Determinants summary

Definition: The determinant of a square matrix A is denoted $\det(A)$ or $|A|$. It is defined as follows.

- If A is a 1×1 matrix, say $A = (a_{11})$, then $\det(A) = a_{11}$.
- If A is a 2×2 matrix, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$.
- If A is a 3×3 matrix, say $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

- If A is an $n \times n$ matrix, say $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{31} & a_{32} & \cdots a_{3n} & \end{pmatrix}$, then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n},$$

  where C_{ij} is the ij^{th} *cofactor* of A, defined as $C_{ij} = (-1)^{i+j}\det(M_{ij}$, and where M_{ij} is the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A; it is called the ij^{th} *minor*.

Properties of the determinant

(∗) **1.** $\det(I) = 1$

(∗) 2. If A is a triangular matrix, then $\det(A)$ equals the product of the diagonal entries.

3. A invertible $\iff$ $\det(A) \neq 0$.

4. The definition of determinant ('cofactor expansion') can be done using any row or any column (not only the first one). That is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad \text{(using row } i\text{)}$$
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad \text{(using column } j\text{)}$$

5. $\det(A) = \det(A^T)$

6. If A has a row or a column of zeros, then $\det(A) = 0$.

7. If A has two identical rows or two identical columns, then $\det(A) = 0$.

8. If A and B are $n \times n$ matrices, $\det(AB) = \det(A)\det(B)$.

9. If we interchange two rows or two columns of A, $\det(A)$ gets multiplied by (-1).

10. If we multiply a row or a column of A by a number, $\det(A)$ gets multiplied by that number.

11. If we add a multiple of a row (column) to another row (column), $\det(A)$ does not change.

Exercises:

1. $\begin{vmatrix} 1 & 3 & 5 & -2 \\ -3 & 2 & -1 & 1 \\ 2 & 4 & 3 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix}$

2. $\begin{vmatrix} 6 & 4 & -5 & -2 \\ -3 & 3 & -1 & 1 \\ 4 & 6 & 4 & 0 \\ 1 & 0 & -9 & -3 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 8 & -5 & -2 & 1 \\ -1 & 3 & -0 & 2 & -2 \\ 4 & 2 & 4 & 0 & 2 \\ 1 & 0 & -5 & 2 & 1 \\ 3 & 2 & -1 & 4 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 4 & 4 & -2 \\ 4 & 2 & 1 & 1 \\ 1 & 4 & 2 & 3 \\ 2 & 1 & -3 & 1 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 2 & 3 & -2 \\ -3 & -3 & 1 & 1 \\ 4 & 6 & 4 & -2 \\ 2 & 0 & -1 & 0 \end{vmatrix}$

6. $\begin{vmatrix} 6 & 8 & -5 & -2 & 1 \\ -4 & 5 & -0 & 2 & -3 \\ 4 & -6 & 4 & 1 & 2 \\ 2 & 1 & -5 & 1 & 1 \\ 1 & -2 & 3 & -2 & 3 \end{vmatrix}$