $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -4 \\ -1 & -3 & 3 \end{bmatrix}$

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5. (10 points) Find the determinant of the matrix $\begin{bmatrix} 2 & 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$

6. (16 points) Use Cramer's rule to find the solution of the system

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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7. (14 points) When x and y are real numbers, we know that $(x + y)(x - y) = x^2 - y^2$. However, this is not true for matrices:
- (a) Give an example of two 2×2 matrices A and B (not necessarily different) such that $(A + B)(A - B) \neq A^2 - B^2$
 - (b) Find the correct expansion of $(A + B)(A - B)$.

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8. (10 points (bonus)) Let A be an $n \times m$ matrix with $n > m$, and suppose that the $m \times m$ matrix $A^T A$ is invertible. Consider the matrix $B = A(A^T A)^{-1} A^T$.
- (a) Show that $B^2 = B$.
 - (b) Show that $BA = A$.